Main Theorem (Proper Base Change Theorem): Let $f : X \to S$ be a proper morphism of schemes and let

$$\begin{array}{ccc} X_T & \xrightarrow{f'} & T \\ {\scriptstyle g'}\downarrow & \square & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & S \end{array}$$

be a cartesian diagram. Let $\mathscr{F}$ be a torsion sheaf on X. Then the *base change homomorphism*

$$g^* \left(R^i f_* \mathscr{F}\right) \xrightarrow{\sim} R^i f'_* \left(g'^* \mathscr{F}\right)$$

is an isomorphism.

First we define the **base change homomorphism**: Let

$$\begin{array}{ccc} Y & \xrightarrow{f'} & T \\ {\scriptstyle g'}\downarrow & \circlearrowleft & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & S \end{array}$$

be a commutative diagram of schemes. Let $\mathscr{F}$ be a sheaf on X. Then we have the map

$$\mathscr{F} \to g'_* g'^* \mathscr{F}$$

Apply the functor f_*, we get

$$f_* \mathscr{F} \to f_* g'_* g'^* \mathscr{F} = g_* f'_* g'_* \mathscr{F}$$

By the adjointness of the functors g_* and g^*, for every i, we have

$$g^* \left(R^i f_* \mathscr{F}\right) \to R^i f'_* \left(g'^* \mathscr{F}\right)$$

called the *base change homomorphism*.

We here only prove the theorem under the supplementary hypothesis:

(1) $\mathscr{F}$ is a sheaf of ${}^{\mathbb{Z}}/_{n\mathbb{Z}}$-modules with n invertible on S.

(2) The morphism $f : X \to S$ is projective.

The non-projective case can be reduced to the projective case by using Chow's lemma.

Remark 1. The proof for the finite morphism (instead of proper morphism) is much easier than the proof of the proper base change theorem [cf. Frietag, Étale Cohomology and the Weil Conjecture, I.3.5]. We here assume the result without proof.

Remark 2. If the base change theorem holds for $f_1 : X' \to S$ and $f_2 : X \to X'$, then it also holds for $f_1 \circ f_2 : X \to S$.

proof: It suffices to prove that it induces isomorphisms on all stalks. By the limit theorem we may assume T and S are affine and T is finitely generated S-scheme, hence we can restrict to the geometric points $t \in T$ that are closed in their fibers.
For $t : \text{Spec } k \to T$ being a geometric point, let $T(t) = \text{Spec } \widetilde{\mathcal{O}}_{T,t}$, where $\widetilde{\mathcal{O}}_{T,t} = \varinjlim \mathcal{O}_T(U)$ with U runs through all étale neighborhood of t. If $\mathscr{G}$ is a sheaf on X_T and $\mathscr{G}(t)$ is its inverse image on $X_T \times_T T(t)$, then in fact we have $(R^i f'_* \mathscr{G})_t \cong H^i(X_T \times_T T(t), \mathscr{G}(t))$ (which is proved by the limit theorem). Therefore it suffices to show in the following cases:

(i) $T = \text{Spec } K \to \text{Spec } k = S$, where K is a finite algebraic extension field of a separably closed field k.

(ii) $S = \text{Spec } A$ with A a strictly Henselian ring, $T = \text{Spec } k$ with k the reside field, and $T \to S$ is the natural map.

For the second case, let $s : T \to S$, the theorem is equivalent to saying $H^i(X, \mathscr{F}) = H^i(X_s, \mathscr{F}_s)$, where $X_s = X \times \text{Spec } k, \mathscr{F}_s = \mathscr{F}|_{X_s}$. We first consider the case $\dim X_s \leq 1$.

Claim: Let $\dim X_s \leq 1$ and n be invertible in $\mathcal{O}_X$. The natural map

$$H^i\left(X, {}^{\mathbb{Z}}/_{n\mathbb{Z}}\right) \to H^i\left(X_s, {}^{\mathbb{Z}}/_{n\mathbb{Z}}\right)$$

is bijective for $i = 0$ and surjective for $i > 0$.

Recall that for $i \geq 3$, $H^i\left(X_s, {}^{\mathbb{Z}}/_{n\mathbb{Z}}\right) = 0$. For $i = 0$, the Zariski connectedness theorem for a proper scheme over a Henselian ring says that the number of connected components of X and X_s is the same.
For $i = 1$, it is enuogh to prove that every Galois étale covering space of the special fiber can be extended to a Galois étale covering space of X. Let $X_n = X \times \text{Spec }\left({}^{A}/_{\mathfrak{m}^n}\right)$. Then the functor $t(X_n) \to t(X_{n-1})$ is an equivalence of categories. Therefore a Galois étale covering space of the special fiber $X_s = X_1$ can be extended to X_n inductively, and then can be extended to a Galois covering space of the formal completion of X along X_s. By the Grothendieck existence theorem: one can lift infinitesimal deformations of a scheme to a deformation, we see that a Galois covering space of the special fiber can be extended to a Galois covering space of $X \times \text{Spec }\left(\widehat{A}\right)$, where $\widehat{A} = \varprojlim {}^{A}/_{\mathfrak{m}^n}$.
Now consider a functor F on the category of A-algebras B assigning $F(B)$ to be the set of isomorphism classes of étale Galois ${}^{\mathbb{Z}}/_{n\mathbb{Z}}$-covering spaces of $X \times \text{Spec } B$. In fact, F is *locally of finite presentation*, that is, $F\left(\varinjlim B_n\right) = \varinjlim F(B_n)$ [cf. Algebraic approximation of structures over complete local rings. Inst. Hautes Etudes sci. Publ. Math. 36 (1969), 23-58.]. Recall that

Theorem 3 (Artin Approximation Theorem)**.** Let R be a field and let A be the henselization of an R-algebra of finite type at a prime ideal, let $\mathfrak{m}$ be a proper ideal of A and let $\widehat{A}$ be the $\mathfrak{m}$-adic completion of A, and let $F : (A\text{-algebras}) \to (\text{Sets})$ be a locally of finite presentation functor. Then for any $n \in \mathbb{N}$ and any $\overline{\xi} \in F\left(\widehat{A}\right)$, there is a $\xi \in F(A)$ such that

$$\overline{\xi} \equiv \xi \bmod \mathfrak{m}^n.$$

By the Artin approximation theorem, there is a Galois covering space of X that agrees over $X \times \text{Spec}\left({}^{A}/_{\mathfrak{m}^n}\right)$ with the covering space constructed for $X \times \text{Spec}\left(\widehat{A}\right)$.
For $i = 2$, consider the Kummer sequence

$$\begin{array}{ccccc} \text{Pic}X & \longrightarrow & H^2(X, \mu_n) & & \\ \downarrow & \circlearrowleft & \downarrow & & \\ \text{Pic}X_s & \twoheadrightarrow & H^2(X_s, \mu_n) & \longrightarrow & H^2\left(X_s, \mathcal{O}^*_{X_s}\right) \end{array}$$

Since k is separably algebraically closed, the Henselian ring A contains a primitive n-th root of unity whose image in k is also a primitive n-th root of unity. Hence we may identify the constant sheaf ${}^{\mathbb{Z}}/_{n\mathbb{Z}}$ with the sheaf μ_n of n-th roots of unity. Hence it suffices to show that every line bundle on the special fiber X_s can be extended to all of X. Similarly to the previous case, it is enough to show that every line bundle on $X_n = X \times \text{Spec}\left({}^{A}/_{\mathfrak{m}^n}\right)$ can be extended to X_{n+1}. We may consider the structure sheaf $\mathcal{O}_{X_n}$ as a sheaf on X_s (w.r.t. the Zariski topology). Let $\mathscr{I} = \ker\left(\mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n}\right)$. Then the sequence

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathcal{O}^*_{X_{n+1}} \longrightarrow \mathcal{O}^*_{X_n} \longrightarrow 1$$

$$\alpha \longmapsto 1 + \alpha$$

is exact. Now the sheaf $\mathscr{I}$ is coherent and $\dim X_s \leq 1$, the cohomology $H^2(X_s, \mathscr{I})$ vanished. Hence $H^1\left(X_s, \mathcal{O}^*_{X_n}\right) \to H^1\left(X_s, \mathcal{O}^*_{X_{n+1}}\right)$ is surjective. The claim is proved.
Now apply the result to schemes $X' \xrightarrow{p} X$ finite over X, then we see that the map

$$H^i(X, \mathscr{F}) \to H^i(X_s, \mathscr{F}_s)$$

is bijective for $i = 0$ and surjective for $i > 0$ for all sheaves $\mathscr{F}$ that are isomorphic to a finite direct sum of sheaves of the form $p_*\left(\left({}^{\mathbb{Z}}/_{n\mathbb{Z}}\right)_{X'}\right)$. Recall that every constructible sheaf on X is a subsheaf of a finite direct sum of sheaves of the form $p_*\left(\left({}^{\mathbb{Z}}/_{n\mathbb{Z}}\right)_{X'}\right)$. Therefore we get that every constructible sheaf $\mathscr{F}$ (of ${}^{\mathbb{Z}}/_{n\mathbb{Z}}$-modules) on X is a subsheaf of a constructible sheaf $\mathscr{G}$ with the map

$$H^i(X, \mathscr{G}) \to H^i(X_s, \mathscr{G}_s)$$

being bijective for $i = 0$ and surjective for $i > 0$. Consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(X,\mathscr{F}) & \longrightarrow & H^0(X,\mathscr{G}) & \longrightarrow & H^0\left(X, \mathscr{G}/\mathscr{F}\right) \\ & & \downarrow & & \wr\downarrow & & \downarrow \\ 0 & \longrightarrow & H^0(X_s,\mathscr{F}_s) & \longrightarrow & H^0(X_s,\mathscr{G}_s) & \longrightarrow & H^0\left(X_s, \left(\mathscr{G}/\mathscr{F}\right)_s\right) \end{array}$$

Clearly, the map

$$H^0(X,\mathscr{F}) \to H^0(X_s,\mathscr{F}_s)$$

is injective. This holds for all constructible sheaves, and hence it holds also for the map

$$H^0\left(X, \mathscr{G}/\mathscr{F}\right) \to H^0\left(X_s, \left(\mathscr{G}/\mathscr{F}\right)_s\right)$$

Then we conclude that the map

$$H^0(X,\mathscr{F}) \to H^0(X_s,\mathscr{F}_s)$$

is bijective for constructible sheaves $\mathscr{F}$, and hence for all sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules (such sheaves is a filtered direct limit of its constructible subsheaves). Suppose that we have proved the map

$$H^{m-1}(X,\mathscr{F}) \to H^{m-1}(X_s,\mathscr{F}_s)$$

is bijective for some $m \in \mathbb{N}$. We may assume $\mathscr{F}$ is constructible. Embedding $\mathscr{F}$ in a torsion sheaf $\mathscr{G}$ for which the map

$$H^i(X,\mathscr{G}) \to H^i(X_s,\mathscr{G}_s)$$

is surjective for all i. Consider the exact sequence $0 \to \mathscr{G} \to \mathscr{J} \to \mathscr{K} \to 0$ where $\mathscr{J}$ is injective in the category of $\mathbb{Z}/n\mathbb{Z}$-modules. Then

$$\begin{array}{ccccccc} H^{m-1}(X,\mathscr{J}) & \longrightarrow & H^{m-1}(X,\mathscr{K}) & \longrightarrow & H^m(X,\mathscr{G}) & \longrightarrow & 0 \\ \wr\downarrow & & \wr\downarrow & & \downarrow & & \\ H^{m-1}(X_s,\mathscr{J}_s) & \longrightarrow & H^{m-1}(X_s,\mathscr{K}_s) & \longrightarrow & H^m(X_s,\mathscr{G}_s) & & \end{array}$$

gives the injectivity of $H^m(X,\mathscr{G}) \to H^m(X_s,\mathscr{G}_s)$ (and thus isomorphism). Consider the long exact sequences

$$\begin{array}{ccccccccc} H^{m-1}(X,\mathscr{G}) & \longrightarrow & H^{m-1}\left(\mathscr{G}/\mathscr{F}\right) & \longrightarrow & H^m(X,\mathscr{F}) & \longrightarrow & H^p(X,\mathscr{G}) & \longrightarrow & H^m\left(\mathscr{G}/\mathscr{F}\right) \\ \wr\downarrow & & \wr\downarrow & & \downarrow & & \wr\downarrow & & \downarrow \\ H^{m-1}(X_s,\mathscr{G}_s) & \longrightarrow & H^{m-1}\left(X_s, \left(\mathscr{G}/\mathscr{F}\right)_s\right) & \longrightarrow & H^m(X_s,\mathscr{F}_s) & \longrightarrow & H^m(X_s,\mathscr{G}_s) & \longrightarrow & H^m\left(X_s, \left(\mathscr{G}/\mathscr{F}\right)_s\right) \end{array}$$

First we see that the map $H^m(X, \mathscr{F}) \to H^m(X_s, \mathscr{F}_s)$ is injective, which also holds for the sheaf $\mathscr{G}/\mathscr{F}$, and therefore the map is bijective.
Now we have showed the case for $f : X \to S$ whose geometric fibers are at most one-dimensional, hence the theorem holds for those map f which can be written as products of proper maps with at most one-dimensional fibers, in particular, for $X = \mathbb{P}^1_S \times \ldots \times \mathbb{P}^1_S$. Now we prove the base change theorem for $f : \mathbb{P}^n_S \to S$. Consider $p : \mathbb{P}^1_S \times \ldots \times \mathbb{P}^1_S \to \mathbb{P}^n_S$, then the base change theorem holds for sheaves of the form $p_*\mathbb{F}$. Note that the map $\mathscr{G} \to p_* p^* \mathscr{G}$ is injective for every sheaf $\mathscr{G}$ on $\mathbb{P}^n_S$, then we see that $\mathbb{G}$ is quasi-isomorphic to a bounded-below complex of sheaves $\mathscr{F}^\bullet$ for which the base change theorem hold. Hence the base change theorem holds for $\mathscr{G}$. Now every projective morphism $f : X \to S$ can be written as $X \hookrightarrow \mathbb{P}^n_S \to S$, hence the base change theorem holds for arbitrary porjective morphisms. $\square$

Here we state two theorems [cf. Frietag, Étale Cohomology and the Weil Conjecture, I.7.3, I.7.4]:

Theorem 4 (Smooth Base Change Theorem)**.** Let

$$\begin{array}{ccc} X_T & \xrightarrow{f'} & T \\ {\scriptstyle g'}\downarrow & \square & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & S \end{array}$$

be a cartesian diagram of schemes. The base change homomorphism

$$g^* R^i f_* \mathscr{F} \to R^i f'_* (g'^* \mathscr{F})$$

is an isomorphism if

(1) g is smooth

(2) $\mathscr{F}$ is a torsion sheaf with torsion relatively prime to the residue characteristic of the scheme S.

Theorem 5 (Acyclicity Theorem)**.** Let $A \to B$ be a smooth homomorphism of strict Henselian rings. Then the morphism $g : \text{Spec } B \to \text{Spec } A$ is acyclic, that is, for all torsion sheaves $\mathscr{F}$ on Spec A with torsion relatively prime to the residue characteristic of A,we have

(i) $\mathscr{F} \to g_* g^* \mathscr{F}$ is an isomorphism.

(ii) $R^i g_* (g^* \mathscr{F}) = 0$ for all $i > 0$.