Final presentation 1 (6/18) Chern class. 蔡以恒

<u>Ref</u> Grothedieck – La théorie des classes de Chern.

- X : non-singular quasi-projective variety.
- $VB_r(X) := \{$ vector bundle (locally free sheaf) on X of rank $r\}$.

Lemma 1. Let $E \in VB_r(X)$ and $\xi_E \in A(\mathbb{P}(E))$ associated to $\mathcal{O}_{\mathbb{P}(E)}(1)$.

Then $A(\mathbb{P}(E))$ is a free $A(X)$-mod on $\{1, \xi_E, \cdots, \xi_E^{r-1}\}$

via $\pi^* : A(X) \to A(\mathbb{P}(E))$ where $\pi : \mathbb{P}(E) \to X$ is the projection.

pf : First, we prove $A(\mathbb{P}(E))$ is an $A(X)$-mod gen. by $\{1, \xi_E, \cdots, \xi_E^{r-1}\}$.

Let U be an affine open subset of X s.t. $E|_U \simeq \mathcal{O}_X|_U^{\oplus r}$.

Consider the diagram

$$\begin{tikzcd} A(Z) \arrow[r] & A(\mathbb{P}(E)) \arrow[r] & A(U \times \mathbb{P}^r) \arrow[r] & 0 \\ A(Y) \arrow[r, "i_*"] \arrow[u, "\pi^*"] & A(X) \arrow[r, "j^*"] \arrow[u, "\pi^*"] & A(U) \arrow[r] \arrow[u, "\pi^*"] & 0 \end{tikzcd}$$

By induction on $\dim(X)$ and $A(U \times \mathbb{P}^n)$ is a free $A(U)$-mod on $\mathcal{O}(1)$,

$A(\mathbb{P}(E))$ is an $A(X)$-mod gen. by $\{1, \xi_E, \cdots, \xi_E^{r-1}\}$.

Next, consider

$$\begin{tikzcd} & A(\mathbb{P}(E)) \arrow[r] & A(\mathbb{P}_{\mathbb{P}}(\pi^*E)) \\ 0 \arrow[r] & A(X) \arrow[r, "\pi^*"] \arrow[u, "\pi^*"] & A(\mathbb{P}(E)) \arrow[u] \end{tikzcd} \quad \& \quad \begin{tikzcd} A(\mathbb{P}_{\mathbb{P}}(\pi^*E)) \arrow[r] & A(\mathbb{P}_{\mathbb{P}}(E')) \\ A(\mathbb{P}(E)) \arrow[r] \arrow[u] & A(\mathbb{P}(E)) \arrow[u] \end{tikzcd}$$

$(\mathcal{O}(-1) \hookrightarrow \pi^*E \Rightarrow E' := \pi^*E / \mathcal{O}(-1)$ with $rk\, E' = r-1)$

By induction on r and diagram chasing, the statement holds. $\square$

<u>Def</u> (i-th Chern class) Let $E \in VB_r(X)$. By Lemma 1, there are unique

$c_i(E) \in A^i(X)$, $i = 0, 1, \cdots, r$ satisfying $\sum_{i=0}^{r} \pi^*(c_i(E)) \cdot \xi_E^{r-i} = 0$, $c_0(E) = 1$.

Then $c_i(E)$ is called the i-th Chern class.

(total Chern class) $c(E) := \sum_{i=0}^{r} c_i(E) = 1 + c_1(E) + \cdots + c_r(E) \in A(X)$.

(Chern polynomial) $c_t(E) := \sum_{i=0}^{r} c_i(E) t^i = 1 + c_1(E)t + \cdots + c_r(E)t^r \in A(X)[t]$.

Prop Let $E \in VB_r(X)$. Then we have the following properties

(a) (Normalization) If $rk(E) = 1$ i.e. $E = \mathcal{O}(D)$ for some $D \in Div(X)$, $c_t(E) = 1 + [D]t$.

(b) (Funtoriality) If $f: X' \to X$ is flat, $c_t(f^*E) = f^*(c_t(E))$.

(c) (Additivity) If $0 \to E' \to E \to E'' \to 0$ is a short exact sequence of vector bundles on X, $c_t(E) = c_t(E') \cdot c_t(E'')$.

pf: (a) Use the notation in Lemma 1. Note $\pi : \mathbb{P}(E) \xrightarrow{\sim} X$ and $\pi_* \mathcal{O}_{\mathbb{P}(E)}(1) = \check{E}$

By definition, $\xi_{\check{E}} + \pi^*(c_1(E)) = 0 \Rightarrow c_1(E) = [D]$.

(b) Let $F = f^*E$. Then

$$\begin{tikzcd} \mathbb{P}(F) \arrow[r, "\bar{f}"] \arrow[d, "\bar{\pi}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{P}(E) \arrow[d, "\pi"] \\ X' \arrow[r, "f"'] & X \end{tikzcd}$$

$$\Rightarrow 0 = \bar{f}^*\left(\sum_{i=0}^{r} \pi^*(c_i(E)) \cdot \xi_{\check{E}}^{r-i}\right) = \sum_{i=0}^{r} \bar{f}^*\pi^*(c_i(E)) \cdot \bar{f}^*\xi_{\check{E}}^{r-i} = \sum_{i=0}^{r} \bar{\pi}^* f^*(c_i(E)) \cdot \xi_{\check{F}}^{r-i}$$

Thus, $c_i(f^*E) = f^*(c_i(E))$. $\square$

To prove Prop (c), we need the following two lemmas.

Lemma 2 (Splitting principle) Let $E \in VB_r(X)$. Then $\exists f: X' \to X$ flat s.t.

(1) $f^*: A(X) \to A(Y)$ injective.

(2) $\exists$ filtration $f^*E = E_r \supset E_{r-1} \supset \cdots \supset E_1 \supset E_0 = 0$ with $L_i = E_i/E_{i-1} \in VB_1(X')$.

pf: Induction on r. When $r = 1$, take $f = id \Rightarrow$ the statement holds.

Let $r > 1$. Consider $\pi^*: A(X) \to A(\mathbb{P}(E))$ & $0 \to E'_{r-1} \to \pi^*\check{E} \to \mathcal{O}_{\mathbb{P}(E)}(1) \to 0$

By induction hypothesis, $\exists\, p: X' \to \mathbb{P}(E)$ s.t. p^*: inj. & $p^*E'_{r-1} \supset E'_{r-2} \supset \cdots \supset 0$

Thus, set $f = \pi \circ p : X' \to X$, we have $f^* = p^* \circ \pi^*$ inj. &

$$f^*\check{E} = p^*\pi^*\check{E} \supset p^*E'_{r-1} \supset E_{r-2} \supset \cdots \supset 0.$$

Note if we have a filtration $E = E_r \supset E_{r-1} \supset \cdots \supset E_1 \supset E_0 = 0$ with $L_i = E_i/E_{i-1} \in VB_1(X)$, then $\check{E} = E_r' \supset E_{r-1}' \supset \cdots \supset E_1' \supset E_0' = 0$ with $L_i' = E_i'/E_{i-1}' = \check{L}_i$.

$\left(E = E_r/0 \twoheadrightarrow E_r/E_1 \twoheadrightarrow E_r/E_2 \twoheadrightarrow \cdots\ E_r/E_r = 0 \right.$

$\left. \Rightarrow \check{E} = (E_r/0)^\vee \supset E_{r-1}' := (E_r/E_1)^\vee \supset \cdots \supset E_0' := \check{0} \text{ and } E_i'/E_{i-1}' = \check{L}_i \right)$ $\square$

<u>rmk</u> Lemma 2 works for a finite set of vector bundles.

Lemma 3 Let $E = E_r \supset E_{r-1} \supset \cdots \supset E_1 \supset E_0 = 0$ a filtration with $L_i = E_i/E_{i-1} \in VB_1(X)$.

Let $0 \neq s \in \Gamma(E)$ s.t. $Y_i := \{x \in X \mid s(x) \in E_{i-1,x} \otimes k(x)\}$: non-sing for $i = 1, \cdots, r$.

Then $\prod_{i=j}^{r} \xi_i = (f_j)_*(Y_j)\ \ \forall\, 1 \leq j \leq r$

where $f_j : Y_j \hookrightarrow X$ the closed immersion and $\xi_i \in A(\mathbb{P}(E))$ is associated to L_i.

In particular, if s is nowhere vanishing, $\prod_{i=1}^{r} \xi_i = 0$.

pf: Induction on j. For $j = r$, $\xi_r = [Y_r] = (f_r)_*(Y_r)$ by definition.

Let $j < r$, consider $Y_j \overset{i}{\hookrightarrow} Y_{j+1} \overset{f_{j+1}}{\hookrightarrow} X$ with $f_j = f_{j+1} \circ i$.

Note $s|_{Y_{j+1}} \in \Gamma(E_i|_{Y_{j+1}}) \Rightarrow f_{j+1}^* \xi_j = i_* Y_j$.

Then $\prod_{i=j}^{r} \xi_i = \xi_j \cdot (f_{j+1})_*(Y_{j+1}) = (f_{j+1})_*(f_{i+1}^* \xi_j \cdot Y_{j+1}) = (f_{j+1})_* i_* Y_j = (f_j)_* Y_j$. $\square$

Cor Suppose we have a filtration $E = E_r \supset E_{r-1} \supset \cdots \supset E_1 \supset E_0 = 0$ with $L_i = E_i/E_{i-1} \in VB_1(X)$.

and $\xi_i \in A(\mathbb{P}(E))$ is associated to L_i. Then $c_t(E) = \prod_{i=1}^{r} c_t(L_i)$.

pf: Consider $\pi : \mathbb{P}(E) \to X$. Let $F = \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^* E$, $F_i = \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^* E_i$

Then we have a filtration $F = F_r \supset F_{r-1} \cdots \supset 0$ with $L_i' = F_i/F_{i-1} \leftrightarrow \xi_E + \xi_i'$

Note $\mathcal{O}_{\mathbb{P}(E)}(-1) \hookrightarrow \pi^* E \Rightarrow \mathcal{O}_{\mathbb{P}(E)} \hookrightarrow \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^* E = F \rightsquigarrow s \in \Gamma(F)$.

We want to apply Lemma 3. Since the conditions in Lemma 3 can be checked locally,

we may assume $X = \operatorname{Spec} A$, $E_i = \tilde{A}^{\oplus n-i}$. Then $\mathbb{P}(E) = \mathbb{P}_A^n = \operatorname{Proj} \underbrace{A[x_0, \cdots, x_{n-1}]}_{=S}$

and $\mathcal{O}_{\mathbb{P}(E)} \hookrightarrow \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^* E$ is given by $S \hookrightarrow S(1)^{\oplus n} = \bigoplus_{i=0}^{n-1} S(1) e_i$

Thus, s is $\sum_{i=0}^{n-1} x_i e_i$ and $Y_i = V(x_{i-1}, \cdots, x_{n-1})$: non-sing.

Then Lemma 3 $\Rightarrow 0 = \prod_{i=1}^{r} (\xi_{\mathcal{E}} + \xi_i')$. Note $\xi_i = \pi^* \xi_i$.

Thus, $\pi^* c_i(E) = e_i(\xi_1', \cdots, \xi_r') = \pi^* e_i(\xi_1, \cdots, \xi_r) \Rightarrow c_t(E) = \prod_{i=1}^{r} c_t(L_i)$.

where e_i is the i-th elementary symmetric polynomial of r variables. □

Return to the proof of (c) :

By Lemma 2, it suffices to show $c_t(f^*E) = c_t(f^*E')\, c_t(f^*E'')$

Note that if we have filtrations of f^*E' and f^*E'' with line bundles L_i', L_j'' resp.

then f^*E filters with line bundles L_i', L_j'.

(Say $f^*E' = E_r' \supset E_{r-1}' \supset \cdots \supset 0$ & $f^*E'' = E_r'' \supset E_{r-1}'' \supset \cdots \supset 0$, then

$f^*E = E_r = \tilde{E}_r \supset \tilde{E}_{r-1} \supset \cdots \supset \tilde{E}_1 \supset \tilde{E}_0 = E_r' \supset E_{r-1}' \supset \cdots \supset 0$ where $\tilde{E}_i$ = lifting of E_i'')

Thus, Cor $\Rightarrow c_t(f^*E) = \prod_i c_t(L_i') \prod_j c_t(L_j') = c_t(f^*E')\, c_t(f^*E'')$ □

rmk The Chern class is characterized by Prop (a) ~ (c).

Examples Let $c_t(E) = \prod_{i=1}^{r} (1 + a_i t) \in VB_r(X)$ and $F = \prod_{j=1}^{s} (1 + b_j t) \in VB_s(X)$.

(1) $c_t(E \otimes F) = \prod_{i,j} (1 + (a_i + b_j)t)$

(2) $c_t(\Lambda^p E) = \prod_{1 \le i_1 < \cdots < i_p \le r} (1 + (a_{i_1} + \cdots + a_{i_p})t)$

(3) $c_t(\check{E}) = c_{-t}(E)$.

Def Formally, let $c_t(E) = \prod_{i=1}^{r} (1 + a_i t)$ for some $a_i \in A^1(X)$

(exponential Chern charactor) $ch(E) := \sum_{i=1}^{r} e^{a_i}$ where $e^x = 1 + x + \frac{1}{2}x^2 + \cdots$.

(Todd class) $td(E) := \prod_{i=1}^{r} a_i / (1 - e^{-a_i})$ where $\frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 \cdots$

Next, we want to define the Chern class on the Grothendieck group $K(X)$.

Recall $K(X) \simeq K_1(X) := \mathcal{F}(VB(X)) \big/ (E - E' - E'' \mid 0 \to E' \to E \to E'' \to 0 \text{ SES})$

(See [Hartshorne] Ex II. 6.10 & Ex IV. 6.9)

Thus the map c_i can be extended to $K(X) \to A^i(X)$ for $i = 0, \cdots, r$.

Similarly, we have $c_t : K(X) \to A(X)[t]$ and $ch : K(X) \to A(X) \otimes \mathbb{Q}$.

Def Let $f : X' \to X$ be proper. Define the additive map $f_! : K(X') \to K(X)$ by

$$f_!(E) = \sum (-1)^i R^i f_*(E).$$

Thm (G.R.R.) Let $f : X \to Y$ be smooth morphism between proj. var. Then, for $x \in K(X)$,

$$ch(f_!(x)) = f_*(ch(x) . td(\mathcal{T}_f)) \in A(Y) \otimes \mathbb{Q}$$ where $\mathcal{T}_f$ is the relative tangent sheaf of f.

$$\left(ch(f_!(x)) . td(\mathcal{T}_Y) = f_*(ch(x) . td(\mathcal{T}_X)) \right)$$