Final presentation 2 (1/3) GRR for $X \times \mathbb{P}^n \to X$. 蔡以恆

Thm (GRR) Let $f: X \to Y$ be a smooth morphism between projective varieties.

Then $ch(f_!(x)) \cdot td(Y) = f_*(ch(x) \cdot td(X)) \quad \forall x \in K(X)$.

Lemma Let $f: X \to Y$ and $f': X' \to Y'$ be proper. Let $x \in K(X)$, $x' \in K(X')$.

If GRR holds for (f, x) and for (f', x'), then it holds for $(f \times f', x \otimes x')$.

By Lemma, to prove GRR for $X \times \mathbb{P}^n \to X$, it suffices to show

- $K(X) \otimes K(\mathbb{P}^n) \twoheadrightarrow K(X \times \mathbb{P}^n)$ &
- GRR holds for $\mathbb{P}^n \to pt$ i.e. $\chi(\mathbb{P}^n, \mathcal{F}) = \deg(ch(\mathcal{F})\, td(\mathbb{P}^n))_n \ \forall \mathcal{F} \in Coh(\mathbb{P}^n)$.

Prop Let $X' \subset X$ be a closed subvariety and $U = X \setminus X'$. Then we have a

exact sequence $K(X') \to K(X) \to K(U) \to 0$. ([H] Ex. II.6.10)

Prop $p^*: K(X) \to K(X \times \mathbb{A}^1)$ is bijective.

pf: 1° Consider $X \times \{0\} \overset{i}{\hookrightarrow} X \times \mathbb{A}^1$, then $0 \to \mathcal{O}_{X \times \mathbb{A}^1} \xrightarrow{t} \mathcal{O}_{X \times \mathbb{A}^1} \to \underset{\mathcal{O}_X}{\overset{}{i_* \mathcal{O}_X}} \to 0$

(Locally, $\underbrace{A \otimes_k k[t]}_{A[t]} \to \underbrace{A \otimes_k k[t]/(t)}_{A}$ with $\ker = \operatorname{im}(A[t] \xrightarrow{t} A[t])$).

Therefore, $0 = Tor_{p-1}^{\mathcal{O}_{X \times \mathbb{A}^1}}(\mathcal{O}_{X \times \mathbb{A}^1}, \mathcal{F}) \leftarrow Tor_p^{\mathcal{O}_{X \times \mathbb{A}^1}}(i_* \mathcal{O}_X, \mathcal{F}) \leftarrow Tor_p^{\mathcal{O}_{X \times \mathbb{A}^1}}(\mathcal{O}_{X \times \mathbb{A}^1}, \mathcal{F}) = 0$

for $\mathcal{F} \in Coh(X \times \mathbb{A}^1)$ and $p \geq 2$. $\Rightarrow Tor_p^{\mathcal{O}_{X \times \mathbb{A}^1}}(i_* \mathcal{O}_X, \mathcal{F}) = 0$

Define $\pi: K(X \times \mathbb{A}^1) \to K(X)$, $\mathcal{F} \mapsto Tor_0(\mathcal{O}_X, \mathcal{F}) - Tor_1(\mathcal{O}_X, \mathcal{F})$

Clearly, $\pi \circ p^* = id$, which proves p^* is inj.

2° Now, prove $K(X) = K(X \times \mathbb{A}^1)$ by induction on $\dim X$.

Consider

$$\begin{tikzcd} K(X') \arrow[r] \arrow[d, "\wr"] & K(X) \arrow[r] \arrow[d, hook] & K(U) \arrow[r] \arrow[d, hook] & 0 \\ K(X' \times \mathbb{A}^1) \arrow[r] & K(X \times \mathbb{A}^1) \arrow[r] & K(U \times \mathbb{A}^1) \arrow[r] & 0 \end{tikzcd} \quad (U = X \setminus X')$$

Note $z \in K(X \times \mathbb{A}^1)$ with $z_{U \times \mathbb{A}^1} \in K(U) \Rightarrow z \in K(X)$.

Thus, we may assume $X = \operatorname{Spec} A$.

3° claim : Let Z be a variety and $K' = \langle [\mathcal{O}_T] \mid T : \text{irr. subvar. of } Z \rangle$.

Then $K' = K(Z)$.

pf : By induction on $\dim Z$, we may assume Z is irr. and affine.

Let $\mathcal{F} \in \operatorname{Coh}(Z)$ and $\mathcal{F}^t$ be the torsion subsheaf of $\mathcal{F}$.

If $\mathcal{F} = \mathcal{F}^t$, then $\operatorname{supp} \mathcal{F} \subsetneq Z$.

Thus, by induction on $\dim Z$, any torsion coherent sheaf is in K'.

Now, assume $\mathcal{F}^t \subsetneq \mathcal{F}$, then consider $0 \to \mathcal{F}^t \to \mathcal{F} \to \mathcal{G} \to 0$

Since $\mathcal{G}$ is torsion free, $0 \to \mathcal{G} \to \mathcal{O}_Z^{\oplus m} \to \mathcal{G}' \to 0$ for some $m \in \mathbb{N}$.

(Module version : Let $M : R\text{-mod}$ and $K = \operatorname{Frac} R$.

Then $M \otimes_R K = K^{\oplus m}$ for some $m \in \mathbb{N}$. Let $M = Rx_1 + \cdots + Rx_r$

and $x_i = \sum_{j=1}^{m} \frac{a_{ij}}{b_{ij}} e_j \Rightarrow M \hookrightarrow R\frac{e_1}{b} \oplus \cdots \oplus R\frac{e_m}{b}$ with $b = \prod b_{ij}$)

Note $\mathcal{G} \otimes_{\mathcal{O}_Z} R(Z) = R(Z)^{\oplus m} = \mathcal{O}_Z^{\oplus m} \otimes R(Z) \Rightarrow \mathcal{G}' \otimes R(Z) = 0$

Thus, $\mathcal{G}' = (\mathcal{G}')^t$ and $[\mathcal{F}] = [\mathcal{F}_t] + [\mathcal{G}] = [\mathcal{F}_t] + [\mathcal{O}_Z^{\oplus m}] - [\mathcal{G}'] \in K'$

4° It suffices to show $[\mathcal{O}_T] \in K(X)$ for all $T \subset X \times \mathbb{A}^1$ irr. Let $T = \mathrm{Spec}\, A[t]/\mathfrak{p}$.

By induction on $\dim X$, we may assume $A \cap \mathfrak{p} = 0$.

(If $A \cap \mathfrak{p} \neq 0$, then $[\mathcal{O}_T] \in K(X' \times \mathbb{A}^1) = K(X') \Rightarrow [\mathcal{O}_T] \in K(X)$

where $X' = \mathrm{Spec}\, A/A\cap\mathfrak{p}$ with $\dim X' < \dim X$).

Let $S = A \setminus \{0\}$, $K = S^{-1}A$. Since $\mathfrak{p} \cap S = \emptyset$, $\mathfrak{p} = \mathfrak{p}' \cap A[t]$ for some $\mathfrak{p}' \in \mathrm{Spec}\, K[t]$

Write $\mathfrak{p}' = K[t]\, P(t)$ for some $P(t) \in A[t] \Rightarrow \mathfrak{q} := A[t]P(t) \subset \mathfrak{p} \subset A[t]$

Note $S^{-1}\mathfrak{q} = S^{-1}\mathfrak{p} \Rightarrow a \cdot (\mathfrak{p}/\mathfrak{q}) = 0$ for some $a \in S$. Let $T' = \mathrm{Spec}\, A[t]/\mathfrak{q}$.

Then $0 \to \mathfrak{p}/\mathfrak{q} \to A[t]/\mathfrak{q} \to A[t]/\mathfrak{p} \to 0$ & $0 \to A[t] \xrightarrow{P(t)} A[t] \to A[t]/\mathfrak{q} \to 0$

imply $[\mathcal{O}_T] = [\mathcal{O}_{T'}] - [\widetilde{\mathfrak{p}/\mathfrak{q}}] = -[\widetilde{\mathfrak{p}/\mathfrak{q}}] \in K(X' \times \mathbb{A}^1) = K(X') \Rightarrow [\mathcal{O}_T] \in K(X)$

where $X' = \mathrm{Spec}\, A/(a)$ $\square$

Cor $K(X) = K(X \times \mathbb{A}^n)$

Prop. $K(X) \otimes K(\mathbb{P}^n) \twoheadrightarrow K(X \times \mathbb{P}^n)$

pf: Induct on n. Let $n > 0$, H: hyperplane of $\mathbb{P}^n$ and $U = \mathbb{P}^n \setminus H \simeq \mathbb{A}^n$.

Consider
$$\begin{tikzcd} K(X)\otimes K(H) \arrow[r] \arrow[d, two heads] & K(X)\otimes K(\mathbb{P}^n) \arrow[r] \arrow[d] & K(X)\otimes K(U) \arrow[r] \arrow[d, "\wr", "\text{Cor}"'] & 0 \\ K(X\times H) \arrow[r] & K(X \times \mathbb{P}^n) \arrow[r] & K(X \times U) \arrow[r] & 0 \end{tikzcd}$$

Therefore, $K(X) \otimes K(\mathbb{P}^n) \twoheadrightarrow K(X \times \mathbb{P}^n)$ $\square$

Prop $\chi(\mathbb{P}^n, \mathcal{F}) = \deg(ch(\mathcal{F})\, td(\mathbb{P}^n))_n \ \forall \mathcal{F} \in Coh(\mathbb{P}^n)$.

pf: · (Euler) $0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}(1)^{n+1} \to \mathcal{T}_{\mathbb{P}^n} \to 0 \Rightarrow td(\mathbb{P}^n) = td(\mathcal{O}(1))^{n+1} = \left(\frac{x}{1-e^{-x}}\right)^{n+1}$, $x := c_1(\mathcal{O}(1))$.

· (Hilbert Syzygy) $[\mathcal{F}]$ = sum of $\mathcal{O}(r)$'s $\Rightarrow$ May assume $\mathcal{F} = \mathcal{O}(r)$, then $ch(\mathcal{F}) = e^{rx}$.

Combining the two facts above, we get $\deg(ch(\mathcal{F})\,td(\mathbb{P}^n))_n = \deg\left(\frac{e^{rx}\cdot x^{n+1}}{(1-e^{-x})^{n+1}}\right)_n$

$$= \operatorname*{res}_{x=0}\left(\frac{e^{rx}}{(1-e^{-x})^{n+1}}\right) = \operatorname*{res}_{y=0}\left(\frac{(1-y)^{-r-1}}{y^{n+1}}\right) = \binom{r+1+n-1}{n} = \binom{n+r}{n}$$

On the other hand, $\chi(\mathbb{P}^n, \mathcal{O}(r)) = h^0(\mathcal{O}(r)) + (-1)^n h^n(\mathcal{O}(r))$

$$= \begin{cases} \binom{n+r}{n} & \text{if } r \geq 0 \\ \binom{-r-1}{n} & \text{if } r \leq -n-1 \\ 0 & \text{if } -n-1 < r < 0 \end{cases} = \binom{n+r}{n}$$

□

Next, we prove two lemmas for GRR for closed immersion.

Lemma Let $Y \overset{i}{\hookrightarrow} X$ non-sing. variety of codim 1. Then GRR holds for (i, i^*x), $x \in K(X)$

i.e. $i_*(ch(y).\,td(Y)) = ch(i_*(y)).\,td(X)$ for $y = i^*x$.

pf: Consider $0 \to \mathcal{T}_Y \to \mathcal{T}_X|_Y \to \mathcal{N}_{Y/X} \to 0$, then $i^*(td(X)) = td(Y).\,td(\mathcal{N}_{Y/X})$

Note $i_*(ch(y).\,td(\mathcal{N}_{Y/X})^{-1}) = i_*(ch(y).\,td(Y).\,i^*td(X)^{-1}) = i_*(ch(y).\,td(Y)).\,td(X)^{-1}$

$$\parallel$$

$i_*(i^*ch(x).\,i^*td(\mathcal{O}(Y))^{-1}) = ch(x).\,td(\mathcal{O}(Y))^{-1}.\,i_*(\mathcal{O}_Y)$ $\quad (\mathcal{N}_{Y/X} = i^*\mathcal{O}(Y))$

Thus, $i_*(ch(y).\,td(Y)) = ch(x).\,td(\mathcal{O}(Y))^{-1}.\,i_*(\mathcal{O}_Y).\,td(X)$

Also, $ch(i_*(y)) = ch(i_*i^*x) = ch(x.\,i_*(\mathcal{O}_Y)) = ch(x.\,(1-[\mathcal{O}(Y)]^{-1}))$

$$= ch(x)[1-e^{-[Y]}] = ch(x).\,td(\mathcal{O}(Y))^{-1}.\,i_*(\mathcal{O}_Y)$$

Combining the two equations, we prove the lemma. □

Def For $[\mathcal{F}], [\mathcal{G}] \in K(X)$, $[\mathcal{F}].[\mathcal{G}] := \sum_{i \geq 0} (-1)^i [\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})]$.

Lemma Let $Y_1, \cdots, Y_m$ be non-singular subvarieties of X s.t Y_i meets $Y_1 \cap \cdots \cap Y_{i-1}$ transversaly for $i = 2, \cdots, m$. Then $[\mathcal{O}_{Y_1 \cap \cdots \cap Y_m}] = \prod_{i=1}^{m} [\mathcal{O}_{Y_i}]$ in $K(X)$.

pf: By induction, we may assume $m=2$. If $Y \cap Z = \phi$, then $[\mathcal{O}_{Y \cap Z}] = 0 = [\mathcal{O}_Y].[\mathcal{O}_Z]$.

Now, pick $a \in Y \cap Z$ with local equation $\begin{cases} f_1, \cdots, f_p & \text{for } Y \\ g_1, \cdots, g_q & \text{for } Z. \end{cases}$

Then we have Koszul complex of $\mathcal{O}_Y$: $0 \to \wedge^p E \to \cdots \to \wedge^2 E \to E \to \mathcal{O}_X$

and $\{f_i\}$ forms a regular seq. of $\mathcal{O}_{Z,a} = \mathcal{O}_{X,a} / \langle g_1, \cdots, g_q \rangle$.

Thus, $0 \to \mathcal{O}_Z \otimes \wedge^p E \to \cdots \to \mathcal{O}_Z \otimes \wedge^2 E \to \mathcal{O}_Z \otimes E \to \mathcal{O}_Z$ & $\mathcal{O}_{Z,a} / \langle f_1, \cdots, f_p \rangle = \mathcal{O}_{Y \cap Z, a}$.

$\rightsquigarrow [\mathcal{O}_Y].[\mathcal{O}_Z] = \sum (-1)^i \mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z) = \mathrm{Tor}_0^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z) = [\mathcal{O}_{Y \cap Z}]$ $\square$

Cor Let $Y \subset X$ be a non-sing. hyperplane section and $k = \dim X$.

Then $(1 - [\mathcal{O}(Y)])^{k+1} = 0$

pf: Note $[\mathcal{O}_{Y_1}] = [\mathcal{O}_{Y_2}]$ $\forall Y_1, Y_2$: hyperplane sections.

Then $0 = [\mathcal{O}_Y]^{k+1} = (1 - [\mathcal{O}(Y)]^{-1})^{k+1}$ $\square$

Ref Borel, A. and J.-P. Serre. Le théoréme de Riemann-Roch.