1.5 The Lefschetz Theorem on $(1,1)$-classes

As an application of Serre vanishing theorem, we will complete our picture of the correspondences among divisors, line bundles, and Chern classes on a complex submanifold of projective space. First, we have the

Proposition. Let $M \subseteq \mathbb{P}^N$ be a submanifold. Then every line bundle on M is of the form $L = [D]$ for some divisor D, i.e.,

$$\mathrm{Pic}(M) \cong \frac{\mathrm{Div}(M)}{\text{linear equivalence}}.$$

Proof. To prove this, we have to show that every line bundle on M has a global meromorphic section. To find such a section, let H denote the restriction to M of the hyperplane bundle on $\mathbb{P}^N$. We will show that for $\mu \gg 0$, $L + \mu H$ has a nontrivial global meromorphic section s; then if t is any global holomorphic section of $[H]$ over M, s/t^μ will be a global meromorphic section of L as desired.

We proceed by induction on $n = \dim M$: assume that for every submanifold $V \subseteq \mathbb{P}^n$ of dimension less that n and every line bundle $L \to V$, $H^0(V, \mathcal{O}(L + \mu H)) \neq 0$ for $\mu \gg 0$. By Bertini's theorem we can find a hyperplane $\mathbb{P}^{N-1} \subset \mathbb{P}^N$ with $V = \mathbb{P}^{N-1} \cap M$ smooth; we consider the exact sheaf sequence

$$0 \longrightarrow \mathcal{O}_M(L + (\mu - 1)H) \longrightarrow \mathcal{O}_M(L + \mu H) \longrightarrow \mathcal{O}_V(L + \mu H) \longrightarrow 0.$$

For $\mu \gg 0$ we have both $H^0(V, \mathcal{O}(L + \mu H)) \neq 0$ by induction and

$$H^0(M, \mathcal{O}(L + \mu H)) \longrightarrow H^0(V, \mathcal{O}(L + \mu H)) \longrightarrow H^1(M, \mathcal{O}(L + (\mu - 1)H)) = 0$$

by Serre vanishing theorem. Thus $H^0(M, \mathcal{O}(L + \mu H)) \neq 0$, and the result is proved. ■

We now consider for a moment the general problem of analytic cycles. On a compact Kähler manifold M, the Hodge decomposition

$$H^r(M, \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(M)$$

on complex cohomology gives a slightly coarser decomposition of real cohomology

$$H^r(M, \mathbb{R}) = \bigoplus_{\substack{p+q=r \\ p \leq q}} (H^{p,q}(M) \oplus H^{q,p}(M)) \cap H^r(M, \mathbb{R}).$$

A natural question to ask is whether we can characterize geometrically the classes in homology that are Poincaré dual to classes in one of these factors.

Definition. We say a homology class $\gamma \in H_{2p}(M, \mathbb{Z})$ is analytic if it is a rational linear combination of fundamental classes of analytic subvarieties of M; dually, we say a cohomology class is analytic if its Poincaré dual is.

Note that if $V \subset M$ is an analytic subvariety of codimension p and ψ any $(2n-2p)$-form on M

$$\int_V \psi = \int_V \psi^{n-p,n-p}.$$

Thus if η is the harmonic form on M representing the cohomology class η_V and ψ any harmonic form,

$$\int_M \psi \wedge \eta = \int_V \psi = \int_V \psi^{n-p,n-p} = \int_M \psi \wedge \eta^{p,p},$$

i.e., $\eta = \eta^{p,p}$, and so we see that any analytic cohomology class of degree $2p$ is of pure type (p,p). The famous Hodge Conjecture asserts that the converse is also true: On $M \subseteq \mathbb{P}^N$ a submanifold of projective space every rational cohomology class of type (p,p) is analytic. The only case which has been proved in general is the case $p = 1$.

Theorem (Lefschetz Theorem on $(1,1)$-classes)**.** For $M \subseteq \mathbb{P}^N$ a submanifold, every cohomology class

$$\gamma \in H^{1,1}(M) \cap H^2(M, \mathbb{Z})$$

is analytic; in fact $\gamma = \eta_D$ for some divisor D on M.

Here, we write $H^2(M,\mathbb{Z})$ for its image in $H^2(M,\mathbb{R})$.

Proof. Consider again the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0$$

and the associated cohomology sequence

$$H^1(M, \mathcal{O}^*) \xrightarrow{c_1} H^2(M, \mathbb{Z}) \xrightarrow{i_*} H^2(M, \mathcal{O}) \cong H^{0,2}(M).$$

We claim that the map i_* is given by first mapping $H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{C})$ and then projecting onto the $(0,2)$-factor of $H^2(M,\mathbb{C})$ in the Hodge decomposition; i.e., that the diagram

$$
\begin{array}{ccc}
H^2(M,\mathbb{Z}) & \xrightarrow{i_*} & H^2(M,\mathcal{O}) \\
\downarrow & & \\
H^2(M,\mathbb{C}) & & \downarrow \\
\downarrow & & \\
H^2_{\mathrm{dR}}(M,\mathbb{C}) & \longrightarrow & H^{0,2}_{\overline{\partial}}(M)
\end{array}
$$

commutes. (The map $\pi^{0,2}$ is defined on the form level, since for $\omega = \omega^{2,0} + \omega^{1,1} + \omega^{0,2} \in Z^2_d(M)$, $\overline{\partial}\omega^{0,2} = (d\omega)^{0,3} = 0$). To see this, let $z = \{z_{\alpha\beta\gamma}\} \in Z^2(M,\mathbb{C})$; to find the image of z under the de Rham isomorphism, we take $f_{\alpha\beta} \in A^0(U_\alpha \cap U_\beta)$ such that

$$z_{\alpha\beta\gamma} = f_{\beta\gamma} + f_{\gamma\alpha} + f_{\alpha\beta} \quad \text{ in } U_\alpha \cap U_\beta \cap U_\gamma;$$

since $z_{\alpha\beta\gamma}$ is constant, $df_{\beta\gamma} + df_{\gamma\alpha} + df_{\alpha\beta} = 0$, so $\{df_{\alpha\beta}\} \in Z^1(M, \mathscr{Z}^1_d)$ and we can find $\omega_\alpha \in A^1(U_\alpha)$ such that

$$df_{\alpha\beta} = \omega_\beta - \omega_\alpha \quad \text{ in } U_\alpha \cap U_\beta.$$

The global 2-form $d\omega_\alpha = d\omega_\beta$ then represents the image of z in $H^2_{\mathrm{dR}}(M,\mathbb{C})$. On the other hand, take the image of $i_* z$ under the Dolbeault isomorphim; we write

$$z_{\alpha\beta\gamma} = f_{\beta\gamma} + f_{\gamma\alpha} + f_{\alpha\beta}, \quad \overline{\partial} f_{\alpha\beta} = \omega^{0,1}_\beta - \omega^{0,1}_\alpha,$$

and we see that $\overline{\partial}\omega^{0,1}_\alpha = (d\omega_\alpha)^{0,2}$ represents z in $H^{0,2}_{\overline{\partial}}(M)$.

Now, given $\gamma \in H^{1,1}(M) \cap H^2(M,\mathbb{Z})$, we have $i_*(\gamma) = 0$, and hence $\gamma = c_1(L)$ for some line bundle $L \in H^1(M,\mathcal{O}^*)$. Writing $L = [D]$ for some divisor D,

$$\gamma = c_1([D]) = \eta_D.$$

■

2 The Kodaira Embedding Theorem

We will be concerned in this section with determining exactly when a compact complex manifold is an algebraic variety, i.e., when it can be embedded in projective space.

Recall that we have a basic dictionary

$$\left\{ \begin{array}{l} \text{nondegenerate maps} \\ f : M \to \mathbb{P}^N, \text{ modulo} \\ \text{projective transformations} \end{array} \right\} \rightleftarrows \left\{ \begin{array}{l} \text{line bundles } L \to M \\ \text{with } E \subseteq H^0(M,\mathcal{O}(L)) \\ \text{such that } |E| \text{ has no base points} \end{array} \right\}$$

where the choice of homogeneous coordinates on $\mathbb{P}^N$ corresponds to the choice of basis $s_0, \ldots, s_N$ for E. We will often write ι_L for $\iota_{H^0(M,\mathcal{O}(L))}$ and ι_D for $\iota_{[D]}$.

We may state the question as: Given $L \to M$ a holomorphic line bundle, when is $\iota_L : M \to \mathbb{P}^N$ an embedding? First, in order for ι_L to be well-defined the linear system $|L|$ cannot have any base points, i.e., for each $x \in M$ the restriction map

$$H^0(M, \mathcal{O}(L)) \xrightarrow{r_x} L_x$$

must be surjective. Granted this, ι_L will be an embedding if

1. ι_{L} is one-to-one. Clearly this is the case if and only if for all x and y in M, there exists a section $s \in H^0(M, \mathcal{O}(L))$ vanishing at x but not at y i.e., if and only if the restriction map

   (♠) $$H^0(M, \mathcal{O}(L)) \xrightarrow{r_{x,y}} L_x \oplus L_y$$

   is surjective for all $x \neq y \in M$. Note that if L satisfies this condition, then $|L|$ must be base-point-free.

2. ι_L has nonzero differential everywhere. If φ_α is a trivialization of L near x, then this is the case if and only if for all $v^* \in T_x^*(M)$, there exists $s \in H^0(M, \mathcal{O}(L))$ with $s_\alpha(x) = 0$ and $ds_\alpha(x) = v^*$, where $s_\alpha = \varphi_\alpha^* s$. Let $\mathscr{I}_x \subset \mathcal{O}$ denote the sheaf of holomorphic functions on M vanishing at x, and let $\mathscr{I}_x(L)$ be the sheaf of sections of L vanishing at x. If s is any section of $\mathscr{I}_x(L)$ defined near x, and $\varphi_\alpha, \varphi_\beta$ are trivializations of L in a neighborhood U of x, then writing $s_\alpha = \varphi_\alpha^* s$, $s_\beta = \varphi_\beta^* s$, $s_\alpha = g_{\alpha\beta} s_\beta$, we have

   $$ds_\alpha = ds_\beta \cdot g_{\alpha\beta} + dg_{\alpha\beta} \cdot s_\beta = ds_\beta \cdot g_{\alpha\beta} \quad \text{at } x.$$

   Thus we have a well-defined sheaf map

   $$d_x : \mathscr{I}_x(L) \longrightarrow T_x^{*\prime} \otimes L_x$$

   and condition 2 can be stated as requiring that the map

   (♣) $$H^0(M, \mathscr{I}_x(L)) \xrightarrow{d_x} T_x^{*\prime} \otimes L_x$$

   be surjective for all $x \in M$.

Note that (♣) is the limiting case of (♠) when $y \to x$.

Theorem (Kodaira Embedding Theorem)**.** Let M be a compact complex manifold and $L \to M$ a positive line bundle. Then there exists k_0 such that for $k \geq k_0$, the map

$$\iota_{L^k} : M \longrightarrow \mathbb{P}^N$$

is well-defined and is an embedding of M.

Proof. We want to prove that there exists k_0 such that

1. The restriction map
$$H^0(M, \mathcal{O}(L^k)) \xrightarrow{r_{x,y}} L_x^k \oplus L_y^k$$
is surjective for all $x \neq y \in M$, $k \geq k_0$; and

2. The differential map
$$H^0(M, \mathscr{I}_x(L^k)) \xrightarrow{d_x} T_x^{*\prime} \otimes L_x^k$$
is surjective for all $x \in M$, $k \geq k_0$.

To prove assertion 1, let $\widetilde{M} \xrightarrow{\pi} M$ denote the blow-up of M at both x and y, $E_x = \pi^{-1}(x)$ and $E_y = \pi^{-1}(y)$ the exceptional divisors of the blow-up; for notational convenience, let E denote the divisor $E_x + E_y$. (Here we are tacitly assuming that $n = \dim(M) \geq 2$; in case M is a Riemann surface, all the arguments that follow will be valid for $\tilde{M} = M$, $\pi = \text{id}$.)

Consider the pullback map on sections

$$\pi^* : H^0(M, \mathcal{O}_M(L^k)) \to H^0(\widetilde{M}, \mathcal{O}_{\widetilde{M}}(\pi^* L^k)).$$

For any global section $\widetilde{\sigma}$ of $\pi^* L$, the section of L^k given by σ over $M - \{x, y\}$ extends by Hartogs' theorem to a global section $\sigma \in H^0\left(M, \mathcal{O}\left(L^k\right)\right)$, and so we see that π^* is an isomorphism. Furthermore, by definition $\pi^* L^k$ is trivial along E_x and E_y, i.e.,

$$(\pi^* L^k)\big|_{E_x} = E_x \times L_x^k, \quad (\pi^* L^k)\big|_{E} = E_y \times L_y^k,$$

so that

$$H^0(E, \mathcal{O}_E(\pi^* L^k)) \cong L_x^k \oplus L_y^k,$$

and if r_E denotes the restricition map to E, the diagram

$$\begin{array}{ccc}
H^0(\widetilde{M}, \mathcal{O}_{\widetilde{M}}(\pi^* L^k)) & \xrightarrow{r_E} & H^0(E, \mathcal{O}_E(\pi^* L^k)) \\
\uparrow & & \| \\
H^0(M, \mathcal{O}(L^k)) & \xrightarrow{r_{x,y}} & L_x^k \oplus L_y^k
\end{array}$$

commutes. Thus to prove assertion 1 for x and y, we have to show the map r_E is surjective.

Now, on $\widetilde{M}$ we have the exact sheaf sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{M}}(\pi^* L^k - E) \longrightarrow \mathcal{O}_{\widetilde{M}}(\pi^* L^k) \xrightarrow{r_E} \mathcal{O}_E(\pi^* L^k) \longrightarrow 0.$$

Choose k_1 such that $L^{k_1} - K_M$ is positive on M. In fact, we can choose k_2 such that $\pi^* L^k - nE$ is positive on $\widetilde{M}$ for $k \geq k_2$, which will be proved in the next proposition. Note that

$$K_{\widetilde{M}} = \pi^* K_M + (n-1)E.$$

So for $k \geq k_0 = k_1 + k_2$,

$$\mathcal{O}_{\widetilde{M}}(\pi^* L^k - E) = \Omega^n_{\widetilde{M}}(\pi^* L^k - E - K_{\widetilde{M}}) = \Omega^n_{\widetilde{M}}((\pi^* L^{k_1} - K_{\widetilde{M}}) + (\pi^* L^{k'} - E))$$

with $k' \geq k_2$. Now by hypothesis, $\pi^* L^{k'} - nE$ has a positive definite curvature form on $\widetilde{M}$; $L^{k_1} - K_M$ has a positive curvature form on M, and so $\pi^* L^{k_1} - \pi^* K_M$ has a positive semidefinite one on $\widetilde{M}$. Thus the line bundle

$$(\pi^* L^{k_1} - K_{\widetilde{M}}) + (\pi^* L^{k'} - E) = (\pi^* L^{k_1} - \pi^* K_M) + (\pi^* L^{k'} - nE)$$

is positive on $\widetilde{M}$, and by the Kodiara vanishing theorem,

$$H^1(\widetilde{M}, \mathcal{O}_{\widetilde{M}}(\pi^* L^k - E)) = H^1(\widetilde{M}, \Omega^n_{\widetilde{M}}((\pi^* L^{k_1} - K_{\widetilde{M}}) + (\pi^* L^{k'} - E))) = 0$$

for $k \geq k_0$. Hence the map

$$r_E : H^0(\widetilde{M}, \mathcal{O}_{\widetilde{M}}(\pi^* L^k)) \longrightarrow H^0(E, \mathcal{O}_E(\pi^* L^k))$$

is surjective for $k \geq k_0$, and so assertion 1 is proved for x and y.

Assertion 2 is proved similarly. Let $\widetilde{M} \xrightarrow{\pi} M$ now denote the blow-up of M at x, $E = \pi^{-1}(x)$ the exceptional divisor. Again, the pullback map

$$\pi^* : H^0(M, \mathcal{O}_M(L^k)) \longrightarrow H^0(\widetilde{M}, \mathcal{O}_{\widetilde{M}}(\pi^* L^k))$$

is an isomorphism. Further, if $\sigma \in H^0(M, \mathcal{O}_M(L^k))$, then $\sigma(x) = 0$ if and only if $\widetilde{\sigma} = \pi^* \sigma$ vanishes on E; thus π^* restricts to give an isomorphism

$$\pi^* : H^0(M, \mathscr{I}_x(L^k)) \longrightarrow H^0(\widetilde{M}, \mathcal{O}_{\widetilde{M}}(\pi^* L^k - E)).$$

As before, we can identify

$$H^0(E, \mathcal{O}_E(\pi^* L^k - E)) = L_x^k \otimes H^0(E, \mathcal{O}_E(-E)) \cong L_x^k \otimes T_x^{*\prime},$$

and the diagram

$$\begin{array}{ccc} H^0(\widetilde{M}, \mathcal{O}_{\widetilde{M}}(\pi^* L^k - E)) & \xrightarrow{r_E} & H^0(E, \mathcal{O}_E(\pi^* L^k - E)) \\ \uparrow & & \| \\ H^0(M, \mathscr{I}_x(L^k)) & \xrightarrow{\quad d_x \quad} & L_x^k \otimes T_x^{*\prime} \end{array}$$

commutes. Thus we must prove that r_E is surjective for $k \gg 0$.

On $\widetilde{M}$, there is an exact sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{M}}(\pi^* L^k - 2E) \longrightarrow \mathcal{O}_{\widetilde{M}}(\pi^* L^k - E) \xrightarrow{r_E} \mathcal{O}_E(\pi^* L^k - E) \longrightarrow 0.$$

Again, choose k_1 such that $L^{k_1} - K_M$ is positive on M and k_2 such that $\pi^* L^{k'} - (n+1)E$ is positive on $\widetilde{M}$ for $k' \geq k_2$. For $k \geq k_0 = k_1 + k_2$,

$$\mathcal{O}_{\widetilde{M}}(\pi^* L^k - 2E) = \Omega^n_{\widetilde{M}}((\pi^* L^{k_1} - \pi^* K_M) \otimes (\pi^* L^{k'} - (n+1)E))$$

with $k' \geq k_2$. It follows by the Kodaira vanishing theorem that

$$H^1(\widetilde{M}, \mathcal{O}_{\widetilde{M}}(\pi^* L^k - 2E)) = 0$$

for $k \geq k_0$; hence r_E is surjective on global sections and assertion 2 is proved for arbitrary fixed x.

All that remains now to be proved is that we can find one value of k_0 such that assertions 1 and 2 hold for all choices of x and y and all $k \geq k_0$. But clearly if ι_{L^k} is defined at x and y and $\iota_{L^k}(x) \neq \iota_{L^k}(y)$, the same will be true for x' near x and y' near y, and likewise if ι_{L^k} is smooth at x it will be smooth at x' near x and separate points $x' \neq x''$, near x. Since M is compact, then, the result follows. ■

Proposition. Let $\widetilde{M} \xrightarrow{\pi} M$ denote the blow-up of M at x, and E the exceptional set of π. Suppose $L \to M$ is a positive line bundle. Then for any m, $\pi^* L^k - mE$ is positive for $k \gg 0$.

Proof. It suffices to show the statement holds for $m = 1$. Consider the complex structure of a blowup by using a explicit chart. Let $z_1, \ldots, z_n$ be a local coordinate in an open coordinate U of x. Then

$$\tilde{U} := \pi^{-1}(U) = \left\{(z, l) \in U \times \mathbb{P}^{n-1} \mid z_i l_j = z_j l_i\right\}.$$

For $i = 1, \ldots, n$, $\tilde{U}_i = \tilde{U} \backslash \{(l_i = 0)\}$ forms an open cover of U, and one can endow $\tilde{U}_i$ with coordinates

$$z(i)_j = \begin{cases} \dfrac{z_j}{z_i} = \dfrac{l_j}{l_i} & j \neq i \\ z_i & j = i \end{cases}$$

This gives the chart $\left(\tilde{U}_i, \varphi_i\right)$ where φ_i is given by

$$\begin{aligned} \varphi_i : & \tilde{U}_i \longrightarrow \mathbb{C}^n \\ & (z, l) \longmapsto \left(\frac{z_1}{z_i}, \ldots, z_i, \ldots, \frac{z_n}{z_i}\right) = (z(i)_1, \ldots, z(i)_i, \ldots, z(i)_n) \end{aligned}$$

Then for $i < j$, the change of coordinates is given by

$$\begin{aligned} & \varphi_j \circ \varphi_i^{-1}\big|_{U_j \cap U_i} (z(i)_1, \ldots, z(i)_i, \ldots, z(i)_j, \ldots, z(i)_n) \\ & = \left(\frac{z(i)_1}{z(i)_j} \ldots, \frac{1}{z(j)_i}, \ldots, z(i)_i z(i)_j, \ldots, \frac{z(i)_n}{z(i)_j}\right) \end{aligned}$$

Since the transition functions of the line bundle $[E]$ on $\tilde{M}$ are given by

$$g_{ij} = z(j)_i = \frac{z_i}{z_j} = \frac{l_i}{l_j} \quad \text{in } \tilde{U}_i \cap \tilde{U}_j,$$

one can realize $[E]$ on $\tilde{U}$ by identifying the fibre at (z, l) with the complex line in $\mathbb{C}^n$ passing through $(l_1, \ldots, l_n)$.

Now we are going to construct a hermitian metric h on $[E]$.

1. Since the fibre of the line bundle $[E]|_{\tilde{U}_i}$ over a point (z, l) can be identified with the complex line $\{\lambda (l_1, \ldots, l_n) \mid \lambda \in \mathbb{C}\}$, we might let h_1 be the metric on $[E]|_{\tilde{U}}$ given by $|(l_1, \ldots, l_n)|^2$

2. Let h_2 be the metric on $[E]|_{\tilde{M} \backslash E}$ such that $h_2(\sigma) \equiv 1$, where $\sigma \in H^0(\tilde{M}, E)$ is the global section of $[E]|_{\tilde{M}}$ corresponding to E

For $\epsilon > 0, U_\epsilon := \{z \in U \mid \|z\| < \epsilon\}$ and $\tilde{U}_\epsilon := \pi^{-1}(U_\epsilon)$. Let ρ_1, ρ_2 be a partion of unity relative to the cover $\left\{\tilde{U}_{2\epsilon}, \tilde{M} \backslash \tilde{U}_\epsilon\right\}$ of $\tilde{M}$. Then we can define a global hermitian metric h which is given by

$$h = \rho_1 h_1 + \rho_2 h_2$$

We will compute the positivity of the first Chern class of $[-E]$ with the metric h.

1. On $\tilde{M}\backslash\tilde{U}_{2\epsilon}, \rho_2 \equiv 1$ and hence $h(\sigma) \equiv 1$, i.e. in the trivialization above $h_\alpha |\sigma_\alpha|^2 = 1$, and

$$c_1([-E]) = dd^c \log \frac{1}{|\sigma|^2} = 0$$

since $\log \frac{1}{|\sigma|^2}$ is a harmonic function.

2. On $\tilde{U}_\epsilon, \rho_2 \equiv 0$. Denote

$$\pi' : \tilde{U}_\epsilon \longrightarrow \mathbb{P}^{n-1}$$
$$(z, l) \longmapsto l$$

Let F be the line bundle on $\mathbb{P}^{n-1}$ given by $\{\lambda (l_1, \ldots, l_n) \mid \lambda \in \mathbb{C}\}$. Recall that F is the dual of a hyperplane bundle. Since the pullback of F via π^* is just $[E]$, one can see that

$$c_1([-E]) = dd^c \log \|l\|^2 = (\pi')^* \omega_{FS}$$

where ω_{FS} is the associated $(1,1)$-form of the Fubini-Study metric. Therefore, $c_1(-E)$ is semi-positive on U_ϵ. In particular, it is positive on E since $\pi'|_E$ gives an isomorphism.

To sum up,

$$c_1(-E) = \begin{cases} 0 & \text{on } \tilde{M}\backslash\tilde{U}_{2\epsilon} \\ \geq 0 & \text{on } \tilde{U}_\epsilon \\ > 0 & \text{on } T_{1,0}(E)_z \subset T_{1,0}(\tilde{M})_z \quad \forall z \in E \end{cases}$$

We then turn to compute the positivity of $\pi^* L^k$. Note that for any $x \in E$ and $v \in T(\tilde{M})_x$,

$$c_1(\pi^* L)(v, \bar{v}) = c_1(L)(\pi_* v, \overline{\pi_* v}) \geq 0$$

$$c_1(\pi^* L) = \pi^* c_1(L)$$

and equality holds if and only if $\pi^* v = 0$. We conclude that

$$c_1(\pi^* L) = \begin{cases} = 0 & \text{on } T_{1,0}(E)_z \subset T_{1,0}(\tilde{M})_z \quad \forall z \in E \\ > 0 & \text{everywhere else} \end{cases}$$

Consequently, $c_1\left(\pi^* L^k \otimes (-E)\right) = kc_1(\pi^* L) - c_1(E)$ is positive on $\tilde{U}_\epsilon$ and on $\tilde{M}\backslash\tilde{U}_{2\epsilon}$ for ϵ small enough. Furthermore, since $\tilde{U}_{2\epsilon}\backslash\tilde{U}_\epsilon$ is relatively compact, $-c_1(E)$ is bounded below and $c_1(\pi^* L)$ is strictly positive on this region, then for k large enough $\pi^* L^k \otimes (-E)$ is a positive line bundle on $\tilde{M}$

■

Remark. According to Kodaira embedding theorem, we see that ampleness and positivity coincide on a compact Kähler complex manifold.

Definition. We say that a line bundle $L \to M$ over an algebraic variety is very ample if $H^0(M, \mathcal{O}(L))$ gives an embedding $M \to \mathbb{P}^N$, i.e., if there exists an embedding $f : M \hookrightarrow \mathbb{P}^N$ such that $L = f^*H$.

The following known result can be derived from the Kodaira Embedding theorem.

Corollary. If $E \to M$ is any line bundle and $L \to M$ a positive line bundle, then for $k \gg 0$, the line bundle $L^k + E$ is very ample.

In conclusion, we can give a some what more intrinsic restatement of the theorem:

Theorem (Kodaira Embedding Theorem)**.** A compact complex manifold M is an algebraic variety if and only if it has a closed, positive $(1,1)$-form ω whose cohomology class $[\omega]$ is rational.

Proof. If M is algebraic, say $i : M \to \mathbb{P}^N$, then $L = i^*[H]$ induces a closed $(1,1)$-form ω such that

$$[\omega] = c_1(L) \in H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{Q}).$$

Since $[H] \to \mathbb{P}^N$ is positive, ω is positive.

If $[\omega] \in H^2(M, \mathbb{Q})$, then for some $k \in \mathbb{N}$, $[k\omega] \in H^2(M, \mathbb{Z})$; in the exact sequence

$$H^1(M, \mathcal{O}^*) \longrightarrow H^2(M, \mathbb{Z}) \xrightarrow{i_*} H^2(M, \mathcal{O}),$$

$i_*([k\omega]) = 0$, and so there exists a holomorphic line bundle $L \to M$ with $c_1(L) = [k\omega]$. The line bundle L will then be positive. ■

A metric whose $(1,1)$-form is rational is called a Hodge metric.

Corollary. If M, M' are algebraic varieties, then $M \times M'$ is.

Proof. If ω, ω' are closed, integral, positive $(1,1)$-forms on M, M', respectively, and $\pi : M \times M' \to M$, $\pi' : M \times M' \to M'$ are the projection maps, then $\pi^*\omega + \pi'^*\omega'$ is again closed, integral, and positive of type $(1,1)$. ■

Corollary. If M is an algebraic variety, $\widetilde{M} \xrightarrow{\pi} M$ the blow-up of M at a point x, then $\widetilde{M}$ is algebraic.

Proof. We have seen in the course of the proof of the embedding theorem that if $L \to M$ is positive and $E = \pi^{-1}(x)$, then $\pi^* L^k - E$ is positive for $k \gg 0$. ■

Corollary. If $\widetilde{M} \xrightarrow{\pi} M$ is a finite unbranched covering of compact complex manifolds, then M is algebraic if and only if $\widetilde{M}$ is.

Proof. Clearly, if $L \to M$ is positive, then $c_1(\pi^* L) = \pi^* c_1(L)$ implies that $\pi^* L$ is positive. Conversely, say $\widetilde{\omega}$ is an integral, positive $(1,1)$-form on $\widetilde{M}$. For any $p \in M$, we have isomorphisms of a neighborhood U of p in M with neighborhoods U_i of the points $q_i \in \pi^{-1}(p)$; we can define a $(1,1)$-form ω on M by

$$\omega(p) = \sum_{q \in \pi^{-1}(p)} \widetilde{\omega}(q).$$

Then ω is closed and of type $(1,1)$, and if $\eta \in H^{2n-2}_{\mathrm{dR}}(M)$ is any integral cohomology class, then

$$\int_M \omega \wedge \eta = \frac{1}{m} \int_{\widetilde{M}} \widetilde{\omega} \wedge \pi^* \eta \in \mathbb{Q},$$

where m is the number of sheets of the cover. Thus $[\omega']$ is rational. ■