X : variety over $k = \overline{k}$.

<u>Def</u> • {Cycle of codimension r on X} $= Z^r(X) = \bigoplus \mathbb{Z} \cdot Y_i$

$Y_i \subseteq X$: irreducible closed subvariety of codim $= r$.

If $Z \subseteq X$: closed subscheme of codimension r, then we associate

$$[Z]^r = \sum_{\substack{Z_i : \text{codim} = r \\ \text{irreducible} \\ \text{components}}} \left(\text{length}_{\mathcal{O}_{X,Z_i}} \mathcal{O}_{Z,Z_i}\right) Z_i \in Z^r(X).$$

• (push forward)

$f : X \to X'$: morphism of varieties. $Y \subseteq X$: subvariety.

Define $f_* Y = \begin{cases} 0 & \text{if } \dim f(Y) < \dim Y. \\ [f(Y) : K(f(Y))] \cdot \overline{f(Y)} & \text{if } \dim f(Y) = \dim Y. \end{cases}$

$\xrightarrow[\text{extend}]{\text{linearly}}$ $f_* : Z^k(X) \to Z^{k+?}(X')$

<u>Fact</u> : $f : X \to Y$, $g : Y \to Z$

Then, $g_* \circ f_* = (g \circ f)_* : Z^k(X) \to Z^{k+?}(Z)$. (degree : product!)

• (Rational equivalence)

$V \subseteq X$: subvariety. $f : \tilde{V} \to V$: normalization. Then, $\tilde{V}$ satisfies $\overset{\text{in Hartshorne}}{(*)}$.

For $D \sim D'$: Weil divisor on $\tilde{V}$, define $f_* D$ and $f_* D'$ are

<u>rational equivalence</u>.

- (Chow group)

$$A^r(X) := Z^r(X) \big/ \langle \text{rational equivalence} \rangle .$$

$$\rightsquigarrow \bigoplus_{r=0}^{\dim X} A^r(X) = A(X) \ : \text{Chow group.}$$

<u>Fact</u>: • $A^0(X) = \mathbb{Z} \cdot [X]$

• $A^r(X) = 0$ for $r > \dim X$ (convention)

- (Intersection)

Suppose $V, W \subseteq X$: subvarieties <u>intersect properly</u>, i.e. every irreducible component of $V \cap W$ has codim = codim V + codim W

$(\dim(Y \cap Z) \leq \dim(Y) + \dim(Z) - \dim(X).)$

Then, we define $$V \cdot W = \sum_{\substack{Z:\text{irreducible} \\ \text{component} \\ \text{of } V\cap W}} \left(\operatorname{length}_{\mathcal{O}_{X,Z}} \mathcal{O}_{V\cap W, Z} \right) Z .$$

<u>Formula</u> (Serre)

$$\operatorname{length}_{\mathcal{O}_{X,Z}} \mathcal{O}_{V\cap W, Z} = \sum_i (-1)^i \operatorname{length}_{\mathcal{O}_{X,Z}} \operatorname{Tor}_i^{\mathcal{O}_{X,Z}} (\mathcal{O}_{W,Z}, \mathcal{O}_{V,Z}) .$$

$$\xrightarrow[\text{linearly}]{\text{extend}} Z^r(X) \times Z^s(X) \longrightarrow Z^{r+s}(X)$$

Prop (pushforward and $\underset{rat}{\sim}$)

$p: X \to Y$. Suppose $\alpha \underset{rat}{\sim} \beta \in Z^k(X)$, then $f_* \alpha \underset{rat}{\sim} f_* \beta$ on Y.

pf: Show: $\alpha \underset{rat}{\sim} 0$ in X $\Rightarrow$ $f_* \alpha \underset{rat}{\sim} 0$ in Y

Suppose $i: W \hookrightarrow X$: closed immersion, codim $W = k-1$.

$f \in R(W)^*$: rational function.

We want to show $p_* i_* \operatorname{div}(f) \sim 0$ on Y.

$W' = \overline{p(W)}$

$$\begin{tikzcd} W \arrow[r, "i"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ W' \arrow[r, "i'"] & Y \end{tikzcd}$$

$p_* i_* \operatorname{div}(f) = i'_* p'_* \operatorname{div}(f)$.

Case 1: $\operatorname{codim}(W') > k$, then $p'_* \operatorname{div}(f) = 0$.

Case 2: $\operatorname{codim}(W') = k$. Claim: $(p')_* \operatorname{div}(f) = 0$.

$\eta \in W'$: generic point, then

$$\begin{tikzcd} W_\eta \arrow[r] \arrow[d, "c"'] & W \arrow[d, "p'"] \\ \operatorname{Spec}(k(\eta)) \arrow[r] & W' \end{tikzcd}$$

$k(W_\eta) = k(W)$

$f_\eta \longleftrightarrow f$

Also, $\{$closed point ξ of $W_\eta\}$ $\overset{1-1}{\longleftrightarrow}$ $\left\{ \begin{array}{l} \text{closed integral subscheme } \underset{\dim = k}{Z_\xi} \subseteq W \\ p'(Z_\xi) = W' \end{array} \right\}$

multiplicity of Z_ξ in $\operatorname{div}(f)$ = $\operatorname{ord}_{W_{\eta,\xi}}(f_\eta)$

On the other hand, multiplicity of W' in $p'_* \operatorname{div}(f)$ = multiplicity of $[\operatorname{Spec}(k(\eta))]$ in $C_* \operatorname{div}(f_\eta)$.

Thus, $\sum_{\xi: \text{closed in } W_\eta} [k(\xi) : k(\eta)] \operatorname{ord}_{\mathcal{O}_{W_{\eta,\xi}}}(f_\eta) = 0$.

Case 3 : $\operatorname{codim}(W') = k-1$, $p' : W \longrightarrow W'$: dominant map, $f \in R(W)^*$.

Claim : $(p')_* \operatorname{div}(f) = \operatorname{div}(g)$, $g = \underbrace{N_{R(W)/R(W')}}_{\text{norm.}}(f)$

subpf: $Z \subseteq W'$: codim $= 1$ closed subvariety.

$\xi \in Z$: generic point. Locally : $\operatorname{Spec}(A) \longrightarrow \operatorname{Spec}(R)$

$\rightsquigarrow R \longrightarrow A$: finite homomorphism

$\longrightarrow$ finite field extension L / K, $f \in L$, $g = N(f) \in K$

$Z \leftrightarrow p \in R$: prime ideal with $\dim(R_p) = 1$

The coefficient of $[Z]$ in $\operatorname{div}(g)$ is $\operatorname{ord}_{R_p}(g)$ $\|$

in $p'_* \operatorname{div}(f)$ is $\sum_{q \text{ lying over } p} [\kappa(q) : \kappa(p)] \operatorname{ord}_{A_q}(f)$.

#

$\Longrightarrow f_* : A^k(X) \longrightarrow A^{k+?}(Y)$.

Def (flat pullback)

$f : X \longrightarrow Y$: flat. $Z \subseteq Y$: codim $= r$ irreducible subvariety.

Define $f^*[Z] = [f^{-1}(Z)]^{r+?} \in A^{r+?}(X)$.

$\xrightarrow[\text{linearly}]{\text{extend}} f^* : Z^k(Y) \longrightarrow Z^k(X)$.

For general morphism, $f : X \longrightarrow Y$, $Z \subseteq Y$: codim $= r$, irreducible subvariety.

Define $f^*[Z] = pr_{1*}\left(\Gamma_f \,.\, pr_2^{-1}(Y)\right)$.

$$\begin{tikzcd} & X \times Y \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ X & & Y \end{tikzcd}$$

Fact : $f: X \to Y$, $g: Y \to Z$: flat.

Then, $f^* \circ g^* = (g \circ f)^* : Z^k(Z) \to Z^k(X)$.

$$\left(\begin{array}{l} \text{local flat ring homomorphism } A \to B \to C : \\ \quad \mathrm{length}_C (C/m_A C) = \mathrm{length}_C (C/m_B C) \cdot \mathrm{length}_B (B/m_A B) \end{array} \right)$$

Prop (pull back and $\underset{rat}{\sim}$)

Let $f: X \to Y$: flat morphism. $\alpha \underset{rat}{\sim} \beta \in Z^k(Y)$, then $f^*\alpha \underset{rat}{\sim} f^*\beta$ on X.

pf: Show $\alpha \underset{rat}{\sim} 0$ on X, then $f_*\alpha \underset{rat}{\sim} 0$ on Y.

Suppose $i : W \hookrightarrow Y$: closed immersion, $\operatorname{codim} W = k-1$, $g \in R(W)^*$.

We need to show $f^* i_* \operatorname{div}(g) \underset{rat}{\sim} 0$ on X.

$$\begin{tikzcd} W \times_Y X \arrow[r, hook, "i'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ W \arrow[r, hook, "i"'] & Y \end{tikzcd}$$

$f^* i_* \operatorname{div}(g) = i'_* (f')^* \operatorname{div}(g)$ (by cohomology & base change)

$$(f')^* \operatorname{div}(g) = \sum_{\substack{X_j \subseteq X \\ \text{irreducible}}} n_j \, i'_* \operatorname{div}_{X_j} (g \circ f|_{X_j}), \quad n_j = \text{multiplicity of } X_j \text{ in } X.$$

#

$\Rightarrow f^* : A^k(Y) \to A^k(X)$

$\mathcal{F}$: coherent $\mathcal{O}_X$-module, $\mathrm{supp}(\mathcal{F})$: codim $= r$

Def The r-cycle associated to $\mathcal{F}$ $:= [\mathcal{F}]^r = \sum \left(\mathrm{length}_{\mathcal{O}_{X,Z_i}} \mathcal{F}_{\xi_i} \right) [Z_i]$

ξ_i : generic point of Z_i, sum over $Z_i \subseteq \mathrm{supp}(\mathcal{F})$ irreducible closed subvariety of codim Z_i

Fact: • $[Z]^r = [\mathcal{O}_Z]^r$

• $f_*[\mathcal{F}]^r = [f_*\mathcal{F}]^r$, $f_*[Z]^r = [f_*\mathcal{O}_Z]^r$

• $f^*[\mathcal{F}]^r = [f^*\mathcal{F}]^r$, $f^*[Z]^r = [f^*\mathcal{O}_Z]^r$

• (formulae de Serre)

$$\underset{Z^r(X)}{\overset{\cap}{W}} . \underset{Z^s(X)}{\overset{\cap}{V}} = \sum_i (-1)^i \left[\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_W, \mathcal{O}_V) \right]^{r+s}$$

Prop (projection formula)

$f: X \longrightarrow Y$: flat. $\alpha \in Z^r(X)$, $\beta \in Z^s(Y)$

Suppose α and $f^*\beta$ intersect properly, then $f_*(\alpha)$ and β intersect properly and $f_*\alpha . \beta = f_*(\alpha . f^*\beta)$.

pf: Enough to show $\alpha = [V]$, $\beta = [W]$

$f^{-1}(W)$: codimension s in X since f is flat.

$V . f^{-1}(W)$ intersect properly.

$V \xrightarrow{\phi} f(V)$.

- If ϕ has general fiber dimension $= a$.

  $a > 0 \Rightarrow f_*[V] = 0$. (and thus $f_* V$ and W intersect properly)

  $f_*([V].f^*[W]) \overset{?}{=} 0$ since every fiber of ϕ has dimension $\geq a$

  $\rightsquigarrow$ every irreducible component Z of $V \cap f^{-1}(W)$ has fiber dim $\geq a$ over $f(Z)$.

  $\rightsquigarrow$ done!

- Suppose ϕ is generic finite.

  •• $Z \subseteq f(V) \cap W$: irreducible component

  $Z_i \subseteq V \cap f^{-1}(W)$: irreducible component dominant Z,

  then Z_i has dimension $\dim(X) - s - r$.

  $\rightsquigarrow \dim(Z) \leq \dim(X) - s - r$

  $\rightsquigarrow f(V)$ and W intersect properly.

and $V \xrightarrow{\phi} f(V)$ has finite fiber over generic point ξ of Z.

•• general projection formula:

$$Rf_*(\mathcal{O}_V \underset{\mathcal{O}_X}{\otimes} Lf^*\mathcal{O}_W) = Rf_* \mathcal{O}_V \underset{\mathcal{O}_Y}{\otimes} \mathcal{O}_W.$$

f: flat $\rightsquigarrow Lf^*\mathcal{O}_W = f^*\mathcal{O}_W$

$f|_V$ is finite in an open neighborhood of ξ $\rightsquigarrow (Rf_* \mathcal{F})_\xi = f_* \mathcal{F}_\xi$.

$$\rightsquigarrow \left(f_* \operatorname{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V, f^*\mathcal{O}_W)\right)_\xi = \left(\operatorname{Tor}_i^{\mathcal{O}_Y}(f_*\mathcal{O}_V, \mathcal{O}_W)\right)_\xi.$$

Finally, use Serre formula, $f^*[W] = [f^*\mathcal{O}_W]$. $f_*[V] = [f_*\mathcal{O}_V]$.

$$f_*([V].f^*[W]) = \sum (-1)^i [f_* \operatorname{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V, f^*\mathcal{O}_W)]$$

$$\parallel$$

$$\sum (-1)^i [\operatorname{Tor}_i^{\mathcal{O}_Y}(f_*\mathcal{O}_V, \mathcal{O}_W)] = [f_* V].[W].$$

#

Prop (f^* & intersection)

$f: X \longrightarrow Y$: flat. $\alpha \in Z^r(X)$, $\beta \in Z^s(X)$. Suppose α intersect β properly. then $(f^*\alpha).(f^*\beta) = f^*(\alpha.\beta)$.

pf: counting dimension $\rightsquigarrow$ $f^*\alpha$ intersect $f^*\beta$ properly.

Now, use Serre formula and $f^* \operatorname{Tor}_i^{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) = \operatorname{Tor}_i^{\mathcal{O}_X}(f^*\mathcal{F}, f^*\mathcal{G})$

#

Prop (intersection and $\underset{rat}{\sim}$)

X : non-singular projective variety. $\alpha \in Z^r(X)$, $\beta \in Z^s(X)$.

Suppose α intersect β properly and $\alpha \underset{rat}{\sim} 0$. Then, $\alpha.\beta \underset{rat}{\sim} 0$.

pf: (sketch) Lemma $Z \in Z^k(X)$, $Z \underset{rat}{\sim} 0 \iff \exists\, W \in Z^k(\mathbb{P}^1 \times X)$ and $a, b \in \mathbb{P}^1$ s.t. $W = Z_a - Z_b$

($\Rightarrow$) rational function gives $X \dashrightarrow \mathbb{P}^1$, ($\Leftarrow$) $f: W \longrightarrow \mathbb{P}^1 \in R(W)^*$ (view as)

$g = N(f)$: norm $R(W)/R(W')$, $W' = pr_2(W)$.

$\rightsquigarrow \operatorname{div}(g) = pr_{2*} \operatorname{div}(f)$.

Now, write $\alpha = \sum_i [W_{i,a_i}] - [W_{i,b_i}]$. Then, compute $\alpha.[V]$.

#

(c.f. stack project 43.25.)

$$\Longrightarrow A^r(X) \times A^s(X) \xrightarrow{\ \cdot\ } A^{r+s}(X)$$

Prop $U, V, W \subseteq X$: closed subvariety.

U, V, W intersect properly pairwise. Then, $U.(V.W) = (U.V).W$.

pf: Use Serre formula, we have

$$U.(V.W) = \sum (-1)^{i+j} [\mathrm{Tor}_j(\mathcal{O}_U, \mathrm{Tor}_i(\mathcal{O}_V, \mathcal{O}_W))]$$

$$(U.V).W = \sum (-1)^{i+j} [\mathrm{Tor}_j(\mathrm{Tor}_i(\mathcal{O}_U, \mathcal{O}_V), \mathcal{O}_W)].$$

Now, use spectral sequence (bounded by dimension ⇝ degenerate)

$$E_2^{p,q} = \mathrm{Tor}_{-p}(\mathcal{O}_U, \mathrm{Tor}_{-q}(\mathcal{O}_V, \mathcal{O}_W)) \Longrightarrow H^{p+q}(\mathcal{O}_U \otimes_{\mathcal{O}_X} \mathcal{O}_V \otimes_{\mathcal{O}_X} \mathcal{O}_W).$$

#

Conclusion:

$A(X) = \bigoplus_{r=0}^{\dim X} A^r(X)$ is a commutative ring with identity.

$(f: X \to Y)$

$f_*: A(X) \longrightarrow A(Y)$ is a graded group homomorphism

$f^*: A(Y) \longrightarrow A(X)$ is a ring homomorphism

(projection formula) $f_*(x . f^*y) = f_*(x) . y$ for $x \in A(X)$, $y \in A(Y)$.

Prop For general map $f: X \to Y$, we define $f^*\alpha = pr_{1*}(\Gamma_f . pr_2^*(\alpha))$.

$$\begin{tikzcd} & X \times Y \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ X & & Y \end{tikzcd}$$

- $pr_{1*}(\Gamma_f . \alpha . \beta) = pr_{1*}(\Gamma_f . pr_1^* pr_{1*}(\Gamma_f . \alpha) . \beta)$
  $= pr_{1*}(pr_1^* pr_{1*}(\Gamma_f . \alpha) . \Gamma_f . \beta)$
  $= pr_{1*}(\Gamma_f . \alpha) . pr_{1*}(\Gamma_f . \beta)$ ← projection formula for flat!

- $f_*(\alpha . pr_{1*}(\Gamma_f . pr_2^*(\beta))) = f_*(pr_{1*}(pr_1^*\alpha . \Gamma_f . pr_2^*(\beta)))$
  $= pr_{2*}(pr_1^*\alpha . \Gamma_f . pr_2^*(\beta))$
  $= pr_{2*}(pr_1^*\alpha . \Gamma_f) . \beta$
  $= f_*(\alpha) . \beta$

#

Fact (1) $Y, Z \in A(X)$. $\Delta : X \longrightarrow X \times X$

Then, $Y . Z = \Delta^*(Y \times Z)$ as cycle in X.

(2) If X is non-singular, then $\mathrm{Pic}(X) \cong A^1(X)$.

(3) The projection $p : X \times \mathbb{A}^n \longrightarrow X$ gives an isomorphism

$$A(X) \xrightarrow{p^*} A(X \times \mathbb{A}^n)$$

(4) $Y \subseteq X$: non-singular closed subvariety. $U = X \setminus Y$.

Then, we have exact sequence, $i : Y \hookrightarrow X$, $j : U \hookrightarrow X$,

$$A(Y) \xrightarrow{i_*} A(X) \xrightarrow{j^*} A(U) \longrightarrow 0 .$$

(5) Use (3)+(4), we may compute

$$A(\mathbb{P}^n) \simeq \mathbb{Z}[H] / [H]^{n+1} ,$$ H : degree 1 hyperplane class in $\mathbb{P}^n$.

(6) $\mathcal{E}$: locally free sheaf of rank $= r$ on X. $\mathbb{P}(\mathcal{E}) \xrightarrow{\pi} X$: projective bundle.

Let $\xi \in A^1(\mathbb{P}(\mathcal{E}))$ corresponding to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

Then, $\pi^* : A(X) \longrightarrow A(\mathbb{P}(\mathcal{E}))$ makes $A(\mathbb{P}(\mathcal{E}))$ into a

free $A(X)$-module generated by $\{1, \xi, \xi^2, \dots, \xi^{r-1}\}$.

Note that π^* is injective by projection formula.