Definition 1.

(1) A locally of finite type morphism $f : X \to Y$ of schemes is called **unramified** at $x \in X$ if $\mathcal{O}_{X,x} \big/ \mathfrak{m}_{f(x)}\mathcal{O}_{X,x}$ is a finite separable field extension of $k\left(f\left(y\right)\right)$.

(2) A morphism $f : X \to Y$ of schemes is called **étale** if f is locally of finite type, flat and unramified.

Proposition 2. Let A be a finite algebra over a field k. TFAE:

(a) A is separable over k.

(b) $\overline{A} = A \otimes_k \overline{k}$ is isomorphic to a finite product of copies of $\overline{k}$.

(c) A is isomorphism to a finite product of separable field extension of k.

(d) The discrimiant of any basis of A over k is nonzero.

proof: (a)⇒(b): $\overline{A}$ has only finitely many prime ideals and they are all maximal. By the assumption, their intersection is zero. The conclusion follows from Chinese remainder theorem.
(b)⇒(c): By Chinese remainder theorem, $A \big/ \mathfrak{J}$ is isomorphic to a finite product $\prod k_i$ of finite field extension of k, where $\mathfrak{J}$ is the Jacobson radical of A. Then $\mathrm{Hom}_k\left(A, \overline{k}\right)$ has $\sum [k_i : k]_s$ elements. Since $\mathrm{Hom}_k\left(A, \overline{k}\right) = \mathrm{Hom}_{\overline{k}}\left(\overline{A}, \overline{k}\right)$, which has $[\overline{A} : \overline{k}]$ elements by the assumption, we have

$$[A : k] = [\overline{A} : \overline{k}] = \sum [k_i : k]_s \leq \sum [k_i : k] = \left[A \big/ \mathfrak{J} : k\right] \leq [A : k]$$

The equality holds, which implies k_i are separable.
(c)⇒(d): If $A = \prod k_i$ with k_i separable field extension of k, then $\mathrm{disc}\left(A\right) = \prod \mathrm{disc}\left(k_i\right)$, which is nonzero since k_i are separable.
(d)⇒(a): If $x \in \mathfrak{J}\left(\overline{A}\right)$, then xa is nilpotent for any $a \in \overline{A}$, and hence $\mathrm{Tr}_{\overline{A}/\overline{k}}\left(xa\right) = 0$. Note that the discriminant of A and $\overline{A}$ are the same, therefore $x = 0$. □

Recall the Hensel's lemma in nunber theory:

Theorem 3. Let A be a complete discrete valuation ring with k the residue field and $f \in A[T]$ monic. If $\overline{f} = g_0 h_0 \in k[T]$ for some $g_0, h_0 \in k[T]$ monic coprime, then $f = gh$ for some $g, h \in A[T]$ monic with $\overline{g} = g_0, \overline{h} = h_0$.

Definition 4. A local ring A is called **Henselian** if the conclusion of Hensel's lemma holds.

Theorem 5. Let A be a local ring and x be the closed point of $X = \mathrm{Spec}\ A$. TFAE:

(a) A is Henselian.

(b) Any finite A-algebra B is a direct product of local rings $B = \prod B_i$.

(c) If $f : Y \to X$ is quasi-finite and separable, then $Y = Y_0 \sqcup \ldots \sqcup Y_n$, where $x \notin f(Y_0)$ and for $i \geq 1$, $Y_i = \text{Spec } B_i$ is finite over X, where B_i are local rings.

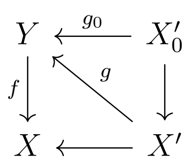
(d) If $f : Y \to X$ is étale and there is a point $y \in Y$ such that $f(y) = x$ and $k(y) = k(x)$, then f has a section $s : X \to Y$.

(e) Let $f_1, \ldots, f_n \in A[T_1, \ldots, T_n]$. If there exists an $a = (a_1, \ldots, a_n) \in k^n$ such that $\overline{f_i}(a) = 0$ and $\det\left(\left(\frac{\partial \overline{f_i}}{\partial T_j}\right)(a)\right) \neq 0$, then there is a $b \in A^n$ such that $\bar{b} = a$ and $f_i(b) = 0$.

[c.f. Étale Cohomology, Milne, p.32]

Proposition 6. Any complete local ring A is Henselian.

proof: Let B be an étale A-algebra, and suppose that there is a section $s_0 : B \to k$. Write $A_r = A/\mathfrak{m}^{r+1}$. It suffices to show that there exist compatible sections $s_r : B \to A_r$, then they induce a section $s : B \to A$. It is clear for $r = 0$, and for $r > 0$, the existence of s_r follows from the existence of s_{r-1} and the following fact: Given an X-morphism $g_0 : X'_0 \to Y$, there is an X-morphism $g : X' \to Y$ such that the diagram commutes



[c.f. EGA.IV.17][c.f. Milne, p.30] □

A ring A is a subring of its completion $\widehat{A}$, hence any local ring A is a subring of Henselian ring. We define the Henselization of A to be the Henselian ring A^h with a local homomorphism $i : A \to A^h$ such that for any other local homomorphism from A to a Henselian local ring factors through i uniquely. It is clear that the Henselization is unique if it exists. To prove the existence of the Henselization, we introduce the étale neighborhood.

Definition 7. An **étale neighborhood** of a local ring A is a pair $(B, \mathfrak{q})$ where B is an étale A-algebra and $\mathfrak{q}$ is a prime ideal of B lying over $\mathfrak{m}$ such that the induced map $k \to k(\mathfrak{q})$ is an isomorphism.

Lemma 8.

(a) If $(B, \mathfrak{q})$ and $(B', \mathfrak{q}')$ are étale neighborhoods of A with Spec B' connected, then there is at most one A-homomorphism $f : B \to B'$ such that $f^{-1}(\mathfrak{q}') = \mathfrak{q}$.

(b) Let $(B, \mathfrak{q})$ and $(B', \mathfrak{q}')$ be étale neighborhoods of A. Then there is an étale neighborhood $(B'', \mathfrak{q}'')$ of A with Spec B'' connected and A-homomorphisms $f : B \to B''$, $f' : B' \to B''$ such that $f^{-1}(\mathfrak{q}'') = \mathfrak{q}$, $f'^{-1}(\mathfrak{q}'') = \mathfrak{q}'$.

proof: (a) Use the fact: Let $f, g : Y' \to Y$ be X-morphisms with Y' connected and Y étale separated over X. If there existsa point $y' \in Y'$ such that $f(y') = g(y') = y$ and the maps $k(y) \to k(y')$ induced by f, g coincide, then $f = g$.
(b) Let $C = B \otimes_A B'$. Then we have a map $C \to k$ induced by $B \to k$ and $B' \to k$. Let $\mathfrak{q}''$ be the kernel. Take $c \notin \mathfrak{q}''$ and let $B'' = C_c$. Then $(B'', \mathfrak{q}''B'')$ is as desired. □

Corollary 9. For any local ring A, the Henselization A^h exists.

proof: The étale neighborhoods of A with connected sepctra form a filtered direct system. Define $\left(A^h, \mathfrak{m}^h\right) = \varinjlim (B, \mathfrak{q})$. Then A^h is a local A-algebra with maximal ideal $\mathfrak{m}^h$ and ${A^h}/{\mathfrak{m}^h} = k$, and it is indeed a Henselian ring. □

Definition 10. Let X be a scheme and let $x \in X$. An **étale neighborhood** of x is a pair (Y, y) where Y is an étale X-scheme and $y \in Y$ is mapped to x such that $k(x) = k(y)$.

Similarly, the connected étale neighborhoods of x form a filtered system and $\varinjlim \Gamma(Y, \mathcal{O}_Y) = \mathcal{O}^h_{X,x}$.

Definition 11. A Henselian ring A is **strictly Henselian** if the residue field of A is separably algebraically closed.

Some of above conclusion can be rewrittent for strictly Henselian rings. The **strict Henselization** of A is a pair $\left(A^{sh}, i\right)$, where A^{sh} is a strictly Henselian ring and $i : A \to A^{sh}$ is a local homomorphism such that for any other local homomorphism from A to a strictly Henselization factors through i.

Definition 12. Let X be a scheme and $\overline{x} : \text{Spec } k \to X$ a geometric point of X, where k is a separably closed field. An **étale neighborhood** of $\overline{x}$ is a commutative diagram

$$\begin{array}{ccc} \text{Spec } k & \longrightarrow & U \\ & \underset{\overline{x}}{\searrow} & \downarrow \\ & & X \end{array}$$

with $U \to X$ being étale.

Similarly $\mathcal{O}^{sh}_{X,x} = \varinjlim \Gamma(U, \mathcal{O}_U)$ where the limit is taken over all étale neighborhood of $\overline{x}$.

Proposition 13.

(a) A composite of étale morphisms is étale.

(b) An étale morphism $X \to Y$ remains étale after an arbitrary base extension $Z \to Y$.

(c) Given morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$. If $g \circ f$ and g are étale, then so is f.

In the version of rings:

(c′) Let $A \to B \to C$ be ring extensions. If B and C are étale over A, then C is étale over B.

For a scheme X, denote $\text{Ét}(X)$ to be the category of all étale extensions of X, considered as a full subcategory of all X-schemes. All of morphisms in $\text{Ét}(X)$ are étale by (c). Similarly define $\text{Ét}(A)$ for a ring A.

Definition 14. A presheaf $\mathscr{F}$ on $\text{Ét}(X)$ of abelian groups is a contravariant functor

$$\mathscr{F} : \text{Ét}(X) \to (\text{Ab})$$

A presheaf $\mathscr{F}$ on $\text{Ét}(A)$ of abelain groups is a covariant functor

$$\mathscr{F} : \text{Ét}(A) \to (\text{Ab})$$

Definition 15. A finite family $\mathscr{B} = \left(U_i \xrightarrow{\phi_i} U, i \in I\right)$ of étale morphisms is called an **étale covering** of a scheme U if $U = \bigcup_{i \in I} \phi_i(U_i)$.

Definition 16. A presheaf $\mathscr{F}$ is called a sheaf if the sequence

$$\mathscr{F}(U) \to \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \times_U U_j)$$

is exact for all coverings $(U_i \to U)$.

Consider the category of all étale coverings of a fixed object $B \in \text{Ét}(A)$. A map between two coverings $\mathscr{B} \to \mathscr{B}'$, where

$$\mathscr{B} = (B \to B_i, i \in I), \quad \mathscr{B}' = \left(B \to B'_j, j \in J\right)$$

is given by a map $\sigma : J \to I$ of the index sets and a family of homomorphisms $B_{\sigma(j)} \to B'_j$. For each covering $\mathscr{B} = (B \to B_i)$, denote $\mathscr{F}(\mathscr{B})$ the set of all families $s_i \in \mathscr{F}(B_i)$ with the above compatibility property.

Lemma 17.

(a) For any two coverings $\mathscr{B}$ and $\mathscr{B}'$, there is a covering $\mathscr{B}''$ together with morphisms $\mathscr{B} \to \mathscr{B}''$ and $\mathscr{B}' \to \mathscr{B}''$.

(b) Two morphisms $\mathscr{B} \rightrightarrows \mathscr{B}'$ induce the same map $\mathscr{F}(\mathscr{B}) \to \mathscr{F}(\mathscr{B}')$.

So we may consider $\widetilde{\mathscr{F}}(B) = \varinjlim_{\mathscr{B}} \mathscr{F}(\mathscr{B})$. In general, $\widetilde{\mathscr{F}}$ is not a sheaf, however, we have the following proposition:

Proposition 18.

(a) If $\mathscr{F}(B) \to \mathscr{F}(\mathscr{B})$ is injective, then $\widetilde{\mathscr{F}}$ is a sheaf.

(b) $\widetilde{\mathscr{F}}(B) \to \widetilde{\mathscr{F}}(\mathscr{B})$ is always injective.

Therefore $\widehat{\mathscr{F}} = \widetilde{\widetilde{\mathscr{F}}}$ is always a sheaf, called the sheaf generated by the presheaf $\mathscr{F}$.
The category $\text{Ét}(X)$ is small, then the collection of all presheaves with the natural transformations as morphisms forms a category. We consider the category of sheaves as a full subcategory.

Fact: The category of presheaves of abelian groups and the category of sheaves of abelian groups are abelian, and every sheaf is a subsheaf of an injective sheaf. The functor $\mathscr{F} \mapsto \mathscr{F}(X)$ (respectively, $\mathscr{F}(A)$) is left exact.

Hence now we define the étale cohomology as

$$H^i_{\text{ét}}(X, \mathscr{F}) = R^i(\mathscr{F} \mapsto \mathscr{F}(X)), \quad H^i_{\text{ét}}(A, \mathscr{F}) = R^i(\mathscr{F} \mapsto \mathscr{F}(A))$$

In particular, if $X = \text{Spec } A$, for any sheaf $\mathscr{F}$ on X, let $\mathscr{F}_0$ be a sheaf on A defined by $\mathscr{F}_0(B) = \mathscr{F}(\text{Spec } B)$. This gives an equivalence of categories, therefore we have

$$H^i_{\text{ét}}(\text{Spec } A, \mathscr{F}) = H^i_{\text{ét}}(A, \mathscr{F}_0)$$

Definition 19. Let $\mathscr{F}$ be a presheaf. Define the **stalk** of $\mathscr{F}$ at the geometric point $\overline{x}$ to be the limit

$$\mathscr{F}_{\overline{x}} = \varinjlim \mathscr{F}(U)$$

where U runs through all étale neighborhoods of $\overline{x}$.

Definition 20. Let $f : X \to Y$ be a morphism of schemes, $\mathscr{F}$ a sheaf on X. The direct image is defined to be

$$(f_*\mathscr{F})(U) = \mathscr{F}(X \times_Y U), \quad U \to Y \text{ étale}$$

which is a sheaf on Y. Note that the functor $\mathscr{F} \mapsto f_*\mathscr{F}$ is left exact, hence we can define the higher derived image $R^i f_*\mathscr{F}$.

Proposition 21. $R^i f_*\mathscr{F}$ agrees with the sheaf generated by the presheaf

$$U \mapsto H^i(X \times_Y U, \mathscr{F}_U)$$

where we denote by $\mathscr{F}_U$ the restriction of $\mathscr{F}$ w.r.t. the étale map $X \times_Y U \to X$.