10 SMOOTH MORPHISMS

Recall. (Flat)
Let $f : X \to Y$ be a morphism of schemes, and let $\mathcal{F}$ be an $\mathcal{O}_x$- module. We say that $\mathcal{F}$ is flat over Y at a point $x \in X$, if the stalk at x is a flat $\mathcal{O}_y$-module, where $y = f(x)$. We say simply $\mathcal{F}$ is flat over Y if it is flat at every point of X.
We say X is flat over Y if $\mathcal{O}_X$ is.

Definition. (Smooth)
Let $f : X \to Y$ be a morphism of schemes of finite type over a field k, we say it's smooth of relative dimension n if:

(1) f is flat.
(2) If $X' \subseteq X, Y' \subseteq Y$ are irreducible components such that $f(X') \subseteq Y'$, then $\dim X' = \dim Y' + n$.
(3) For each point $x \in X$ (closed or not),

$$\dim_{k(x)}(\Omega_{X/Y} \otimes k(x)) = n$$

Proposition. 0.1 If X is integral, (3) is equivalent to $\Omega_{X/Y}$ is locally free of rank n.
Recall **(II 8.9)**: Let A be a noetherian local integral domain, with residue field k and quotient field K. If M is a finitely generated A-module and if $\dim_k M \otimes_A k = \dim_K M \otimes_A K = r$, then M is free of rank r.

Proposition. 0.2 If $Y = \operatorname{Spec} k, k = \overline{k}$, then X is smooth over k if and only if X is regular of dimension n.
Recall **(II 8.8)**: Let B be a local ring containing a field k isomorphic to its residue field. Assume furthermore that k is perfect, and that B is a localization of a finitely generated k-algebra. Then

$$\Omega_{B/k} \text{ is a free } B\text{-module of rank } \dim B \Longleftrightarrow B \text{ is a regular local ring.}$$

In particular, if X is irreducible separated over k, then it's smooth if and only if it's a nonsingular variety.
Recall **(II 8.15)**: Let X be an irreducible separated scheme of finite type over an algebraically closed field k. Then

$$\Omega_{X/k} \text{ is locally free of rank } \dim X \Longleftrightarrow X \text{ is a nonsingular variety over } k.$$

Remark. Over a nonperfect field, the last proposition is false: let k_0 be a field of characteristic $p > 0$, let $k = k_0(t)$, and let $X \subseteq A_k^2$ be the curve defined by $y^2 = x^p - t$, then every local ring of X is a regular local ring, but X is not smooth over k.

Proposition. 1

(a) An open immersion is smooth of relative dimension 0.

(b) If $f : X \to Y$ is smooth of relative dimension n, and $g : Y' \to Y$ is any morphism, then the morphism $f' : X' \to Y'$ obtained by base extension is also smooth of relative dimension n.
(c) If $f : X \to Y$ is smooth of relative dimension n, and $g : Y \to Z$ is smooth of relative dimension m, then $g \circ f : X \to Z$ is smooth of relative dimension $n+m$.
(d) If $f : X \to Z$ is smooth of relative dimension n, and $g : Y \to Z$ is smooth of relative dimension m, then $X \times_Z Y \to Z$ is smooth of relative dimension $n+m$.

Proof. (a) Just check the condition.

(b) (1) By **(III 9.2)**, f' is flat.
(2) Recall **(III 9.6)**: Let $f : X \to Y$ be a flat morphism of schemes of finite type over a field k, and assume that Y is irreducible. Then the following conditions are equivalent:
(i) every irreducible component of X has dimension equal to $\dim Y + n$.
(ii) for any point $y \in Y$ (closed or not), every irreducible component of the fibre X_y has dimension n.

In this case, (2) is equivalent to say every irreducible component of every fibre X_y of f' has dimension n, which, by **(II Ex. 3.20)**, is preserved under base change.
(3) By **(II 8.10)**, $\Omega_{X/Y}$ is stable under base extension, thus (3) holds for f'.

(c) (1) By **(III 9.2)**.
(2) If $X' \subseteq X, Y' \subseteq Y, Z' \subseteq Z$ are irreducible components such that $f(X') \subseteq Y', g(Y') \subseteq Z'$, then $\dim X' = \dim Y' + n = \dim Z' + n + m$.
(3) Use the exact sequence in **(II 8.11)**

$$f^*\Omega_{Y/Z} \longrightarrow \Omega_{X/Z} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

Tensoring $k(x)$ to get:

$$f^*\Omega_{Y/Z} \otimes k(x) \longrightarrow \Omega_{X/Z} \otimes k(x) \longrightarrow \Omega_{X/Y} \otimes k(x) \longrightarrow 0$$

Then $\dim \Omega_{X/Z} \otimes k(x) \leq \dim f^*\Omega_{Y/Z} \otimes k(x) + \dim \Omega_{X/Y} \otimes k(x) = n + m$.
On the other hand, let $z = g(f(x))$, we have $\Omega_{X/Z} \otimes k(x) = \Omega_{X_z/k(z)} \otimes k(x)$, let X' be an irreducible component of X_z containing x, with its reduced induced structure. Then by **(II 8.12)**,

$$\Omega_{X_z/k(z)} \otimes k(x) \longrightarrow \Omega_{X'/k(z)} \otimes k(x)$$

is surjective.
But again by **(III 9.6)**, X' is an integral scheme of finite type over $k(z)$ of dimension $n+m$, thus by **(II 8.6A)**, we see $\Omega_{X'/k(z)}$ is locally generated by at least $n+m$ elements. Combining above, (3) is verified.

(d) Use (b), (c).

Theorem. 2 Let $f : X \to Y$ be a morphism of schemes of finite type over a field k, then f is smooth of relative dimension n if and only if:

(i) f is flat.
(ii) The fibres of f are geometrically regular of equidimension n, i.e., for point $y \in Y$, let $X_{\overline{y}} = X_y \otimes_{k(y)} \overline{k(y)}$, then $X_{\overline{y}}$ is equidimensional of dimension n and regular.

Proof. ($\Rightarrow$) By **Proposition (0.2),(1(b))**.
($\Leftarrow$) (1) By assumption.
(2) By **(III 9.6)**.
(3) By **Proposition (0.2)**, regularity implies $\Omega_{X_{\overline{y}}/\overline{k(y)}}$ is locally free of rank n. By the fact states below, it is equivalent to $\Omega_{X_y/k(y)}$ is locally free of rank n, thereby (3) holds.

Fact. If M is an A-module with A being a local ring, B is a faithfully flat A-algebra, then

$$M \text{ is free} \Longleftrightarrow M \otimes_A B \text{ is free.}$$

Recall. The Zaraski tangent space T_x for a point in a scheme X be the dual of k-vector space m_x/m_x^2. If $f : X \to Y$ is a morphism, $y = f(x)$, then there is a natural induced mapping

$$T_f : T_x \to T_y \otimes_{k(y)} k(x)$$

Lemma. 3-1 Suppose that $R \to S$ is a local homomorphism of Noetherian local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Let M be a flat R-module and N a finite S-module. Let $u : N \to M$ be a map of R-modules. If $u_1 : N/\mathfrak{m}N \to M/\mathfrak{m}M$ is injective then u is injective. In this case $M/u(N)$ is flat over R.

Proof. Claim: $u_k : N/\mathfrak{m}^k N \to M/\mathfrak{m}^k M$ is injective $\forall\, k \in \mathbb{N}$.
The case $k = 1$ is just the assumption. To prove by induction, assuming the case $k = n$ holds. Consider the diagram:

$$\begin{array}{ccccccccc}
 & & M/\mathfrak{m}M \otimes_{R/\mathfrak{m}} \mathfrak{m}^n/\mathfrak{m}^{n+1} & & & & & & \\
 & & || & & & & & & \\
0 & \to & M \otimes_R \mathfrak{m}^n/\mathfrak{m}^{n+1} & \to & M/\mathfrak{m}^{n+1}M & \to & M/\mathfrak{m}^n M & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & N \otimes_R \mathfrak{m}^n/\mathfrak{m}^{n+1} & \to & N/\mathfrak{m}^{n+1}N & \to & N/\mathfrak{m}^n N & \to & 0 \\
 & & || & & & & & & \\
 & & N/\mathfrak{m}N \otimes_{R/\mathfrak{m}} \mathfrak{m}^n/\mathfrak{m}^{n+1} & & & & & &
\end{array}$$

The first and the last map are injective, which imply the middle one is also injective. Therefore the claim is proved by induction.

By Krull's intersection theorem , $\bigcap \mathfrak{m}^n N = 0$, thus the injectivity of $u_n \forall\, n \in \mathbb{N}$ implies u is injective.

To show that $M/u(N)$ is flat over R, it suffices to show that $I \otimes_R M/u(N) \to M/u(N)$ is injective for every ideal I in R. Consider the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & N/IN & \to & M/IM & \to & M/(IM + u(N)) & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & N & \to & M & \to & M/u(N) & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & N \otimes_R I & \to & M \otimes_R I & \to & M/u(N) \otimes_R I & \to & 0
\end{array}$$

$M \otimes_R I \to M$ is injective, then by the snake lemma, it suffices to prove that N/IN injects into M/IM. Note that $R/I \to S/IS$ is a local homomorphism of Noetherian local rings, $N/IN \to M/IM$ is a map of R/I-modules, N/IN is finite over S/IS, and M/IM is flat over R/I and $\overline{u} : N/IN \to M/IM$is injective modulo $\mathfrak{m}$. Thus we may apply the first part of the proof to $\overline{u}$ to conclude.

Definition. Let A be a commutative ring, I be an ideal of A. An A-module M is ideally Hausdorff w.r.t. J if $\forall\, a$ is a finitely generated ideal of A, $a \otimes_A M$ is Hausdorff with J-adic topology.

Lemma. 3-2 Let A be a commutative ring, I, J be ideals of A, M be an A-module, $gr(A)$ be the graded ring with I-adic filtration, $gr(M)$ be the graded $gr(A)$-module associated with M with I-adic topology. Consider the following:

(i) M is a flat A-module.
(ii) $\mathrm{Tor}_1^A(N, M) = 0\, \forall N$ is an A-module annihilated by J.
(iii) M/IM is a flat A/I-module and the canonical map $I \otimes_A M \to IM$ is bijective.
(iv) M/IM is a flat A/I-module and canonical homomorphism $r : gr(A) \otimes_{gr_0(A)} gr_0(M)$ is bijective.
(v) $\forall\, n \in \mathbb{N}, M/I^n$ is a flat A/I^n-module.

Then (i)⇒(ii)⇔(iii)⇒(iv)⇔(v). Furthermore, if A is noetherian and M is ideally Hausdorff, all of them are equivalent.

Proof.

(i)⇒(ii)⇔(iii) Omitted.

(ii)⇔(ii)' (ii)':$\mathrm{Tor}_1^A(N, M) = 0\, \forall\, N$ is an A-module annihilated by some power of J.
(⇐) is trivial, so let's focus on (⇒).
In particular $\mathrm{Tor}_1^A(I^n N/I^{n+1}N, M) = 0\, \forall\, n \in \mathbb{N}$. Then the exact sequence

$$0 \to I^{n+1}N \to I^n N \to I^n N/I^{n+1}N \to 0$$

which induces

$$\mathrm{Tor}_1^A(I^{n+1}N, M) \to \mathrm{Tor}_1^A(I^n N, M) \to \mathrm{Tor}_1^A(I^n N/I^{n+1}N, M)$$

Notice that $\exists\, m \in \mathbb{N}$ s.t. $I^m N = 0$, using descending induction, we have $\mathrm{Tor}_1^A(I^n N, M) = 0\, \forall\, n \le m$, especially $m = 0$.

(ii)⇒(iv) First we have a lemma:
(a) $\mathrm{Tor}_1^A(A/I^n, M) = 0\, \forall\, n \in \mathbb{N}$.
(b) The canonical homomorphism $\theta_n : I^n \otimes_A M \to I^n M$ is bijective.
(c) $r : gr(A) \otimes_{gr_0(A)} gr_0(M)$ is bijective.
Then (a)⇔(b)⇒(c), (b)⇐(c) when I is nilpotent.
Proof. (a)⇔(b) by the exact sequence

$$0 = \mathrm{Tor}_1^A(A/I, M) \to \mathrm{Tor}_1^A(A/I^n, M) \to I^n \otimes M \to M$$

Consider the diagram:

$$\begin{array}{ccccccccc} & & I^{n+1} \otimes_A M & \to & I^n \otimes_A M & \to & I^n/I^{n+1} \otimes_{A/I} M/IM & \to & 0 \\ & & \downarrow \theta_{n+1} & & \downarrow \theta_n & & \downarrow r_n & & \\ 0 & \to & I^{n+1}M & \to & I^n M & \to & gr_n(M) & & \end{array}$$

Where r_n is the canonical surjective map. It's commutative by the definition of r_n. In this case, θ_n is bijective, thus r_n is bijective. For the second statement, notice that $I^n \otimes_A M = I^n M = 0$ for some n, thus using descending induction we see it's true.

Thus by lemma, (ii)⇒(ii)'⇒ r is bijective, also (ii)⇒(iii), a fortiori M/IM is a flat A/I-module, thereby (iv) stands.
Also, (iii) ⇐(iv) when I is nilpotent is also shown by the above process.

(iv)⇔(v) Observe $gr_m(M/I^nM) = \begin{cases} gr_m(M), m < n \\ 0, m \geq n \end{cases}$ For $k \in \mathbb{N}$, let $(\text{iv})_k$ and $(\text{v})_k$ be the statement replacing A, I, M by $A/I^kA, I/I^k, M/I^kM$. Obviously

$$(\text{iv}) \Rightarrow (\text{iv})_k \, \forall \, k \in \mathbb{N}$$
$$(\text{v}) \Rightarrow (\text{v})_k \, \forall \, k \in \mathbb{N}$$

Thus it's sufficient to show that $(\text{iv})_k \Rightarrow (\text{v})_k \forall \, k \in \mathbb{N}$ or also (iv)⇔(v) for the case I is nilpotent, which will be assumed below. Since $M/I^nM \cong M \otimes_A A/I^n$, we know (v) ⇒ (i), more apparently, (i) ⇒ (v), so the statement is proved.

(v)⇒(i) (For the case A is noetherian and M is ideally Hausdorff) It's proved in **Lemma 3-1**.

With above we now conclude:

Lemma. 3 Let $A \to B$ be a local homomorphism of local noetherian rings. Let M be a finitely generated B-module, and let $t \in A$ be a nonunit that is not a zero divisor. Then M is flat over A if and only if t is not a zero divisor in M and M/tM is flat over A/tA.

Proof. Take $I = J = (t)$ in **Lemma 3-1, 3-2**.

Proposition. 4 Let $f : X \to Y$ be a morphism of nonsingular varieties over an algebraically closed field k. Let $n = \dim X - \dim Y$. Then the following are equivalent:

(i) f is smooth of relative dimension n.
(ii) $\Omega_{X/Y}$ is locally free of rank n on X.
(iii) For every closed point $x \in X$, $T_f : T_x \to T_y \otimes_{k(y)} k(x)$ is surjective.

Proof.

(i)⇒(ii) By **Proposition (0.2)**.

(ii)⇒(iii) Use the exact sequence in **(II 8.11)** tensoring $k(x) \cong k$(since x is a closed point), we have:

$$f^*\Omega_{Y/k} \otimes k(x) \xrightarrow{\phi} \Omega_{X/k} \otimes k(x) \longrightarrow \Omega_{X/Y} \otimes k(x) \longrightarrow 0$$

since the dimension of each terms are $\dim Y, \dim X, n$, ϕ is actually an injection.
Recall **(II 8.7)**: Let (B, m) be a local ring with a field $k \cong B/m$ containing in B. Then

$$m \to m/m^2 \xrightarrow{\mathfrak{d}} \Omega_{B/k} \otimes_B k$$
$$b \longmapsto \overline{b} \longmapsto db \otimes 1$$

is an isomorphism.
So ϕ is actually $m_y/m_y^2 \hookrightarrow m_x/m_x^2$, take dual to get our result.

(iii)$\Rightarrow$(i) Recall **(III 9.1A(d))**: M is an A-module, M is flat over A if and only if M_p is flat over $A_p \,\forall\, p \in \operatorname{Spec} A$. So to prove f is flat, we only have to verify $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y} \,\forall\, x \in X$ is a closed point, $y = f(x)$.
Since X, Y both nonsingular, $\mathcal{O}_x, \mathcal{O}_y$ are regular local rings. Also by taking dual on T_f, $m_y/m_y^2 \hookrightarrow m_x/m_x^2$ is injective.
Take a regular system of parameters of $\mathcal{O}_y : (t) = (t_1, \ldots, t_r)$, then (t) forms a part of regular system of parameters of $\mathcal{O}_x$. Since $\mathcal{O}_x/(t)$ is flat over $\mathcal{O}_y/(t) \cong k$, by **Lemma 3** and induction, $\mathcal{O}_x/(t_1, \ldots, t_i)$ is flat over $\mathcal{O}_y/(t_1, \ldots, t_i) \,\forall\, i = 0\; r$, in particular it's already proved f is flat. Proceed similar to (ii)$\Rightarrow$(iii) but backward, we see for x is a closed point,

$$\dim_{k(x)}(\Omega_{X/Y} \otimes k(x)) = n$$

On the other hand, f is flat, by **(III Ex 9.1)**, is also dominant, by **(II 8.6A)**, the generic point ξ has the property:

$$\dim_{k(x)}(\Omega_{X/Y} \otimes k(x)) \geq n$$

In either case, we can conclude $\Omega_{X/Y}$ is a coherent sheaf of rank $\geq n$, so by **(II 8.9)**, it's locally free of rank n, thus f is smooth of relative dimension n.

Lemma. 5 Let $f : X \to Y$ be a dominant morphism of integral schemes of finite type over an algebraically closed field k of characteristic 0. Then there is a nonempty open set $U \subseteq X$ such that $f : U \to Y$ is smooth.

Proof. By **(II 8.16)**, every variety over k has an open dense nonsingular subset, thus we may assume X, Y are nonsingular. Since char $k = 0$, thus perfect, by **(I,4.8A)**, $K(X)$ is a separably generated field extension of $K(Y)$. Therefore by **(II 8.6A)**, $\Omega_{X/Y}$ is free of rank n at the generic point, thus locally free of rank n on some nonempty open set U. By **Proposition 4**, $f : U \to Y$ is smooth.

Remark. The lemma may fail when characteristic of field is not zero. Let char $k = p > 0, k = \overline{k}, f : P_k^1 \to P_k^1$ be the Frobenius morphism , then f is not smooth on any open set.

Proposition. 6 Let $f : X \to Y$ be a morphism of schemes of finite type over an algebraically closed field k of characteristic 0. For any r, let

$$X_r = \{\text{closed points}\, x \in X \mid \operatorname{rank} T_{f,x} \leq r\}$$

Then $\dim \overline{f(X_r)} \leq r$.

Proof. Let Y' be any irreducible component of $\overline{f(X_r)}$, and let X' be an irreducible component of X_r which dominates Y'. Give X' and Y' their reduced induced structures, and consider the induced dominant morphism $f' : X' \to Y'$. By **(Lemma 5)**, there is a nonempty open subset $U' \subseteq X'$ such that $f : U' \to Y'$ is smooth. Let $x \in U' \cap X_r$, then consider the diagram

$$\begin{array}{ccc} T_{x,U'} & \to & T_{x,X} \\ \downarrow T_{f',x} & & \downarrow T_{f,x} \\ T_{y,Y'} & \to & T_{y,Y} \end{array}$$

The horizontal arrows are injective, because U' and Y' are locally closed subschemes of X and Y, respectively. Also $T_{f',x}$ is surjective by **Proposition 4**. Since $\dim T_{f,x} \leq r$, $\dim T_{y,Y'} \leq r$, we get $\dim Y' \leq r$.

Proposition. 7 Let $f : X \to Y$ be a morphism of varieties over an algebraically closed field k of characteristic 0, suppose X is nonsingular. Then there exists a nonempty open subset $V \subseteq Y$ such that $f : f^{-1}V \to V$ is smooth.

Proof. Again may assume Y is nonsingular by **(II 8.16)**. Let $\dim Y = r, X_{r-1} \subseteq X$ as defined in **Proposition 6**, then $\dim \overline{f(X_{r-1})} \leq r-1$, moving it from X, therefore we can assume for every closed point in X, rank $T_f \geq r = \dim Y$, which means they're all surjective, thus f is smooth by **Proposition 4**.

Proposition. 8 A morphism $f : X \to Y$ of schemes of finite type over k is étale if it is smooth of relative dimension 0. It is unramified if for every $x \in X$, letting $y = f(x)$, we have $m_y \mathcal{O}_X = m_x$, and $k(x)$ is a separable algebraic extension of $k(y)$. Show that the following conditions are equivalent:

(i) f is étale.
(ii) f is flat, $\Omega_{X/Y} = 0$.
(iii) f is flat and unramified.

Proof.

(i)⇔(ii) By the definition of smooth.

(iii)⇒(ii) We have

$$m_y/m_y^2 \otimes_{k(y)} k(x) = (m_y \otimes_{\mathcal{O}_y} k(y)) \otimes_{k(y)} k(x) = m_y \otimes_{\mathcal{O}_y} (\mathcal{O}_x/m_y\mathcal{O}_x)$$

Also, $m_y \otimes_A \mathcal{O}_x \cong m_y \mathcal{O}_x = m_x$, thus

$$m_y \otimes (\mathcal{O}_x/m_y\mathcal{O}_x) = (m_y \otimes \mathcal{O}_x)/m_y^2 = m_x/m_x^2$$

Thus the map T_f, x is an isomorphism, i.e. it's smooth of relative dimension 0.

(ii)⇒(iii) Fact: Let $\mathcal{O}_x = B, \mathcal{O}_y = A$, if $\hat{A} \to \hat{B}$ is an isomorphism then f is unramified at x.

Proof. We know $m_y^n \mathcal{O}_x = m_x^n \,\forall\, n \in \mathbb{N}$, and the composition

$$A/m_y^n \to B/m_y^n B \to \hat{A}/m_y^n \hat{A}$$

is an isomorphism, it's left to show $\hat{A} \to \hat{B}$ is injective, in other words,

$$m_y^n \hat{A} \cap B = m_y^n B \,\forall\, n \in \mathbb{N}$$

Notice that $B = A + m_y^n B$ and $m_y^n B \subseteq m_y^n \hat{A}$, so $\forall\, b \in B$, it can be represented as $a + \epsilon, a \in A, b \in m_y^n B$ $m_y^n \hat{A} \cap B \subseteq m_y^n B$.

Conversely, if $b \in m_y^n \hat{A}, a \in m_y \hat{A} \cap A = m_y^n$, thus $b \in m_y^n B$. Thus we only need to prove $\hat{A} \to \hat{B}$ is an isomorphism, but $\mathcal{O}_x = \mathcal{O}_y + m_x^n, m_x^n = m_x^{n+1} + m_y^n \forall\, n \in \mathbb{N}$, thus $\hat{A} \to \hat{B}$ is an isomorphism. Also, by (II 8.6A), $\dim_{k(x)} \Omega_{X/Y} \otimes k(x) \geq$tr.deg $k(x)/k(y)$, equality holds if and only if $k(x)$ is separately extended over $k(y)$. Thus in this case, $k(x)$ is separately extended over $k(y)$ of transcendental degree 0, i.e. a separable algebraic extension.

Recall. A group variety G is a variety G over an algebraically closed field k, together with morphisms $\mu : G \times G \to G, \rho : G \to G$ s.t. $G(k)$, k-rational points of G, becomes a group under the operation induced by μ, with ρ giving the inverses.
We say that a group variety G acts on a variety X if we have a morphism $\theta : G \times X \to X$ which induces a homomorphism $G(k) \to \mathrm{Aut}X$ of groups.
A homogeneous space is a variety X, together with a group variety G acting on it, such that the group $G(k)$ acts transitively on $X(k)$.

Remark. A homogeneous space is necessarily a nonsingular variety.

Theorem. 9 Let X be a homogeneous space with group variety G over an algebraically closed field k of characteristic 0. Let $f : Y \to X, g : Z \to X$ are two morphisms between nonsingular varieties. For $\sigma \in G(k)$, let Y^σ be the same variety with Y but with morphism $\sigma \circ f : Y^\sigma \to X$. Then $\exists\, V \subseteq G$ s.t. $\forall\, \sigma \in V(k)$, $Y^\sigma \times_X Z$ is nonsingular and either empty or has the dimension $\dim Y + \dim Z - \dim X$.

Proof. Define $h : G \times Y \to X$ as the composition of f and $\theta : G \times X \to X$. We first prove that it's smooth. Now nonsingular is equivalent to smooth over k by **Proposition 0.2**, and G is nonsingular by the **Remark** above, therefore $G \times Y$ is nonsingular by **Proposition 1(d)**.
Apply **Proposition 7**, we see that $\exists\, U \subseteq X$ s.t. $h^{-1}(U) \to U$ is smooth.
Now G acts on $G \times Y$ by left multiplication on G; G acts on X by θ, and these two actions are compatible with the morphism h, by construction. Thus $\forall\, \sigma \in G(k), h^{-1}(U^\sigma) \to U^\sigma$ is smooth. Since U^σ cover X, we conclude h is smooth.
Next consider the diagram:

$$\begin{array}{ccccc} W & := & (G \times Y) \times_X Z & \xrightarrow{h'} & Z \\ & & \downarrow g' & & \downarrow g \\ & & G \times Y & \xrightarrow{h} & X \\ & & \downarrow pr_1 & & \\ & & G & & \end{array}$$

Then h' is smooth again by **Proposition 1(b)**, and remind that Z is smooth over k by **Proposition 0.2**, using **Proposition 1(c)**, W is nonsingular. Consider

$$q = pr_1 \circ g' : W \to G$$

Again apply **Proposition 7**, $\exists\, V \subseteq G$ s.t. $q^{-1}(V) \to V$ is smooth.
Therefore $\forall\, \sigma \in V(k), W_{\overline{\sigma}}$ is nonsingular, where the result and notation follow from **Theorem 2**. But $W_{\overline{\sigma}}$ is just W_σ since k is algebraically closed. Also, W_σ actually coincides to $Y^\sigma \times_X Z$, which proves the first statement.
For the second statement, we notice that h is smooth of dimension $\dim G + \dim Y - \dim X$, so is h' by **Proposition 1(b)**, thus

$$\dim W - \dim Z = \dim G + \dim Y - \dim X$$

Also, $q \mid_{q^{-1}(V)}$ is smooth of dimension $\dim W - \dim G$, thus by the definition of smoothness and **(III 9.6)**, $\forall\, \sigma \in V(k)$

$$\dim W_\sigma = \dim W - \dim G = \dim Y + \dim Z - \dim X$$

Corollary. 10 Let X be a nonsingular projective variety over an algebraically closed field k of characteristic 0. Let $\mathfrak{d}$ be a linear system without base points. Then almost every element of $\mathfrak{d}$, considered as a closed subscheme of X, is nonsingular (but maybe reducible).

Proof. Let $f : X \to \mathbb{P}^n$ be the morphism determined by $\mathfrak{d}$, applying **(II 7.8.1)**. Consider $\mathbb{P}^n$ as a homogeneous space under the action of $\mathrm{PGL}(n)$ by **(II 7.1.1)**.
Take an arbitrary hyperplane $H \hookrightarrow \mathbb{P}^n$ and apply **Theorem 9** on it, then for almost every $\sigma \in G(k), X \times_{\mathbb{P}^n} H^\sigma = f^{-1}(H^\sigma)$ is nonsingular. But $f^{-1}(H^\sigma)$ is just some element of $\mathfrak{d}$, thus the result.

Remark. In **(Ex. 11.3)**, if $\dim f(X) \geq 2$, then all the divisors in $\mathfrak{d}$ are connected. Hence almost all of them are irreducible and nonsingular.

Remark. In fact, X is not need to be projective if $\mathfrak{d}$ is finite-dimensional.
In particular, if X is projective, a straightforward and more general statement is that "a general member of $\mathfrak{d}$ can have singularities only at the base points."

Remark. This result fails in characteristic $p > 0$. Take the same example in **Remark of Lemma 5**, the morphism f corresponds to the one-dimensional linear system $\{pP \mid P \in \mathbb{P}\}$. Thus every divisor in $\mathfrak{d}$ is a point with multiplicity p.