<u>Notation</u> For coherent sheaf $\mathcal{F}, \mathcal{G}$ on X,

- $[\mathcal{F}].[\mathcal{G}] = \sum_{i \geq 0} (-1)^i [\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})] \underset{\text{linearly}}{\overset{\text{extend}}{\rightsquigarrow}} K(X) \times K(X) \xrightarrow{\ \cdot\ } K(X)$

- $\lambda^i [\mathcal{F}] := [\Lambda^i \mathcal{F}]$ in $K(X)$. $\underset{\text{linearly}}{\overset{\text{extend}}{\rightsquigarrow}} \lambda_t : K(X) \longrightarrow K(X)[[t]]$
  $\mathcal{G} \longmapsto \lambda_t(\mathcal{G}) = 1 + \sum_{i \geq 1} \lambda^i [\mathcal{G}] t^i$.

- $\lambda_{-1}[\mathcal{F}] = 1 - [\mathcal{F}] + [\Lambda^2 \mathcal{F}] - [\Lambda^3 \mathcal{F}] + \cdots$

<u>Property</u> $0 \to E' \to E \to E'' \to 0$: short exact sequence of vector bundles,

then • $\lambda_t(E) = \lambda_t(E') \cdot \lambda_t(E'')$. • $td(E) = td(E') \cdot td(E'')$.

• $ch(E) = ch(E') \cdot ch(E'')$.

For a morphism $f : Y \to X$

- $f^! : K(X) \longrightarrow K(Y)$: pull back of vector bundle.
- $f_! : K(Y) \longrightarrow K(X)$: $f_!(\mathcal{F}) = \sum_{i \geq 0} (-1)^i R^i f_*(\mathcal{F})$.

<u>Property</u> For $x \in K(X)$, $y \in K(Y)$, we have $f_!(y \cdot f^!(x)) = f_!(y) \cdot x$.

<u>Theorem</u> (GRR)

$f : X \to Y$: proper, X, Y : quasi-projective non-singular varieties over $k = \bar{k}$.

Then, for $x \in K(X)$, we have $f_*(ch(x) . td(\mathcal{T}_X)) = ch(f_!(x)) . td(\mathcal{T}_Y)$.

<u>Lemma</u> $X \xrightarrow{g} Y \xrightarrow{f} Z$: proper between non-singular varities over $k = \bar{k}$.
(quasi-projective) $x \in K(X)$, $y = g_!(x)$

(1) If GRR holds for (g, x) and (f, y), then GRR holds for $(f \circ g, x)$

(2) If GRR holds for $(f \circ g, x)$ and (f, y), and f_* is injective, then GRR holds for (g, x)

(3) $f : Y \to X$, $y \in K(Y)$; If GRR holds for (f, y) and (f', y'),
$f' : Y' \to X'$, $y' \in K(Y')$, then GRR holds for $(f \times f', y \otimes y')$.

<u>Goal</u>: $i : Y \hookrightarrow X$: closed immersion, non-singular, quasi-projective subvariety.

$y \in K(Y)$, then $i_*(ch(y).td(\mathcal{T}_Y)) = ch(i_!(y)).td(\mathcal{T}_X)$.

<u>Reduction 1</u>: $0 \to \mathcal{T}_Y \to \mathcal{T}_X|_Y \to \mathcal{N}_{Y/X} \to 0$ $\left(\begin{array}{l}\mathcal{N}_{Y/X} : \text{normal sheaf} \\ \quad \shortparallel \\ \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y).\end{array}\right)$

$\rightsquigarrow \quad i^* td(\mathcal{T}_X) = td(\mathcal{T}_Y).td(\mathcal{N}_{Y/X})$

$$i_*(ch(y).td(\mathcal{N}_{Y/X})^{-1}) = i_*(ch(y).td(\mathcal{T}_Y).i^*td(\mathcal{T}_X)^{-1})$$

$$= i_*(ch(y).td(\mathcal{T}_Y)).td(\mathcal{T}_X)^{-1} \quad \text{// projection formula}$$

Then, it suffices to show $ch(i_!(y)) = i_*(ch(y).td(\mathcal{N}_{Y/X})^{-1})$.

<u>Prop 1</u>: $Y_1, \dots, Y_m$: non-singular subvarieties of X such that

$Y_{i-1} \cap \cdots \cap Y_1$ intersect Y_i transversally for $i = 2, \dots, n$.

Then, in $K(X)$, we have $[\mathcal{O}_{Y_1 \cap \cdots \cap Y_m}] = \prod_{i=1}^{m} [\mathcal{O}_{Y_i}]$.

pf: By induction, it suffices to show $m = 2$, Y, Z meets transversally.

$$[\mathcal{O}_Y].[\mathcal{O}_Z] = \sum_i (-1)^i [Tor_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)]$$

- If $Y \cap Z = \phi$, then both side $= 0$.
- If $a \in Y \cap Z$, write the local equations $f_1, \dots, f_p$ for Y; $g_1, \dots, g_q$ for Z.

Use Koszul complex of $\mathcal{O}_Y$: $0 \to \wedge^p E \to \cdots \to E \to \mathcal{O}_X$ to compute Tor.

$\overset{\otimes \mathcal{O}_Z}{\rightsquigarrow} \quad 0 \to \mathcal{O}_Z \otimes_K \wedge^p E \to \cdots \to \mathcal{O}_Z \otimes_K E \to \mathcal{O}_Z$

transverally $\Rightarrow$ $\{f_i\}$ form regular sequence of $\mathcal{O}_{Z,a} = \mathcal{O}_{X,a} / \langle g_1, \dots, g_q \rangle$.

$$[\mathcal{O}_Y].[\mathcal{O}_Z] \underset{\substack{\uparrow \\ \text{regular sequence}}}{=} [Tor_0^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)] = [\mathcal{O}_{Y \cap Z}] \text{ since } \mathcal{O}_{Z,a} / \langle f_1, \dots, f_p \rangle = \mathcal{O}_{Y \cap Z}.$$

#

Corollary $Y \subseteq X$: non-singular hyperplane section.

$k = \dim X$. Then, $(1 - [\mathcal{O}(Y)])^{k+1} = 0$.

pf: $[\mathcal{O}_{Y_1}] = [\mathcal{O}_{Y_2}]$ for any two non-singular hyperplane sections of X.

$$0 \to \mathcal{O}(-Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

In $K(X)$, we have $[\mathcal{O}_Y] = [\mathcal{O}_X] - [\mathcal{O}(Y)]^{-1}$

$$\rightsquigarrow 0 \underset{\text{Prop 1}}{=\!=} [\mathcal{O}_Y]^{k+1} = (1 - [\mathcal{O}(Y)]^{-1})^{k+1}$$

\#

Prop 2 : $i : Y \hookrightarrow X$: non-singular subvariety.

For any $y \in K(Y)$, $i^! i_! (y) = y \cdot \lambda_{-1}(\mathcal{N}^*)$.

In particular, $i^! [\mathcal{O}_Y] = \lambda_{-1}(\mathcal{N}^*)$.

pf: By linearity, enough to show $y = [\mathcal{F}]$. $\mathcal{F}$: locally free.

$$i^! i_! [\mathcal{F}] = i^! [i_* \mathcal{F}] = \sum_i (-1)^i [\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_Y)].$$

$$[\mathcal{F}] \cdot \lambda_{-1}(\mathcal{N}^*) = [\mathcal{F}] \cdot (1 - [\mathcal{N}^*] + [\Lambda^2 \mathcal{N}^*] - \cdots)$$

$$= [\mathcal{F}] - [\mathcal{F} \otimes \mathcal{N}^*] + [\mathcal{F} \otimes \Lambda^2 \mathcal{N}^*] - \cdots$$

use $\mathcal{F}$: locally free $\mathrm{Tor}_i = 0$ for $i \geq 1$.

Now, it suffices to show, since $\mathcal{N}^* \simeq \mathcal{J}/\mathcal{J}^2$

(1) $\mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_Y) = \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{J}/\mathcal{J}^2$

(2) $\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_Y) = \mathcal{F} \otimes_{\mathcal{O}_Y} \Lambda^i \mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y)$ $\boxed{\text{(2) use (1)!}}$

(1): $0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$

$$\overset{\otimes \mathcal{F}}{\rightsquigarrow} \quad 0 \to \mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_Y) \xrightarrow{f} \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{\varphi} \mathcal{F}$$

$$(u \otimes v) \longmapsto uv$$

$\mathcal{F}$: support on Y, annihilated by $\mathcal{I}$. $\Rightarrow$ f: isomorphism.

Also, the image of $\mathcal{I}^2 \otimes \mathcal{F}$ in $\mathcal{I} \otimes \mathcal{F}$ is zero

$$\Rightarrow \mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_Y) \simeq \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \simeq \mathcal{I}/\mathcal{I}^2 \otimes_{\mathcal{O}_X} \mathcal{F} \simeq \mathcal{I}/\mathcal{I}^2 \otimes_{\mathcal{O}_Y} \mathcal{F}.$$

(2): local equation for Y: $f_1 \cdots f_p$

$$\xrightarrow[\text{complex}]{\text{Koszul}} 0 \to \mathcal{F} \otimes_k \Lambda^p E \to \mathcal{F} \otimes_k \Lambda^{p-1} E \to \cdots \to \mathcal{F} \otimes_k E \to \mathcal{F}$$

$\{f_i\}$: sections of $\mathcal{I}$, annihilated by $\mathcal{F}$ $\Rightarrow$ differential are zero

$$\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_Y) = \mathcal{F} \otimes_k \Lambda^i E = \mathcal{F} \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y \otimes_k \Lambda^i E)$$

Take $\mathcal{F} = \mathcal{O}_Y$, $i = 1$ $\Rightarrow$ $\mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) = \mathcal{O}_Y \otimes_k E$

Then, $\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_Y) = \mathcal{F} \otimes_{\mathcal{O}_Y} \Lambda^i \mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y)$. #

<u>Prop</u> $Y \overset{i}{\hookrightarrow} X$: codim $= 1$ subvariety. $\mathcal{L} = \mathcal{O}(Y)|_Y$. Then,

(1) $\mathcal{L} = \mathcal{N}_{Y/X}$ $\left(\mathcal{O}(-Y) \simeq \mathcal{I},\ \mathcal{O}(-Y)|_Y = \mathcal{I}/\mathcal{I}^2|_Y \Rightarrow \mathcal{O}(Y)|_Y \simeq (\mathcal{I}/\mathcal{I}^2|_Y)^\vee \right)$.

(2) $[\mathcal{O}_Y] = [\mathcal{O}_X] - [\mathcal{O}(Y)]^{-1}$

(3) $i^* i_* (y) = y(1 - \mathcal{L}^*)$ for $y \in K(Y)$. (use (1) and <u>Prop 2</u>)

<u>Theorem</u> (GRR for divisor)

$i : Y \hookrightarrow X$: codim $= 1$, non-singular variety. $y = i^!(x)$ for some $x \in K(X)$.

then we have $ch(i_!(y)) = i_*(ch(y) . td(\mathcal{N}_{Y/X})^{-1})$.

pf: $$ch(i_! i^!(x)) \underset{\substack{\uparrow \\ \text{projection} \\ \text{formula}}}{=} ch(x . i_!(1)) = ch(x . (1 - [\mathcal{O}(Y)]^{-1}))$$

$$= ch(x) . ch(1 - [\mathcal{O}(Y)]^{-1}) = ch(x) . (1 - e^{-[Y]})$$

$$\boxed{\begin{array}{lll} D : \text{divisor} & c(\mathcal{O}(D)) = 1 + D & \\ & ch(\mathcal{O}(D)) = e^{D} & td(\mathcal{O}(D)) = \dfrac{D}{1 - e^{-D}} \end{array}}$$

$$i_*(ch(y) . td(\mathcal{N}_{Y/X})^{-1}) = i_*(i^* ch(x) . i^* td(\mathcal{O}(Y))^{-1})$$

$\wr$ projection formula

$$= ch(x) . td(\mathcal{O}(Y))^{-1} . i_*(1)$$

$$= ch(x) . td(\mathcal{O}(Y))^{-1} . [Y]$$

Finally, note that $td(\mathcal{O}(Y))^{-1} = [Y] . (1 - e^{-[Y]})^{-1}$

#

<u>Corollary</u> GRR holds for $i: Y \hookrightarrow Y \times \mathbb{P}^n$, P_0: fixed point in $\mathbb{P}^n$.
$a \longmapsto (a, P_0)$

pf: GRR holds for $i: Y \to Y$: identity, so it sufficies to prove

$i: \{a\} \hookrightarrow \mathbb{P}^n$. $\left(\begin{array}{l} f: Y \to X \\ f': Y' \to X' \end{array} : \text{GRR holds} \Rightarrow \underset{\text{GRR holds}}{f \times f': Y \times Y' \to X \times X'} \right)$

$n = 1$: divisor case, ok!

$n > 1$: $i: \{a\} \underset{\substack{\text{ok!} \\ \Uparrow \\ \text{induction}}}{\overset{u}{\hookrightarrow}} \underset{\substack{\shortparallel \\ \text{hyperplane}}}{H} \overset{v}{\hookrightarrow} \mathbb{P}^n$, $K(\{a\}) = \mathbb{Z}$.

Suffices to show GRR for $\underset{\text{divisor case.}}{v: H \hookrightarrow \mathbb{P}^n}$, $u_*(1) \in v^*(K(\mathbb{P}^n))$?.

Pick another hyperplane $Z \subseteq \mathbb{P}^n$, $\underset{\substack{\shortparallel \\ \text{line}}}{L} \subseteq H$ s.t. $\{a\} = Z \cap L \cap H$.

<u>Prop 1</u> $\Rightarrow$ $[\underset{\substack{\shortparallel \\ \{a\}}}{\mathcal{O}_Y}] = [\mathcal{O}_L] . \underbrace{[\mathcal{O}_{H \cap Z}]}_{\substack{\shortparallel \\ 1 - [\mathcal{O}(H \cap Z)]^{-1}}}$.

Z : hyperplane $\rightsquigarrow$ $\mathcal{O}(H \cap Z) = \mathcal{O}(H)|_H$.

<u>Prop (3)</u> $\Rightarrow$ $u_*(1) = [\mathcal{O}_Y] = v^*(v_*[\mathcal{O}_L])$.

#

<u>Corollary</u> $Y \hookrightarrow X$: non-singular subvariety.

If GRR holds for $2\dim(Y) \le \dim(X) - 2$, then GRR holds for general.

pf: GRR holds for $X \to X \times \mathbb{P}^n$. $Y \overset{i}{\hookrightarrow} X \overset{f}{\longrightarrow} X \times \mathbb{P}^n$

pick n large s.t. $\dim(X) + n - 2 \ge 2\dim(Y)$. Then, $Y \to X \times \mathbb{P}^n$ holds.

f_* : injective $\Rightarrow$ $Y \overset{i}{\hookrightarrow} X$ holds!

#

Reduction 2: By above corollary, to prove GRR for $i : Y \hookrightarrow X$, $p = \mathrm{codim}_X(Y)$

we may assume $p \geq \dim(Y) + 2$.

Consider the diagram

$$\begin{tikzcd} Y' \arrow[r, hook, "j"] \arrow[d, "g"'] & X' = Bl_Y X \arrow[d, "f"] \\ Y \arrow[r, hook, "i"'] & X \end{tikzcd}$$

exceptional divisor — Y'

projective bundle — g, fiber $= \mathbb{P}^{p-1}$

$\mathrm{codim}_X Y = p$

$\vartheta \longleftrightarrow \mathcal{O}_X(-Y)$

$\vartheta' \longleftrightarrow \mathcal{O}_{X'}(-Y')$

$\vartheta/\vartheta^2 \longleftrightarrow N^*$

$\vartheta'/\vartheta'^2 \longleftrightarrow L^*$

$\vartheta/\vartheta^2 \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'} \longleftrightarrow \tilde{N}^*$

N : normal bundle of Y in X

$\tilde{N} = g^* N$

L = line bundle in X' corresponding to Y'

we have surjective map $\vartheta/\vartheta^2 \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'} \twoheadrightarrow \vartheta'/\vartheta'^2 \rightsquigarrow L \hookrightarrow \tilde{N}$

$F := \tilde{N}/L$: vector bundle of rank $p-1$.

lemma ⓪ G : rank k vector bundle over X. Then, $ch(\lambda_{-1} G) = c_k(G^*) . td(G^*)^{-1}$.

① $f_*(1) = 1$ for $f_* : A(Y) \to A(X)$ and thus $f_* f^* = id_{A(X)}$.

② $g_*(c_{p-1}(F)) = 1$

③ $f^! i_!(y) = j_!(g^!(y) . \lambda_{-1}[F^*])$ for $y \in K(Y)$

④ $\lambda_{-1}[F^*] \equiv 0 \pmod{1 - L^*}$ if $p \geq \dim Y + 2$.

pf of GRR:

$p \geq \dim Y + 2$, we have $g^!(y) . \lambda_{-1}[F^*] \equiv 0 \pmod{1 - L^*}$.

$j^! j_! (\tilde{y}) = \tilde{y} . (1 - L^*)$ Prop (3)

Apply GRR for divisor $Y' \xhookrightarrow{j} X'$, we have

$$ch(j_!(g^!(y) . \lambda_{-1}[F^*])) = j_*(ch(g^!(y) . \lambda_{-1}[F^*]) . td(L)^{-1})$$

$$\Rightarrow f_*(ch(j_!(g^!(y) . \lambda_{-1}[F^*]))) = f_* j_* (ch(g^!(y) . \lambda_{-1}[F^*]) . td(L^{-1}))$$

$$\begin{array}{ccc} f_*(ch(j_!(g^!(y) . \lambda_{-1}[F^*]))) & = & f_* j_* (ch(g^!(y) . \lambda_{-1}[F^*]) . td(L^{-1})) \\ \| (1) & & \| (2) \\ ch(i_!(y)) & & i_*(ch(y) . td(N)^{-1}) \end{array}$$

(1): $f_*(ch(j_!(g^!(y) . \lambda_{-1}[F^*]))) \overset{\text{lemma}}{\underset{③}{=\!=}} f_*(ch(f^! i_!(y))) = f_* f^* ch(i_!(y))$

$\overset{\text{lemma}}{\underset{①}{=\!=}} ch(i_!(y))$

(2): $ch(g^!(y) . \lambda_{-1}[F^*]) = ch(g^!(y)) . ch(\lambda_{-1}[F^*]) = g^* ch(y) . ch(\lambda_{-1}[F^*])$

$\overset{\text{lemma}}{\underset{⓪}{=\!=}} g^* ch(y) . c_{p-1}(F) . td(F)^{-1}$.

Note that $\tilde{N}/L = F \rightsquigarrow g^* td(N) = td(\tilde{N}) = td(F) . td(L)$

$$\Rightarrow ch(g^!(y) . \lambda_{-1}[F^*]) . td(L)^{-1} = g^*(ch(y)) . c_{p-1}(F) . g^*(td(N)^{-1})$$

$\overset{g_*}{\rightsquigarrow} g_*(ch(g^!(y) . \lambda_{-1}[F^*]) . td(N)^{-1}) \overset{\text{lemma ②}}{\underset{\text{projection formula}}{=\!=}} ch(y) . td(N)^{-1}$

$\overset{i_*}{\rightsquigarrow} \underbrace{i_* g_*}_{= f_* j_*} (ch(g^!(y) . \lambda_{-1}[F^*]) . td(N)^{-1}) = i_*(ch(y) . td(N)^{-1})$

#

lemma⓪ G : rank k vector bundle over X. Then, $ch(\lambda_{-1}G) = c_k(G^*) . td(G^*)^{-1}$.

pf: $c(G) = \prod_{i=1}^{k}(1+a_i)$ by splitting principle.

$$c(G^*) = \prod_{i=1}^{k}(1-a_i) \quad , \quad c(\Lambda^j G) = \prod_{i_1<\cdots<i_j}(1+a_{i_1}+\cdots+a_{i_j})$$

top chern class:

$$C_k(G^*) = (-1)^k a_1 \cdots a_k$$

$$ch(\lambda_{-1}(G)) = \prod_{i=1}^{k}(1-e^{a_i}) = \overbrace{td(G^*)^{-1}}^{\prod_{i=1}^{k}\frac{-a_i}{1-e^{a_i}}} . C_k(G^*)$$

$$\lambda_{-1}(G) = 1 - G + \Lambda^2 G - \Lambda^3 G + \cdots + (-1)^k \Lambda^k G$$

#

lemma① $f_*(1) = 1$ for $f_* : A^*(X') \longrightarrow A^*(X)$. Thus, $f_* f^* = id_{A^*(X)}$.

pf: $f: X' \longrightarrow X$: isomorphism except Y'.

It has local degree 1.

$\rightsquigarrow f_*(1) = 1$.

#

lemma② $g_*(C_{p-1}(F)) = 1$.

pf: $C_1(L) = -[H]$, $H \in \mathbb{P}^{p-1}$: hyperplane. $g: Y' \longrightarrow Y$ $\left(\begin{matrix}\text{codimension decreasing} \\ p-1\end{matrix}\right)$

$g_*([H]^{p-1}) = 1$ and $g_*([H])^i = 0$ for $0 \le i \le p-2$. (counting dimension)

$\tilde{N}/L = F \rightsquigarrow c(\tilde{N}) = c(F) . c(L) = c(F)(1-[H])$

$\rightsquigarrow c(F) = g^* c(N) . \left(1 + [H] + [H]^2 + [H]^3 + \cdots\right)$

$\rightsquigarrow C_{p-1}(F) = [H]^{p-1} + g^*(C_1(N)) . [H]^{p-2} + \cdots + g^*(C_{p-1}(N))$

$$\xrightarrow{g_*} g_*(C_{p-1}(F)) = g_*([H]^{p-1}) + g_*(g^*(C_1(N)) . [H]^{p-2}) + \cdots + g_* g^*(C_{p-1}(N))$$

$= 1$

projection formula

#

lemma ③ For $y \in K(Y)$, $f^! i_!(y) = j_!(g^!(y) . \lambda_{-1}[F^*])$.

pf: $0 \to F^* \to \mathcal{I}/\mathcal{I}^2 \underset{\mathcal{O}_Y}{\otimes} \mathcal{O}_{Y'} \xrightarrow{\mu} \mathcal{I}'/\mathcal{I}'^2 \to 0$

$\quad\quad\quad\quad\quad\quad\quad \updownarrow \quad\quad\quad\quad\quad\quad \updownarrow$

$\quad\quad\quad\quad\quad\quad\quad \tilde{N}^* \quad\quad\quad\quad\quad\; L^*$

By linearity, it suffices to show $y = [\mathcal{G}]$, $\mathcal{G}$: locally free.

$g^!(y) = [\mathcal{G} \underset{\mathcal{O}_Y}{\otimes} \mathcal{O}_{Y'}]$ is locally free on Y'.

$$g^!(y) . \lambda_{-1}[F^*] = \sum_{i \geq 0} (-1)^i \left[\mathcal{G} \underset{\mathcal{O}_Y}{\otimes} \mathcal{O}_{Y'} \underset{\mathcal{O}_{Y'}}{\otimes} \left(\Lambda^i [F^*]\right)\right]$$

$$= \sum_{i \geq 0} (-1)^i \left[\mathcal{G} \underset{\mathcal{O}_Y}{\otimes} \Lambda^i [F^*]\right]$$

$$f^! i_!(y) = \sum_{j \geq 0} (-1)^j \left[\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{G}, \mathcal{O}_{X'})\right]$$

Now, it suffices to show

(1) $\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_{X'}) = \Lambda^i \, \mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_{X'})$ for $i \geq 1$

(2) $[\mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_{X'})] = [F^*]$

(3) $\mathrm{Tor}_j^{\mathcal{O}_X}(\mathcal{G}, \mathcal{O}_{X'}) = \mathcal{G} \underset{\mathcal{O}_Y}{\otimes} \mathrm{Tor}_j^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_{X'})$ for $j \geq 1$

(1) both side vanish outside Y'. Pick $\underset{\text{open}}{U \subseteq X}$ for some $b' \in Y'$.

$b = g(b')$ ($f_1 \cdots f_p$: locally define $U \cap Y$.)

locally coordinate :

$U' = f^{-1}(U) = \{ (x,y) \mid x_i f_j(y) - x_j f_i(y) = 0 \}$

Consider Koszul complex :

$$0 \to \mathcal{O}_{X'} \otimes_k \Lambda^p E \to \cdots \to \mathcal{O}_{X'}$$

Suppose $b' \in U_j' = \{ x_j \neq 0 \}$

$\rightsquigarrow 0 \to \mathcal{O}_{X'} \otimes_k \Lambda^p E' \to \cdots \to \mathcal{O}_{X'}$ where E' has basis $\{e_i'\}$ with differential $d(1 \otimes e_j') = f_j \otimes 1$, $\underset{(i \neq j)}{d(1 \otimes e_i')} = \left(f_i - f_j \frac{x_i}{x_j} \right) \otimes 1$.

cycle : $Z_s = \mathcal{O}_{X'} \otimes_k \Lambda^s (e_1', \ldots, \hat{e}_j', \ldots, e_p')$

boundary : $B_s = f_j \cdot \mathcal{O}_{X'} \otimes_k \Lambda^s (e_1', \ldots, \hat{e}_j', \ldots, e_p')$

$$\rightsquigarrow \mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_{X'}) \simeq \mathcal{O}_{Y'} \otimes_k \Lambda^i (e_1', \ldots, \hat{e}_j', \ldots, e_p')$$

#

(2)

$$0 \to \vartheta \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

$\xrightarrow{\otimes \mathcal{O}_{X'}}$ $0 \to \mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_{X'}) \to \vartheta \otimes_{\mathcal{O}_X} \mathcal{O}_{X'} \xrightarrow{g} \mathcal{O}_{X'}$

$\vartheta \otimes_{\mathcal{O}_X} \mathcal{O}_{X'} \xrightarrow{g} \vartheta' \to 0$.

$\boxed{\mathrm{Im}(g) = \vartheta \mathcal{O}_{X'} = \vartheta'}$

<u>Claim</u> : $g = \mu : \vartheta/\vartheta^2 \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'} \twoheadrightarrow \vartheta'/\vartheta'^2$

$\xrightarrow{\otimes_{\mathcal{O}_{X'}} \mathcal{O}_{Y'}}$

$\boxed{\begin{array}{l} \vartheta' = \mathcal{O}_{X'}(-Y') \text{ : locally free} \\ \rightsquigarrow \mathrm{Tor}_1^{\mathcal{O}_{X'}}(\vartheta', \mathcal{O}_{Y'}) = 0 \end{array}}$

$$\underset{0}{\overset{}{\mathrm{Tor}_1^{\mathcal{O}_{X'}}(\vartheta', \mathcal{O}_{Y'})}} \to \overbrace{\mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_{X'})}^{\text{support on } Y'} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{Y'} \to \vartheta \otimes_{\mathcal{O}_X} \mathcal{O}_{Y'} \to \vartheta' \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{Y'} \to 0$$

$$\mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_{X'}) \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{Y'} = \mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_{X'}), \quad \vartheta \otimes_{\mathcal{O}_X} \mathcal{O}_{Y'} \simeq \vartheta/\vartheta^2 \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'}, \quad \vartheta' \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{Y'} \simeq \vartheta'/\vartheta'^2$$

#

(3): Consider $T(\mathcal{G}, \mathcal{O}_{X'}) = \mathcal{G} \underset{\mathcal{O}_Y}{\otimes} \mathcal{O}_Y \otimes \mathcal{O}_{X'}$: bifunctor.

Compute $L_i T$ (left derived functor of T)

Consider $E_2^{ij} = \mathrm{Tor}_i^{\mathcal{O}_Y}(\mathrm{Tor}_j^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Y), \mathcal{G})$

$E_2'^{ji} = \mathrm{Tor}_j^{\mathcal{O}_X}(\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{G}, \mathcal{O}_Y), \mathcal{O}_{X'})$

$E_2^{ij} = E_2'^{ji} = 0$ for $i > 0$.

$$\rightsquigarrow \underset{\mathcal{G} \otimes \mathrm{Tor}_j^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Y)}{\underset{\|}{E_2^{0j}}} = E_2'^{j0} = \underset{\mathrm{Tor}_j^{\mathcal{O}_X}(\mathcal{G}, \mathcal{O}_{X'})}{\underset{\|}{L_j T(\mathcal{G}, \mathcal{O}_{X'})}}$$

#

lemma ④ $\lambda_{-1}(F^*) \equiv 0 \pmod{1 - L^*}$ if $p \geq \dim(Y) + 2$.

lemma $g = \dim(Y)$, G: vector bundle of rank $p = g + k$ on Y, $(k \geq 0)$

Then, $\Lambda^s([G] - k) = 0$ for $s \geq g + 1$.

pf: ℓ : line bundle of hyperplane section of Y.

$\xrightarrow{\text{Prop}} (1 - [\ell])^{g+1} = 0$ Write $[\ell] = 1 + u \in K(Y)$, $u^{g+1} = 0$

$\Rightarrow [\ell]^n = \sum_{i=0}^{g} \binom{n}{i} u^i$

$\Rightarrow \lambda_t([G].[\ell]^n - k) = \prod_{i=1}^{g} \lambda_t([G].u^i)^{\binom{n}{i}} \cdot (1-t)^{-k}$.

$\xrightarrow{\text{degree } s} \lambda^s([G].[\ell]^n - k) = \sum_{i=1}^{m_s} B_{s,i} P_{s,i}(n)$ for some $B_{s,i} \in K(Y)$, $\underset{\text{integer-valued when } n \gg 0}{P_{s,i}(n) \in \mathbb{Q}[n]}$.

$P_{s,i}(n) = \mathbb{Z}$-linear combination of $\binom{n}{j}$ $\rightsquigarrow \lambda^s([G].[\ell]^n - k) = \sum_{i=0}^{n_s} A_{s,i} \binom{n}{i}$

for some $A_{s,i} \in K(Y)$.

For $n > n_0$, $G \otimes \ell^n$ is g.b.g.s., $G \otimes \ell^n$ contains a trivial fiber of rank k.

$\rightsquigarrow [G].[\ell]^n - k = [G']$ for some fiber G' of rank g.

G' has rank g $\rightsquigarrow \Lambda^s[G'] = 0$ for $s > g+1$.

Now, it suffices to show

if $P(n) = \sum_{i=0}^{m} A_i \binom{n}{i} = 0$ with $A_i \in K(Y)$ for $n > n_0$, then $A_i = 0$ for all i.

By induction on m, consider $P(n+1) - P(n) = \sum_{i=0}^{m} A_i \left(\binom{n+1}{i} - \binom{n}{i}\right)$

$= \sum_{i=0}^{m} A_i \binom{n}{i-1} = \sum_{j=0}^{m-1} A_{j+1} \binom{n}{j} = 0$ for $n > n_0$.

$\Rightarrow A_1, \ldots, A_m = 0$. Also, $A_0 = 0$.

#

pf of lemma④: Let $\mathcal{G}$ be a locally free sheaf of rank k, then we have

(1) $\Lambda^k([\mathcal{G}] - 1) = (-1)^k \lambda_{-1}[\mathcal{G}]$

pf: $\lambda_t([\mathcal{G}] - 1) = \dfrac{\lambda_t[\mathcal{G}]}{\lambda_t(1)} = \lambda_t[\mathcal{G}](1+t)^{-1} = \lambda_t[\mathcal{G}](1 - t + t^2 - t^3 + \cdots)$

$= (1 + [\mathcal{G}]t + [\Lambda^2\mathcal{G}]t^2 + \cdots)(1 - t + t^2 - t^3 + \cdots)$

Now, take degree k coefficient.

#

(2) $\lambda_t([\mathcal{G}].(1 - [L])) \equiv 1 \pmod{1 - L}$

pf: L: invertible sheaf. $\Lambda^i([\mathcal{G}][L]) = [L]^i \Lambda^i[\mathcal{G}]$. $\Rightarrow \Lambda^i([\mathcal{G}][L]) \equiv \Lambda^i[\mathcal{G}]$ $\pmod{1 - L}$

#

Apply (1) to F^*: $(-1)^{p-1} \lambda_{-1}[F^*] = \Lambda^{p-1}([F^*] - 1)$

- $\tilde{N}^*/F^* \simeq L^* \Rightarrow [F^*] - 1 \equiv [\tilde{N}^*] - 2 \pmod{1 - L^*}$

$\longrightarrow (-1)^{p-1} \lambda_{-1}[F^*] = \Lambda^{p-1}([F^*] - 1) \equiv \Lambda^{p-1}([\tilde{N}^*] - 2) \pmod{1 - L^*}$

$$\Lambda^{p-1}([\tilde{N}^*] - 2) = g!\, \Lambda^{p-1}([\tilde{N}^*] - 2) \underset{\text{lemma}}{=} 0$$

#