10 SMOOTH MORPHISMS

Recall. (Flat)
Let \( f : X \to Y \) be a morphism of schemes, and let \( \mathcal{F} \) be an \( \mathcal{O}_X \)-module. We say that \( \mathcal{F} \) is flat over \( Y \) at a point \( x \in X \), if the stalk at \( x \) is a flat \( \mathcal{O}_y \)-module, where \( y = f(x) \). We say simply \( \mathcal{F} \) is flat over \( Y \) if it is flat at every point of \( X \).

We say \( X \) is flat over \( Y \) if \( \mathcal{O}_X \) is.

Definition. (Smooth)
Let \( f : X \to Y \) be a morphism of schemes of finite type over a field \( k \), we say it's smooth of relative dimension \( n \) if:

1. \( f \) is flat.
2. If \( X' \subseteq X, Y' \subseteq Y \) are irreducible components such that \( f(X') \subseteq Y' \), then \( \dim X' = \dim Y' + n \).
3. For each point \( x \in X \) (closed or not), \( \dim_{k(x)}(\Omega_{X/Y} \otimes k(x)) = n \)

Proposition. 0.1 If \( X \) is integral, (3) is equivalent to \( \Omega_{X/Y} \) is locally free of rank \( n \).

Recall (II 8.9): Let \( A \) be a noetherian local integral domain, with residue field \( k \) and quotient field \( K \). If \( M \) is a finitely generated \( A \)-module and if \( \dim_k M \otimes_A k = \dim_K M \otimes_A K = r \), then \( M \) is free of rank \( r \).

Proposition. 0.2 If \( Y = \text{Spec} k, k = \overline{k} \), then \( X \) is smooth over \( k \) if and only if \( X \) is regular of dimension \( n \).

Recall (II 8.8): Let \( B \) be a local ring containing a field \( k \) isomorphic to its residue field. Assume furthermore that \( k \) is perfect, and that \( B \) is a localization of a finitely generated \( k \)-algebra. Then \( \Omega_{B/k} \) is a free \( B \)-module of rank \( \dim B \iff B \) is a regular local ring.

In particular, if \( X \) is irreducible separated over \( k \), then it's smooth if and only if it's a nonsingular variety.

Recall (II 8.15): Let \( X \) be an irreducible separated scheme of finite type over an algebraically closed field \( k \). Then \( \Omega_{X/k} \) is locally free of rank \( \dim X \iff X \) is a nonsingular variety over \( k \).

Remark. Over a nonperfect field, the last proposition is false: let \( k_0 \) be a field of characteristic \( p > 0 \), let \( k = k_0(t) \), and let \( X \subseteq A^2_k \) be the curve defined by \( y^2 = x^p - t \), then every local ring of \( X \) is a regular local ring, but \( X \) is not smooth over \( k \).

Proposition. 1

(a) An open immersion is smooth of relative dimension 0.
(b) If \( f : X \to Y \) is smooth of relative dimension \( n \), and \( g : Y' \to Y \) is any morphism, then the morphism \( f' : X' \to Y' \) obtained by base extension is also smooth of relative dimension \( n \).

(c) If \( f : X \to Y \) is smooth of relative dimension \( n \), and \( g : Y \to Z \) is smooth of relative dimension \( m \), then \( g \circ f : X \to Z \) is smooth of relative dimension \( n + m \).

(d) If \( f : X \to Z \) is smooth of relative dimension \( n \), and \( g : Y \to Z \) is smooth of relative dimension \( n + m \).

**Proof.**

(a) Just check the condition.

(b) (1) By (III 9.2), \( f' \) is flat.

(2) Recall (III 9.6): Let \( f : X \to Y \) be a flat morphism of schemes of finite type over a field \( k \), and assume that \( Y \) is irreducible. Then the following conditions are equivalent:

(i) every irreducible component of \( X \) has dimension equal to \( \dim Y + n \).

(ii) for any point \( y \in Y \) (closed or not), every irreducible component of the fibre \( X_y \) has dimension \( n \).

In this case, (2) is equivalent to say every irreducible component of every fibre \( X_y \) of \( f' \) has dimension \( n \), which, by (II Ex. 3.20), is preserved under base change.

(3) By (II 8.10), \( \Omega_{X/Y} \) is stable under base extension, thus (3) holds for \( f' \).

(c) (1) By (III 9.2).

(2) If \( X' \subseteq X, Y' \subseteq Y, Z' \subseteq Z \) are irreducible components such that \( f(X') \subseteq Y', g(Y') \subseteq Z' \), then \( \dim X' = \dim Y' + n = \dim Z' + n + m \).

(3) Use the exact sequence in (II 8.11)

\[
f^* \Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0
\]

Tensoring \( k(x) \) to get:

\[
f^* \Omega_{Y/Z} \otimes k(x) \to \Omega_{X/Z} \otimes k(x) \to \Omega_{X/Y} \otimes k(x) \to 0
\]

Then \( \dim \Omega_{X/Z} \otimes k(x) \leq \dim f^* \Omega_{Y/Z} \otimes k(x) + \dim \Omega_{X/Y} \otimes k(x) = n + m \).

On the other hand, let \( z = g(f(x)) \), we have \( \Omega_{X/Z} \otimes k(x) = \Omega_{X_z/k(z)} \otimes k(x) \), let \( X' \) be an irreducible component of \( X_z \) containing \( x \), with its reduced induced structure. Then by (II 8.12),

\[
\Omega_{X_z/k(z)} \otimes k(x) \to \Omega_{X'/k(z)} \otimes k(x)
\]

is surjective.

But again by (III 9.6), \( X' \) is an integral scheme of finite type over \( k(z) \) of dimension \( n + m \), thus by (II 8.6A), we see \( \Omega_{X'/k(z)} \) is locally generated by at least \( n + m \) elements. Combining above, (3) is verified.

(d) Use (b), (c).

**Theorem 2** Let \( f : X \to Y \) be a morphism of schemes of finite type over a field \( k \), then \( f \) is smooth of relative dimension \( n \) if and only if:

(i) \( f \) is flat.

(ii) The fibres of \( f \) are geometrically regular of equidimension \( n \), i.e., for point \( y \in Y \), let \( X_y = X_{k(y)} \), then \( X_y \) is equidimensional of dimension \( n \) and regular.
Proof.  ($\Rightarrow$) By Proposition (0.2), (1(b)).

($\Leftarrow$) (1) By assumption.
(2) By (III 9.6).
(3) By Proposition (0.2), regularity implies $\Omega_{X/k(y)}$ is locally free of rank $n$.
By the fact states below, it is equivalent to $\Omega_{X/k(y)}$ is locally free of rank $n$, thereby (3) holds.

Fact. If $M$ is an $A$-module with $A$ being a local ring, $B$ is a faithfully flat $A$-algebra, then

$$M \text{ free } \iff M \otimes_A B \text{ is free.}$$

Recall. The Zaraski tangent space $T_x$ for a point in a scheme $X$ be the dual of $k$-vector space $m_x/m_x^2$. If $f : X \to Y$ is a morphism, $y = f(x)$, then there is a natural induced mapping

$$T_f : T_x \to T_y \otimes_{k(y)} k(x)$$

Lemma. 3-1 Suppose that $R \to S$ is a local homomorphism of Noetherian local rings. Denote $m$ the maximal ideal of $R$. Let $M$ be a flat $R$-module and $N$ a finite $S$-module. Let $u : N \to M$ be a map of $R$-modules. If $u_1 : N/mN \to M/mM$ is injective then $u$ is injective. In this case $M/u(N)$ is flat over $R$.

Proof. Claim: $u_k : N/m^kN \to M/m^kM$ is injective $\forall \ k \in \mathbb{N}$.
The case $k = 1$ is just the assumption. To prove by induction, assuming the case $k = n$ holds. Consider the diagram:

$$
\begin{array}{cccccc}
0 & \to & M/IM & \to & M/(IM + u(N)) & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & \\
N/IN & \to & M/IM & \to & M/IM + u(N) & \to & 0 \\
\end{array}
$$

The first and the last map are injective, which imply the middle one is also injective. Therefore the claim is proved by induction.

By Krull’s intersection theorem, $\bigcap m^nN = 0$, thus the injectivity of $u_n \forall n \in \mathbb{N}$ implies $u$ is injective.
To show that $M/u(N)$ is flat over $R$, it suffices to show that $I \otimes_R M/u(N) \to M/u(N)$ is injective for every ideal $I$ in $R$. Consider the diagram

$$
\begin{array}{cccccc}
0 & \to & M/IM & \to & M/(IM + u(N)) & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & \\
N/IN & \to & M/IM & \to & M/IM + u(N) & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & \\
0 & \to & M & \to & M/u(N) & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & \\
N \otimes_R I & \to & M \otimes_R I & \to & M/u(N) \otimes_R I & \to & 0 \\
\end{array}
$$
\[ M \otimes_R I \to M \] is injective, then by the snake lemma, it suffices to prove that \( N/IN \) injects into \( M/IM \). Note that \( R/I \to S/IS \) is a local homomorphism of Noetherian local rings, \( N/IN \to M/IM \) is a map of \( R/I \)-modules, \( N/IN \) is finite over \( S/IS \), and \( M/IM \) is flat over \( R/I \) and \( \pi : N/IN \to M/IM \) is injective modulo \( m \). Thus we may apply the first part of the proof to \( \pi \) to conclude.

**Definition.** Let \( A \) be a commutative ring, \( I \) be an ideal of \( A \). An \( A \)-module \( M \) is ideally Hausdorff w.r.t. \( J \) if \( \forall a \) is a finitely generated ideal of \( A \), \( a \otimes_A M \) is Hausdorff with \( J \)-adic topology.

**Lemma. 3-2** Let \( A \) be a commutative ring, \( I, J \) be ideals of \( A \), \( M \) be an \( A \)-module, \( gr(A) \) be the graded ring with \( I \)-adic filtration, \( gr(M) \) be the graded \( gr(A) \)-module associated with \( M \) with \( I \)-adic topology. Consider the following:

(i) \( M \) is a flat \( A \)-module.

(ii) \( \text{Tor}^1_A(N,M) = 0 \ \forall N \) is an \( A \)-module annihilated by \( J \).

(iii) \( M/IM \) is a flat \( A/I \)-module and the canonical map \( I \otimes_A M \to IM \) is bijective.

(iv) \( M/IM \) is a flat \( A/I \)-module and canonical homomorphism \( r : gr(A) \otimes_{gr_0(A)} gr_0(M) \) is bijective.

(v) \( \forall n \in \mathbb{N}, M/I^n \) is a flat \( A/I^n \)-module.

Then (i)\( \Rightarrow \) (ii)\( \Leftrightarrow \) (iii)\( \Rightarrow \) (iv)\( \Leftrightarrow \) (v). Furthermore, if \( A \) is noetherian and \( M \) is ideally Hausdorff, all of them are equivalent.

**Proof.**

(i)\( \Rightarrow \) (ii)\( \Leftrightarrow \) (iii) Omitted.

(ii)\( \Leftrightarrow \) (ii)’: \( \text{Tor}^1_A(N,M) = 0 \ \forall N \) is an \( A \)-module annihilated by some power of \( J \). 

(\( \Leftarrow \)) is trivial, so let’s focus on (\( \Rightarrow \)).

In particular \( \text{Tor}^1_I(I^nN/I^{n+1}N,M) = 0 \ \forall n \in \mathbb{N} \). Then the exact sequence

\[
0 \to I^{n+1}N \to I^nN \to I^nN/I^{n+1}N \to 0
\]

which induces

\[
\text{Tor}^1_I(I^{n+1}N,M) \to \text{Tor}^1_A(I^nN,M) \to \text{Tor}^1_A(I^nN/I^{n+1}N,M)
\]

Notice that \( \exists m \in \mathbb{N} \) s.t. \( I^mN = 0 \), using descending induction, we have \( \text{Tor}^1_A(I^nN,M) = 0 \ \forall n \leq m \), especially \( m = 0 \).

(ii)\( \Rightarrow \) (iv) First we have a lemma:

(a) \( \text{Tor}_1^A(A/I^n,M) = 0 \ \forall n \in \mathbb{N} \).

(b) The canonical homomorphism \( \theta_n : I^n \otimes_A M \to I^nM \) is bijective.

(c) \( r : gr(A) \otimes_{gr_0(A)} gr_0(M) \) is bijective.

Then (a)\( \Leftrightarrow \) (b)\( \Rightarrow \) (c), (b)\( \Leftrightarrow \) (c) when \( I \) is nilpotent.

**Proof.** (a)\( \Leftrightarrow \) (b) by the exact sequence

\[
0 = \text{Tor}_1^A(A/I,M) \to \text{Tor}_1^A(A/I^n,M) \to I^n \otimes M \to M
\]

Consider the diagram:

\[
\begin{array}{cccccc}
I^{n+1} \otimes_A M & \to & I^n \otimes_A M & \to & I^n/I^{n+1} \otimes_{A/I} M/IM & \to & 0 \\
\downarrow \theta_{n+1} & & \downarrow \theta_n & & \downarrow r_n & & \\
0 & \to & I^{n+1}M & \to & I^nM & \to & gr_n(M)
\end{array}
\]
Where \( r_n \) is the canonical surjective map. It’s commutative by the definition of \( r_n \). In this case, \( \theta_n \) is bijective, thus \( r_n \) is bijective. For the second statement, notice that \( I^n \otimes_A M = I^n M = 0 \) for some \( n \), thus using descending induction we see it’s true.

Thus by lemma, \((ii) \Rightarrow (ii)’ \Rightarrow r \) is bijective, also \((ii) \Rightarrow (iii) \), a fortiori \( M/IM \) is a flat \( A/I \)-module, thereby \((iv) \) stands.

Also, \((iii) \Leftrightarrow (iv) \) when \( I \) is nilpotent is also shown by the above process.

\[ (iv) \Leftrightarrow (v) \]

Observe \( gr_m(M/I^nM) = \begin{cases} gr_m(M), m < n \\ 0, m \geq n \end{cases} \)

For \( k \in \mathbb{N} \), let \((iv)_k \) and \((v)_k \) be the statement replacing \( A,I,M \) by \( A/I^kA,I/I^k,M/I^kM \). Obviously

\[ (iv) \Rightarrow (iv)_k \forall k \in \mathbb{N} \]
\[ (v) \Rightarrow (v)_k \forall k \in \mathbb{N} \]

Thus it’s sufficient to show that \((iv)_k \Rightarrow (v)_k \forall k \in \mathbb{N} \) or also \((iv) \Leftrightarrow (v) \) for the case \( I \) is nilpotent, which will be assumed below. Since \( M/I^nM \cong M \otimes_A A/I^n \), we know \((v) \Rightarrow (i) \), more apparently, \((i) \Rightarrow (v) \), so the statement is proved.

\[ (v) \Rightarrow (i) \] (For the case \( A \) is noetherian and \( M \) is ideally Hausdorff) It’s proved in Lemma 3-1.

With above we now conclude:

**Lemma. 3** Let \( A \to B \) be a local homomorphism of local noetherian rings. Let \( M \) be a finitely generated \( B \)-module, and let \( t \in A \) be a nonunit that is not a zero divisor. Then \( M \) is flat over \( A \) if and only if \( t \) is not a zero divisor in \( M \) and \( M/tM \) is flat over \( A/tA \).

**Proof.** Take \( I = J = (t) \) in Lemma 3-1, 3-2.

**Proposition. 4** Let \( f : X \to Y \) be a morphism of nonsingular varieties over an algebraically closed field \( k \). Let \( n = \dim X - \dim Y \). Then the following are equivalent:

(i) \( f \) is smooth of relative dimension \( n \).
(ii) \( \Omega_{X/Y} \) is locally free of rank \( n \) on \( X \).
(iii) For every closed point \( x \in X \), \( T_f : T_x \to T_y \otimes_{k(y)} k(x) \) is surjective.

**Proof.**

(i)\( \Rightarrow \) (ii) By Proposition (0.2).

(ii)\( \Rightarrow \) (iii) Use the exact sequence in (II 8.11) tensoring \( k(x) \cong k \) (since \( x \) is a closed point), we have:

\[ f^*\Omega_{Y/k} \otimes k(x) \xrightarrow{\phi} \Omega_{X/k} \otimes k(x) \to \Omega_{X/Y} \otimes k(x) \to 0 \]

since the dimension of each terms are \( \dim Y, \dim X, n, \phi \) is actually an injection.

Recall (II 8.7): Let \((B, m)\) be a local ring with a field \( k \cong B/m \) containing in \( B \). Then

\[ m \to m/m^2 \xrightarrow{\phi} \Omega_{B/k} \otimes_B k \]
\[ b \mapsto \overline{b} \mapsto db \otimes 1 \]

is an isomorphism.

So \( \phi \) is actually \( m_y/m_y^2 \hookrightarrow m_x/m_x^2 \), take dual to get our result.
(iii)⇒(i) Recall (III 9.1A(d)): $M$ is an $A$-module, $M$ is flat over $A$ if and only if $M_p$ is flat over $A_p \forall p \in \text{Spec } A$. So to prove $f$ is flat, we only have to verify $O_{X,x}$ is flat over $O_{Y,y} \forall x \in X$ is a closed point, $y = f(x)$.

Since $X,Y$ both nonsingular, $O_x, O_y$ are regular local rings. Also by taking dual on $T_f$, $m_y/m_y^2 \hookrightarrow m_x/m_x^2$ is injective.

Take a regular system of parameters of $O_y : (t) = (t_1, \ldots, t_r)$, then $(t)$ forms a part of regular system of parameters of $O_x$. Since $O_x/(t)$ is flat over $O_y/(t) \cong k$, by Lemma 3 and induction, $O_x/(t_1, \ldots, t_i)$ is flat over $O_y/(t_1, \ldots, t_i) \forall i = 0 r$, in particular it’s already proved $f$ is flat. Proceed similar to (ii)⇒(iii) but backward, we see for $x$ is a closed point,

$$\dim_{k(x)}(\Omega_{X/Y} \otimes k(x)) = n$$

On the other hand, $f$ is flat, by (III Ex 9.1), is also dominant, by (II 8.6A), the generic point $\xi$ has the property:

$$\dim_{k(x)}(\Omega_{X/Y} \otimes k(x)) \geq n$$

In either case, we can conclude $\Omega_{X/Y}$ is a coherent sheaf of rank $\geq n$, so by (II 8.9), it’s locally free of rank $n$, thus $f$ is smooth of relative dimension $n$.

**Lemma.** 5 Let $f : X \to Y$ be a dominant morphism of integral schemes of finite type over an algebraically closed field $k$ of characteristic 0. Then there is a nonempty open set $U \subseteq X$ such that $f : U \to Y$ is smooth.

**Proof.** By (II 8.16), every variety over $k$ has an open dense nonsingular subset, thus we may assume $X,Y$ are nonsingular. Since char $k = 0$, thus perfect, by (I,4.8A), $K(X)$ is a separably generated field extension of $K(Y)$. Therefore by (II 8.6A), $\Omega_{X/Y}$ is free of rank $n$ at the generic point, thus locally free of rank $n$ on some nonempty open set $U$. By Proposition 4, $f : U \to Y$ is smooth.

**Remark.** The lemma may fail when characteristic of field is not zero. Let char $k = p > 0, k = \overline{k}, f : P^r_k \to P^r_k$ be the Frobenius morphism, then $f$ is not smooth on any open set.

**Proposition.** 6 Let $f : X \to Y$ be a morphism of schemes of finite type over an algebraically closed field $k$ of characteristic 0. For any $r$, let

$$X_r = \{\text{closed points } x \in X \mid \text{rank } T_{f,x} \leq r\}$$

Then $\dim f(X_r) \leq r$.

**Proof.** Let $Y'$ be any irreducible component of $f(X_r)$, and let $X'$ be an irreducible component of $X_r$ which dominates $Y'$. Give $X'$ and $Y'$ their reduced induced structures, and consider the induced dominant morphism $f' : X' \to Y'$. By (Lemma 5), there is a nonempty open subset $U' \subseteq X'$ such that $f : U' \to Y'$ is smooth. Let $x \in U' \cap X_r$, then consider the diagram

$$\begin{array}{ccc}
T_{x,U'} & \to & T_{x,X} \\
\downarrow T_{f',x} & & \downarrow T_{f,x} \\
T_{y,Y'} & \to & T_{y,Y}
\end{array}$$
The horizontal arrows are injective, because $U'$ and $Y'$ are locally closed subschemes of $X$ and $Y$, respectively. Also $T_{y,x}$ is surjective by Proposition 4. Since $\dim T_{y,x} \leq r$, $\dim T_{y'} \leq r$, we get $\dim Y' \leq r$.

Proposition 7 Let $f : X \to Y$ be a morphism of varieties over an algebraically closed field $k$ of characteristic 0, suppose $X$ is nonsingular. Then there exists a nonempty open subset $V \subseteq Y$ such that $f : f^{-1}V \to V$ is smooth.

Proof. Again may assume $Y$ is nonsingular by (II 8.16). Let $Y = r, X_{r-1} \subseteq X$ as defined in Proposition 6, then $\dim f(X_{r-1}) \leq r - 1$, moving it from $X$, therefore we can assume for every closed point in $X$, $\text{rank} \ T_f \geq r = \dim Y$, which means they're all surjective, thus $f$ is smooth by Proposition 4.

Proposition 8 A morphism $f : X \to Y$ of schemes of finite type over $k$ is étale if it is smooth of relative dimension 0. It is unramified if for every $x \in X$, letting $y = f(x)$, we have $m_y O_X = m_x$, and $k(x)$ is a separable algebraic extension of $k(y)$. Show that the following conditions are equivalent:

(i) $f$ is étale.
(ii) $f$ is flat, $\Omega_{X/Y} = 0$.
(iii) $f$ is flat and unramified.

Proof.
(i)$\iff$(ii) By the definition of smooth.
(iii)$\implies$(ii) We have
\[m_y/m_y^2 \otimes_{k(y)} k(x) = (m_y \otimes_{O_y} k(y)) \otimes_{k(y)} k(x) = m_y \otimes_{O_y} (O_x/m_y O_x)\]
Also, $m_y \otimes_A O_x \cong m_y O_x = m_x$, thus
\[m_y \otimes (O_x/m_y O_x) = (m_y \otimes O_x)/m_y^2 = m_x/m_x^2\]
Thus the map $T_f, x$ is an isomorphism, i.e. it’s smooth of relative dimension 0.
(ii)$\implies$(iii) Fact: Let $O_x = B, O_y = A$, if $\hat{A} \to \hat{B}$ is an isomorphism then $f$ is unramified at $x$.

Proof. We know $m_y^n O_x = m_x^n \forall n \in \mathbb{N}$, and the composition
\[A/m_y^n \to B/m_y^n B \to \hat{A}/m_y^n \hat{A}\]
is an isomorphism, it’s left to show $\hat{A} \to \hat{B}$ is injective, in other words,
\[m_y^n \hat{A} \cap B = m_y^n B \forall n \in \mathbb{N}\]
Notice that $B = A + m_y^n B$ and $m_y^n B \subseteq m_y^n \hat{A}$, so $\forall b \in B$, it can be represented as $a + \epsilon, a \in A, b \in m_y^n B$ $m_y^n \hat{A} \cap B \subseteq m_y^n B$.
Conversely, if $b \in m_y^n A, a \in m_y \hat{A} \cap A = m_y^n$, thus $b \in m_y^n B$. Thus we only need to prove $\hat{A} \to \hat{B}$ is an isomorphism, but $O_x = O_y + m_y^n, m_y^n = m_x^{n+1} + m_y^n \forall n \in \mathbb{N}$, thus $\hat{A} \to \hat{B}$ is an isomorphism. Also, by (II 8.6A), $\dim_{k(x)} \Omega_{X/Y} \otimes k(x) \geq \text{tr.deg} k(x)/k(y)$, equality holds if and only if $k(x)$ is separately extended over $k(y)$. Thus in this case, $k(x)$ is separately extended over $k(y)$ of transcendental degree 0, i.e. a separable algebraic extension.
Recall. A group variety $G$ is a variety $G$ over an algebraically closed field $k$, together with morphisms $\mu : G \times G \to G, \rho : G \to G$ s.t. $G(k)$, $k$-rational points of $G$, becomes a group under the operation induced by $\mu$, with $\rho$ giving the inverses. We say that a group variety $G$ acts on a variety $X$ if we have a morphism $\theta : G \times X \to X$ which induces a homomorphism $G(k) \to \text{Aut}X$ of groups.

A homogeneous space is a variety $X$, together with a group variety $G$ acting on it, such that the group $G(k)$ acts transitively on $X(k)$.

Remark. A homogeneous space is necessarily a nonsingular variety.

Theorem. 9 Let $X$ be a homogeneous space with group variety $G$ over an algebraically closed field $k$ of characteristic 0. Let $f : Y \to X, g : Z \to X$ are two morphisms between nonsingular varieties. For $\sigma \in G(k)$, let $Y^\sigma$ be the same variety with $Y$ but with morphism $\sigma \circ f : Y^\sigma \to X$. Then $\exists V \subseteq G$ s.t. $\forall \sigma \in V(k), Y^\sigma \times_X Z$ is nonsingular and either empty or has the dimension $\dim Y + \dim Z - \dim X$.

Proof. Define $h : G \times Y \to X$ as the composition of $f$ and $\theta : G \times X \to X$. We first prove that it’s smooth. Now nonsingular is equivalent to smooth over $k$ by Proposition 0.2, and $G$ is nonsingular by the Remark above, therefore $G \times Y$ is nonsingular by Proposition 1(d).

Apply Proposition 7, we see that $\exists U \subseteq X$ s.t. $h^{-1}(U) \to U$ is smooth.

Now $G$ acts on $G \times Y$ by left multiplication on $G$; $G$ acts on $X$ by $\theta$, and these two actions are compatible with the morphism $h$, by construction. Thus $\forall \sigma \in G(k), h^{-1}(U^\sigma) \to U^\sigma$ is smooth. Since $U^\sigma$ cover $X$, we conclude $h$ is smooth.

Next consider the diagram:

$$
\begin{align*}
W & := (G \times Y) \times_X Z \xrightarrow{h'} Z \\
\downarrow g' & \quad \downarrow g \\
G \times Y & \xrightarrow{h} X \\
\downarrow pr_1 & \\
G &
\end{align*}
$$

Then $h'$ is smooth again by Proposition 1(b), and remind that $Z$ is smooth over $k$ by Proposition 0.2, using Proposition 1(c), $W$ is nonsingular. Consider $q = pr_1 \circ g' : W \to G$.

Again apply Proposition 7, $\exists V \subseteq G$ s.t. $q^{-1}(V) \to V$ is smooth.

Therefore $\forall \sigma \in V(k), W_\sigma$ is nonsingular, where the result and notation follow from Theorem 2. But $W_\sigma$ is just $W_\sigma$ since $k$ is algebraically closed. Also, $W_\sigma$ actually coincides to $Y^\sigma \times_X Z$, which proves the first statement.

For the second statement, we notice that $h$ is smooth of dimension $\dim G + \dim Y - \dim X$, so is $h'$ by Proposition 1(b), thus

$$
\dim W - \dim Z = \dim G + \dim Y - \dim X
$$

Also, $q_{|q^{-1}(V)}$ is smooth of dimension $\dim W - \dim G$, thus by the definition of smoothness and (III 9.6), $\forall \sigma \in V(k)$

$$
\dim W_\sigma = \dim W - \dim G = \dim Y + \dim Z - \dim X
$$
Corollary. 10 Let $X$ be a nonsingular projective variety over an algebraically closed field $k$ of characteristic 0. Let $\mathfrak{d}$ be a linear system without base points. Then almost every element of $\mathfrak{d}$, considered as a closed subscheme of $X$, is nonsingular (but maybe reducible).

Proof. Let $f : X \to \mathbb{P}^n$ be the morphism determined by $\mathfrak{d}$, applying (II 7.8.1). Consider $\mathbb{P}^n$ as a homogeneous space under the action of $\text{PGL}(n)$ by (II 7.1.1). Take an arbitrary hyperplane $H \hookrightarrow \mathbb{P}^n$ and apply Theorem 9 on it, then for almost every $\sigma \in G(k), X \times_{\mathbb{P}^n} H^\sigma = f^{-1}(H^\sigma)$ is nonsingular. But $f^{-1}(H^\sigma)$ is just some element of $\mathfrak{d}$, thus the result.

Remark. In (Ex. 11.3), if $\dim f(X) \geq 2$, then all the divisors in $\mathfrak{d}$ are connected. Hence almost all of them are irreducible and nonsingular.

Remark. In fact, $X$ is not need to be projective if $\mathfrak{d}$ is finite-dimensional. In particular, if $X$ is projective, a straightforward and more general statement is that "a general member of $\mathfrak{d}$ can have singularities only at the base points."

Remark. This result fails in characteristic $p > 0$. Take the same example in Remark of Lemma 5, the morphism $f$ corresponds to the one-dimensional linear system $\{pP \mid P \in \mathbb{P}\}$. Thus every divisor in $\mathfrak{d}$ is a point with multiplicity $p$. 