III.12 The Semicontinuity Theorem

Yu-Ting, Huang

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In this section, our goal is to study how does the cohomology along the fiber $H^i(X_y, \mathscr{F}_y)$ vary as a function of $y \in Y$. We will prove the two main results: "semicontinuity theorem" and "cohomology and base change".

Assumption. A: noetherian ring; $Y = \operatorname{Spec} A$; $f : X \to Y$ is a projective morphism; $\mathscr{F}$: coherent sheaf on X, flat over Y.

Definition. For each A-module M, define $T^i(M) = H^i(X, \mathscr{F} \otimes_A M)$ for all $i \geq 0$.
T^i is an additive covariant functor from A-modules to A-modules which is exact in the middle. $(T^i)_{i\geq 0}$ forms a δ-functor.

Lemma 1. *Let $C^\bullet$ be a complex of flat A-modules bounded above such that for each i, $h^i(C^\bullet)$ is a finitely generated A-module. Then there exists a complex $L^\bullet$ of finitely generated free A-modules, also bounded above, and a morphism $g : L^\bullet \to C^\bullet$, such that the induced map $h^i(L^\bullet \otimes M) \to h^i(C^\bullet \otimes M)$ is an isomorphism for all i.*

Proof. We will define $L^\bullet$ by induction. For $n >> 0$,$C^n = 0$, we then define $L^n = 0$. Note that when n is large enough, we have the following two properties.

$$h^i(L^\bullet) \xrightarrow{\sim} h^i(C^\bullet) \text{ for } i > n+1, \text{ and } Z^{n+1}(L^\bullet) \to h^{n+1}(C^\bullet) \text{ is surjective.}$$

Now, suppose the above two properties is satisfied for some n, and we will construct L^n, $d : L^n \to L^{n+1}$, and $g : L^n \to C^n$ to propagate these properties one step further.
Let $\overline{x}_1, \ldots \overline{x}_r$ be a set of generators of $h^n(C^\bullet)$. Lift them to $x_1, \ldots x_n \in Z^n(C^\bullet)$. Let $y_{r+1}, \ldots, y_s$ be a set of generators of $g^{-1}(B^{n+1}(C^\bullet))$, and let $g(y_i) = \overline{y}_i \in B^{n+1}(C^\bullet)$. Lift $\overline{y}_i$ to $x_{r+1}, \ldots, x_s \in C^n$.

Now take L^n to be a free A-module generated by $e_1, \ldots, e_s$, where $ge_i = x_i$ for all i. Define

$d : L^n \to L^{n+1}$ by $de_i = 0$ for $1 \le i \le r$ and $de_i = y_i$ for $r+1 \le i \le s$. With this construction, g commutes with d, $h^{n+1}(L^\bullet) \xrightarrow{\sim} h^{n+1}(C^\bullet)$, and $Z^n(L^\bullet) \twoheadrightarrow h^n(C^\bullet)$.

Next, we will prove that

$$h^i(L^\bullet \otimes M) \to h^i(C^\bullet \otimes M)$$

is an isomorphism for all A-modules M. It suffices to prove for finitely generated A-module, since any module is a direct limit of finitely generated ones. So given a finitely generated A-module M, we have the exact sequence:

$$0 \to R \to E \to M \to 0,$$

where E is free, and $R = \ker(E \to M)$.

Since both $C^\bullet$ and $L^\bullet$ are flat, we have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^\bullet \otimes R & \longrightarrow & L^\bullet \otimes E & \longrightarrow & L^\bullet \otimes M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^\bullet \otimes R & \longrightarrow & C^\bullet \otimes E & \longrightarrow & C^\bullet \otimes M & \longrightarrow & 0 \end{array}$$

Applying h^i, we get the commutative diagram of long exact sequences:

$$\begin{array}{ccccccccc} h^i(L^\bullet \otimes R) & \longrightarrow & h^i(L^\bullet \otimes E) & \longrightarrow & h^i(L^\bullet \otimes M) & \longrightarrow & h^{i+1}(L^\bullet \otimes R) & \longrightarrow & h^{i+1}(C^\bullet \otimes E) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ h^i(C^\bullet \otimes R) & \longrightarrow & h^i(C^\bullet \otimes E) & \longrightarrow & h^i(C^\bullet \otimes M) & \longrightarrow & h^{i+1}(C^\bullet \otimes R) & \longrightarrow & h^{i+1}(C^\bullet \otimes E) \end{array}$$

Suppose the result is true for $i+1$, then the first and second vertical arrows on the right are isomorphisms. Since E is free, $h^i(L^\bullet \otimes E) \xrightarrow{\sim} h^i(C^\bullet \otimes E)$. By four lemma, $h^i(L^\bullet \otimes M) \to h^i(C^\bullet \otimes M)$ is an epimorphism, then so is $h^i(L^\bullet \otimes R) \to h^i(C^\bullet \otimes R)$. Now, by five lemma, we can conclude that $h^i(L^\bullet \otimes M) \xrightarrow{\sim} h^i(C^\bullet \otimes M)$. The result follows from the induction. □

Proposition 1. *There exists a complex $L^\bullet$ of finitely generated free A-modules, bounded above such that*

$$T^i(M) \simeq h^i(L^\bullet \otimes M)$$

for any A-module M, any $i \ge 0$, and this gives an isomorphism of δ-functors.

Proof. By our assumptions, $\mathscr{F} \otimes M$ is quasi-coherent on X, and X is noetherian and separated, therefore, we can use Čech cohomology to compute $H^i(X, \mathscr{F} \otimes M)$. Let $\mathfrak{U} = (U_i)$ be an open cover of X. For any $i_0, \ldots, i_p$, we have

$$\Gamma(U_{i_0 \ldots i_p}, \mathscr{F} \otimes M) = \Gamma(U_{i_0 \ldots i_p}, \mathscr{F}) \otimes M.$$

so

$$C^\bullet(\mathfrak{U}, \mathscr{F} \otimes M) = C^\bullet(\mathfrak{U}, \mathscr{F}) \otimes M.$$

Hence,

$$T^i(M) = h^i(C^\bullet(\mathscr{F} \otimes M)) = h^i(C^\bullet(\mathscr{F}) \otimes M).$$

Note that $C^\bullet := C^\bullet(\mathscr{F})$ is bounded above, and $C^\bullet = H^i(X, \mathscr{F})$ is a flat and finitely generated A-module. Then the result follows from the previous lemma. □

Definition. For a complex $N^\bullet$, define

$$W^i(N^\bullet) = \operatorname{coker}(d^{i-1} : N^{i-1} \to N^i).$$

Proposition 2. *The followings are equivalent:*

(i) T^i is left exact;

(ii) $W^i := W^i(L^\bullet)$ is a projective A-module;

(iii) there is a unique finitely generated A-module Q, such that

$$T^i(M) = \operatorname{Hom}_A(Q, M)$$

for all M.

Proof. (i)⇔(ii)

For any A-module M, we have exact sequences:

$$L^{i-1} \to L^i \to W^i \to 0 \text{ ,then } L^{i-1} \otimes M \to L^i \otimes M \to W^i \otimes M \to 0,$$

and

$$L^{i-1} \otimes M \to L^i \otimes M \to W^i(L^\bullet \otimes M) \to 0.$$

This implies $W^i \otimes M = W^i(L^\bullet \otimes M)$. Now, we have an exact sequence:

Suppose $M' \to M$ is injective, then we have the following exact, commutative diagram:

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \downarrow \\
0 & \longrightarrow & T^i(M') & \longrightarrow & W^i \otimes M' & \longrightarrow & L^{i+1} \otimes M' \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & T^i(M) & \longrightarrow & W^i \otimes M & \longrightarrow & L^{i+1} \otimes M
\end{array}$$

Since L^{i+1} is free, the third vertical arrow is injective. By diagram chasing, T^i is left exact if and only if W^i is flat. But since W^i is finitely generated, this is equivalent to that W^i is projective.

(iii)$\Rightarrow$(i) is obvious.

(ii)$\Rightarrow$(iii)

Let $\check{L}^{i+1}$ and $\check{W}^i$ be dual modules. Define $Q := \operatorname{coker}(\check{L}^{i+1} \to \check{W}^i)$, which is a finitely generated A-module. Then for every A-module M, we have an exact sequence

$$0 \to \operatorname{Hom}(Q, M) \to \operatorname{Hom}(\check{W}^i, M) \to \operatorname{Hom}(\check{L}^{i+1}, M).$$

Since W^i is projective and L^{i+1} is free, $\operatorname{Hom}(\check{W}^i, M) = W^i \otimes M$ and $\operatorname{Hom}(\check{L}^{i+1}, M) = L^{i+1} \otimes M$. Hence,

$$T^i(M) = \operatorname{Hom}(Q, M).$$

To see the uniqueness, let Q' be another A-module such that $T^i(M) = \operatorname{Hom}(Q', M)$. Then

$$1 \in \operatorname{Hom}(Q, Q) = \operatorname{Hom}(Q', Q),$$

and

$$1' \in \operatorname{Hom}(Q', Q') = \operatorname{Hom}(Q, Q').$$

This gives an isomorphism of Q and Q'. $\square$

Proposition 3. *For any M, there is a natural map*

$$\varphi : T^i(A) \otimes M \to T^i(M).$$

Furthermore, the following conditions are equivalent:

(i) T^i is right exact;

(ii) φ is an isomorphism for all M;

(iii) φ is surjective for all M.

Proof. For any module M, since T^i is a functor, we have a natural map,

$$M = \mathrm{Hom}(A, M) \xrightarrow{\psi} \mathrm{Hom}(T^i(A), T^i(M))$$

We define φ by

$$\varphi(\sum a_i \otimes m_i) = \sum \psi(m_i)a_i,$$

for $a_i \otimes m_i \in T^i(A) \otimes M$.

(i)⇒(ii)

Again, it suffices to consider finitely generated A-modules M. Then we have an exact sequence:

$$A^s \to A^r \to M \to 0,$$

where A^r and A^s are free A-module

Suppose T^i is right exact, then we have a diagram:

$$\begin{array}{ccccccc} T^i(A) \otimes A^s & \longrightarrow & T^i(A) \otimes A^r & \longrightarrow & T^i(A) \otimes M & \longrightarrow & 0 \\ \downarrow \wr & & \downarrow \wr & & \downarrow \varphi & & \\ T^i(A^s) & \longrightarrow & T^i(A^r) & \longrightarrow & T^i(M) & \longrightarrow & 0 \end{array}$$

Thus, φ is an isomorphism.

(ii)⇒(iii) is obvious.

(iii)⇒(i)

Suppose φ is surjective for all A-modules. Let $M \twoheadrightarrow M''$ be a surjective map. Then we have the following diagram:

$$\begin{array}{ccccc} T^i(A) \otimes M & \longrightarrow & T^i(A) \otimes M' & \longrightarrow & 0 \\ \downarrow \varphi & & \downarrow \varphi & & \\ T^i(M) & \longrightarrow & T^i(M') & & \end{array}$$

This implies $T^i(M) \twoheadrightarrow T^i(M'')$. Hence, T^i is right exact. □

Corollary 1. *The following conditions are equivalent:*

1. *T^i is exact;*

2. T^i is right exact, and $T^i(A)$ is a projective A-module.

Proof. For T^i is right exact, by the previous proposition, we have $T^(A) \otimes M \simeq T^i(M)$ for any M. Therefore, T^i is left exact is and only if $T^i(A)$ is flat. Since $T^i(A)$ is finitely generated, this is equivalent to that $T^i(A)$ is projective. □

Theorem 1. (Semicontinuity) *Let $f : X \to Y$ be a projective morphism of noetherian schemes, $\mathscr{F}$ be a coherent sheaf on X which is flat over Y. Then for $i \geq 0$, the function*

$$h^i(y, \mathscr{F}) = \dim_{k(y)} H^i(X_y, \mathscr{F}_y)$$

is an upper semicontinuous function on Y.

Proof. We may assume that $Y = \operatorname{Spec} A$ with A noetherian. By Corollary III.9.4,

$$H^i(X_y, \mathscr{F}_y) \simeq H^i(X, \mathscr{F} \otimes k(y)) = T^i(k(y)).$$

Note that

$$0 \to T^i(k(y)) \to W^i \otimes k(y) \to L^{I+1} \otimes k(y)$$

and

$$W^i \to L^{i+1} \to W^{i+1} \to 0$$

are exact. So we have an exact sequence:

$$0 \to T^i(k(y)) \to W^i \otimes k(y) \to L^{i+1} \otimes k(y) \to W^{i+1} \otimes k(y) \to 0.$$

Then

$$\begin{aligned} h^i(y, \mathscr{F}) &= \dim_{k(y)} T^i(k(y)) \\ &= \dim_{k(y)}(W^i \otimes k(y)) + \dim_{k(y)}(W^{i+1} \otimes k(y)) - \dim_{k(y)}(L^{i+1} \otimes k(y)). \quad (\star) \end{aligned}$$

By Proposition 5, the first two terms are upper semicontinuous, and the last term is constant since L^{i+1} is free. We then can conclude that $h^i(y, \mathscr{F})$ is upper semicontinuous. □

Corollary 2. (Grauert) *With the same hypotheses as the theorem, suppose that Y is integral, and $h^i(y, \mathscr{F})$ is constant on Y for some i. Then $R^i f_*(\mathscr{F})$ is locally free and the natural map*

$$R^i f_*(\mathscr{F}) \otimes k(y) \to H^i(X_y, \mathscr{F}_y)^\sim$$

is an isomorphism.

Proof. Again, we may assume $Y = \operatorname{Spec} A$ is affine. By our assumption, we can see that $\dim_{k(y)}(W^i \otimes k(y)) + \dim_{k(y)}(W^{i+1} \otimes k(y))$ is constant on Y from ($\star$), and since $\dim_{k(y)}(W^i \otimes k(y))$ and $\dim_{k(y)}(W^{i+1} \otimes k(y))$ are upper semicontinuous, $\dim_{k(y)}(W^i \otimes k(y))$ and $\dim_{k(y)}(W^{i+1} \otimes k(y))$ are constant on Y. By II.8.9, $\widetilde{W^i}$ and $\widetilde{W^{i+1}}$ are locally free on Y, so W^i and W^{i+1} are projective. By Proposition 2, T^i and T^{i+1} are left exact, then T^i is exact. By Corollary 1, $T^i(A)$ is a projective A-module. So $R^i f_*(\mathscr{F}) = T^i(A)$ is a locally free sheaf. By Proposition 3, we have the isomorphism

$$R^i f_*(\mathscr{F}) \otimes k(y) = T^i(A)^\sim \otimes k(y) \xrightarrow{\sim} T^i(k(y))^\sim = H^i(X_y, \mathscr{F}_y)$$

for all $y \in Y$. □

Definition.

1. For any $y \in Y = \operatorname{Spec} A$, define T^i_y as the restriction of the functor T^i to the category of $A_\mathfrak{p}$-modules, where $\mathfrak{p} \subseteq A$ is the prime ideal corresponding to y. Namely, for any $A_\mathfrak{p}$ module N, $T^i_y(N) = h^i(L^\bullet_\mathfrak{p} \otimes N)$.

2. For $y \in Y$, denote $W^i_y = W^i(L^\bullet_\mathfrak{p})$.

3. We say "T^i is left exact at y" to mean T^i_y is left exact, and similarly for right exact and exact.

Here, we recall the results about "cohomology commutes with flat base extension."

Proposition 9.2. ***(b)*** *Let $X \to Y$, let $\mathscr{F}$ be an $\mathscr{O}_X$-module which is flat over Y, and let $u : Y' \to Y$ be any morphism. Let $X' = X \times_Y Y'$, and let $\mathscr{F}' = v^*(\mathscr{F})$. Then $\mathscr{F}'$ is flat over Y'.*

Proposition 9.3 *Let $f : X \to Y$ be a separated morphism of finite type of noetherian schemes, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Let $u : Y' \to Y$ be a flat morphism of noetherian schemes.*

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ \downarrow g & & \downarrow f \\ Y' & \xrightarrow{u} & Y \end{array}$$

Then for all $i \geq 0$ there are natural isomorphisms $u^* R^i f_*(\mathscr{F}) \simeq R^i g_*(v^* \mathscr{F})$.

Now, with our original assumption and let $Y' = \operatorname{Spec} \mathscr{O}_y$ for some $u \in Y$, $\mathfrak{p} \subseteq A$ is a prime ideal corresponding to y, and $\mathscr{F}' = v^* \mathscr{F}$. By Proposition 9.2 (b), $\mathscr{F}'$ is flat over Y', and by Proposition 9.3,

$$T_y^i(M) = H^i(X, \mathscr{F} \otimes M)_{\mathfrak{p}} = \Gamma(Y', u^* R' f_*(\mathscr{F} \otimes M)) = \Gamma(Y', R^i g_*(v^* \mathscr{F} \otimes M)) = H^i(X', \mathscr{F}' \otimes M),$$

for every $A_{\mathfrak{p}}$-module M. Hence, we can apply the previous results that we prove for T^i to T_y^i.

Proposition 4. *If T^i is left exact (respectively, right exact, exact) at some point $y \in Y$, then the same is true for all points y in a suitable open neighborhood U of y.*

Proof.

(i) Note that over a local ring, a module is projective is equivalent to it is free. Therefore, T^i is left exact at y is equivalent to $\widetilde{W}_y^i$ is free. Since $\widetilde{W}^i$ is a coherent sheaf on Y, then there exists an open neighborhood U of Y such that $\widetilde{W}^i|_U$ is free (Ex.II.5.7). This implies T^i is exact at all points of U.

(ii) Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of $A_{\mathfrak{p}}$-modules. We have the long exact sequence for δ-functor T_y^i:

$$\to T_y^i(M) \to T_y^i(M'') \to T_y^{i+1}(M') \to T_y^{i+1}(M) \to .$$

Thus, T_y^i is right exact if and only if T_y^{i+1} is left exact. Then the result follows from (i).

□

Definition. Let Y be a topological space. A function $\varphi : Y \to \mathbf{Z}$ is upper semicontinuous if for each $y \in Y$, there exists an open neighborhood U such that for all $y' \in U$, $\varphi(y') \leq \varphi(y)$.

Proposition 5. *Let $\mathscr{F}$ be a coherent sheaf on a noetherian scheme Y. Then the function*

$$\varphi(y) = \dim_{k(y)}(\mathscr{F}_y \otimes k(y))$$

is upper semicontinuous. (cf. Ex.II.5.8)

Proof. Since the property is local, it can be reduced to the affine case. Let $Y = \operatorname{Spec} A$, and $\mathscr{F} = \widetilde{M}$ for some finitely generated A-module M. For $y \in Y$, suppose that the prime ideal

$\mathfrak{p} \subseteq A$ is corresponding to y. Note that $\mathscr{F}_y \otimes k(y) = \mathscr{F}_y / m_y \mathscr{F}_y$. By Nakayama's lemma, φ is equal to the minimal number of generators of $A_\mathfrak{p}$-module $M_\mathfrak{p}$. Suppose $\varphi(y) = r$ with generators $s_1, \ldots, s_r \in M_\mathfrak{p}$. They then generate M. Thus, for every $y' \in Y$, corresponding to prime ideal $\mathfrak{q} \subseteq A$, $s_1, \ldots, s_r$ also generate $A_\mathfrak{q}$ as an $A_\mathfrak{q}$-module. This implies $\varphi(y') \leq \varphi(y)$. $\square$

Proposition 6. Assume that

$$\varphi : T^i(A) \otimes k(y) \to T^i(k(y))$$

is surjective for some i, y. Then T^i is right exact (and conversely, proved in proposition 3).

Proof. By making a flat base change $\operatorname{Spec} \mathscr{O}_y \to Y$ if necessary, we may assume $y \in Y$ is a closed point, A is a local ring, and $k(y) = A/\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal in A. By Proposition 3, it suffices to show that $\varphi : T^i(A) \otimes M \to T^i(M)$ is surjective for all A-module M, and we may assume M is finitely generated.

First, we consider M with finite length and we will prove by induction. For $length(M) = 1$, i.e. $M = k(y)$, φ is surjective by our assumption. For general M, let

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence, where $length(M'), length(M'') < length(M)$.

Consider the following diagram:

$$\begin{array}{ccccccc} T^i(A) \otimes M' & \longrightarrow & T^i(A) \otimes M & \longrightarrow & T^i(A) \otimes M'' & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ T^i(M') & \longrightarrow & T^i(M) & \longrightarrow & T^i(M'') & & \end{array}$$

By the induction hypothesis, the two outside vertical arrows are surjective, so the middle one is surjective also. Now, let M be any finitely generated A-module. For each n, $M/\mathfrak{m}^n M$ is with finite length. So we have

$$\varphi_n : T^i(A) \otimes M/\mathfrak{m}^n M \to T^i(M/\mathfrak{m}^n M)$$

is surjective. $\ker \varphi_n$ is also with finite length, so $(\ker \varphi_n)$ is an inverse system of modules with descending chain condition over a ring. Then by Example 9.1.2, $(\ker \varphi_n)$ satisfies the Mittag-Leffler condition. By II.9.1,

$$(T^i(A) \otimes M)^\wedge = \varprojlim (T^i(A) \otimes M/\mathfrak{m}^n M) \twoheadrightarrow \varprojlim T^i(M/\mathfrak{m}^n M).$$

By the theorem on formal functions (III.11.1),

$$T^i(M)^\wedge \xrightarrow{\sim} \lim_{\leftarrow} T^i(M/\mathfrak{m}^n M).$$

Since completion is a faithful exact functor for finitely generated A-modules, then

$$\varphi : T^i(A) \otimes M \to T^i(M)$$

is surjective. □

Theorem 2. (Cohomology and Base Change) With the hypotheses above, let y be a point of Y, then

(a) If $\varphi^i(y) : R^i f_*(\mathscr{F}) \otimes k(y) \to H^i(X_y, \mathscr{F}_y)$ is surjective, then it is an isomorphism, and the same is true for all y' in a suitable neighborhood of y.

(b) Assume $\varphi^i(y)$ is surjective. The following are equvalent:

  (i) $\varphi^{i-1}(y)$ is also surjective;

  (ii) $R^i f_*(\mathscr{F})$ is locally free in a neighborhood of y.

Proof.

(a) By proposition 6, T^i_y is right exact. Furthermore, there is a neighborhood U of y such that $T^i_{y'}$ is exact for all $y' \in U$. Then $\varphi^i(y')$ is an isomorphism for all $y' \in U$.

(b) Again, by proposition 6, T^i_y is right exact. By proposition 3 and 6, $\varphi^{i-1}(y)$ is surjective if and only if T^{i-1}_y is right exact. Then T^i_y is exact, so T^i is exact at some neighborhood U of y. This is equivalent to that $T^i_y(A)$ is projective. i.e. (ii).

□