Transcendental Methods

June 17, 2020

0 Preliminaries

Let M be a complex manifold. Denote the holomorphic tangent space and antiholomorphic tangent space at $z \in M$ by $T'_z(M), T''_z(M)$. Also, let $T^{*\prime}_z(M), T^{*\prime\prime}_z(M)$ denote their dual spaces. We write

$$A^n(M) \coloneqq \{\text{complex-valued } n\text{-forms}\} = \bigoplus_{p+q=n} A^{p,q}(M),$$

as a decomposition of forms of type (p,q) where

$$A^{p,q}(M) = \{\varphi \in A^n(M) : \varphi(z) \in \bigwedge^p T^{*\prime}_z(M) \otimes \bigwedge^q T^{*\prime\prime}_z(M) \text{ for all } z \in M\}.$$

Let

$$\pi^{p,q} : A^*(M) \to A^{p,q}(M),\ \pi^r = \bigoplus_{p+q=r} \pi^{p,q} : A^*(M) \to A^r(M)$$

denote the projections and define the operators

$$\partial = \pi^{p+1,q} \circ d : A^{p,q}(M) \to A^{p+1,q}(M),\ \overline{\partial} = \pi^{p,q+1} \circ d : A^{p,q}(M) \to A^{p,q+1}(M).$$

Accordingly, we have $d = \partial + \overline{\partial}$. In terms of local coordinates $z = (z_1, \dots, z_m)$, if we write a form $\varphi \in A^{p,q}(M)$ as

$$\varphi(z) = \sum_{|I|=p,|J|=q} \varphi_{I\overline{J}}(z)\, dz_I \wedge d\overline{z}_J,$$

the operators ∂ and $\overline{\partial}$ are then given by

$$\partial\varphi(z) = \sum_{I,J,k} \frac{\partial}{\partial z_k} \varphi_{I\overline{J}}(z)\, dz_k \wedge dz_I \wedge d\overline{z}_J,$$

$$\overline{\partial}\varphi(z) = \sum_{I,J,k} \frac{\partial}{\partial \overline{z}_k} \varphi_{I\overline{J}}(z)\, d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J.$$

In particular, we say that a form φ of type $(p,0)$ is holomorphic if $\overline{\partial}\varphi = 0$. We define the Dolbeault cohomology groups to be

$$H^{p,q}_{\overline{\partial}}(M) = \frac{Z^{p,q}_{\overline{\partial}}(M)}{\overline{\partial}(A^{p,q-1}(M))}$$

and the de Rham cohomology to be

$$H^r_{\mathrm{dR}}(M) = \frac{Z^r_d(M)}{dA^{r-1}(M)},$$

where $Z^{*,*}_{\overline{\partial}}(M), Z^*_d(M)$ are the $\overline{\partial}$-closed and d-closed forms, respectively.

Let n be the dimension of M. A hermitian metric on M is given by a positive definite hermitian inner product

$$\langle\ ,\ \rangle_z : T'_z(M) \otimes \overline{T'_z(M)} \to \mathbb{C}$$

depending smoothly on z. Writing it in terms of the basis $\{dz_i \otimes d\overline{z}_j\}$, the metric is given by

$$ds^2 = \sum_{i,j} h_{jk}(z) dz_j \otimes d\overline{z}_k.$$

A coframe for the hermitian metric is an n-tuple of forms $(\varphi_1, \ldots, \varphi_m)$ of type $(1,0)$ such that

$$ds^2 = \sum_j \varphi_j \otimes \overline{\varphi}_j.$$

It is clear that coframes always exist locally: we can construct one by applying the Gram-Schmidt process.

Since we have a natural $\mathbb{R}$-linear isomorphism

$$T_{\mathbb{R},z}(M) \to T'_z(M),$$

we see that for a hermitian metric ds^2 on M, $\operatorname{Re} ds^2$ is a Riemannian metric on M. When we speak of distance, area, or volume on a complex manifold with hermitian metric, we always refer to the induced Riemannian metric. We also see that since the quadratic form $\operatorname{Im} ds^2$ is alternating, it represents a real differential form of degree 2;

$$\omega = -\operatorname{Im} ds^2 = \frac{i}{2} \sum_j \varphi_j \wedge \overline{\varphi}_j$$

is called the associated $(1,1)$-form of the metric.

Theorem (Wirtinger)**.** Let M be a complex manifold, ds^2 a hermitian metric on M with associated $(1,1)$-form ω. Let $S \subseteq M$ be a complex submanifold of dimension d. Then

$$\mathrm{Vol}(S) = \frac{1}{d!}\int_S \omega^d.$$

Proof. We only need to do the case when $S = M$. Let $z = (z_1, \ldots, z_n)$ be a local coordinates on M, and $ds^2 = \sum_j \varphi_j \otimes \overline{\varphi}_j$. Write $\varphi_j = \alpha_j + i\beta_j$; then the associated Riemannian metric on M is

$$\mathrm{Re}\, ds^2 = \frac{1}{2}(ds^2 + \overline{ds^2}) = \sum_j \alpha_j \otimes \alpha_j + \beta_j \otimes \beta_j,$$

and the volume element associated to $\mathrm{Re}\, ds^2$ is given by

$$\Phi = \alpha_1 \wedge \beta_1 \wedge \cdots \wedge \alpha_n \wedge \beta_n.$$

On the other hand, we have $\omega = \sum_j \alpha_j \wedge \beta_j$, so that the n^{th} exterior power

$$\omega^n = n!\, \alpha_1 \wedge \beta_1 \wedge \cdots \wedge \alpha_n \wedge \beta_n = n!\, \Phi.$$

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Example (Fubini-Study metric)**.** Let $Z_0, \ldots, Z_n$ be coordinates on $\mathbb{C}^{n+1}$ and $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$ the standard projection map. Let $U \subseteq \mathbb{P}^n$ be an open set and $Z : U \to \mathbb{C}^{n+1} - \{0\}$ a lifting of U; consider the differential form

$$\omega = \frac{i}{2\pi}\partial\overline{\partial} \log \|Z\|^2.$$

Then ω is independent of the lifting chosen; since lifting always exist locally, ω is globally defined differential form in $\mathbb{P}^n$. Clearly ω is of type $(1,1)$. To see that ω is positive, first note that the unitary group $U(n+1)$ acts transitively on $\mathbb{P}^n$ and leaves the form ω invariant, so that ω is positive everywhere if it is positive at one point. At the point $[1 : 0 : \cdots : 0]$,

$$\omega = \frac{i}{2\pi}\sum_j dw_j \wedge d\overline{w}_j > 0,$$

where $w_j = Z_j/Z_0$. This ω defines a hermitian metric on $\mathbb{P}^n$.

0.1 The Hodge Theorem

Let M be a connected, compact complex manifold of complex dimension n. We choose a hermitian metric ds^2 with associated $(1,1)$-form $\omega = \frac{i}{2}\sum_j \varphi_j \wedge \overline{\varphi}_j$ in terms of a unitary coframe $\{\varphi_1, \ldots, \varphi_n\}$. The metric ds^2 induces a hermitian metric on all tensor bundles $T^{*(p,q)}(M)$; the inner product in $T^{*(p,q)}_z(M)$ is given by taking the basis $\{\varphi_I(z) \wedge \overline{\varphi}_J(z)\}_{|I|=p,|J|=q}$ to be orthogonal and of length $\|\varphi_I \wedge \overline{\varphi}_J\|^2 = 2^{p+q}$ (recall that $\|dz_j\|^2 = 2$ on $\mathbb{C}^n$). Let Φ be the volume form, then the global inner product

$$\langle \psi, \eta \rangle = \int_M \langle \psi(z), \eta(z) \rangle \Phi(z)$$

makes the space $A^{p,q}(M)$ into a pre-Hilbert space.

We define the star, or duality operator,

$$* : A^{p,q}(M) \longrightarrow A^{n-p,n-q}(M)$$

by requiring

$$\langle \psi(z), \eta(z) \rangle \Phi(z) = \psi(z) \wedge * \eta(z)$$

for all $\psi \in A^{p,q}(M)$. This is an algebraic operator, which is given locally as follows: if we write $\eta = \sum_{I,J} \eta_{I\overline{J}}\, \varphi_I \wedge \overline{\varphi}_J$, then

$$* \eta = 2^{p+q-n} \sum_{I,J} \varepsilon_{I,J} \overline{\eta}_{I\overline{J}}\, \varphi_{I^\complement} \wedge \overline{\varphi}_{J^\complement},$$

where $I^\complement = \{1, \ldots, n\} - I$, $J^\complement = \{1, \ldots, n\} - J$ and we write $\varepsilon_{I,J}$ for the sign of the permutation

$$(1, \ldots, n, \overline{1}, \ldots, \overline{n}) \longrightarrow (i_1, \ldots, i_p, \overline{j}_1, \ldots, \overline{j}_q, i^\complement_1, \ldots, i^\complement_{n-p}, \overline{j}^\complement_1, \ldots, \overline{j}^\complement_{n-q}).$$

The signs work out so that $* * \eta = (-1)^{p+q} \eta$.

In fact, the space $A^{p,q}(M)$ is complete and $\overline{\partial}$ is bounded, so we can define the adjoint operator

$$\overline{\partial}^* : A^{p,q}(M) \to A^{p,q-1}(M)$$

by requiring that $\langle \overline{\partial}^* \psi, \eta \rangle = \langle \psi, \overline{\partial} \eta \rangle$ for all $\eta \in A^{p,q-1}(M)$. The $\overline{\partial}$-Laplacian is defined by $\triangle_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^* \overline{\partial}$. Differential forms satisfying the Laplace equation $\triangle_{\overline{\partial}} \psi = 0$ are called harmonic forms; the space of harmonic forms of type (p, q) is denoted $\mathcal{H}^{p,q}(M)$ and called the harmonic space.

Theorem (Hodge). Using the notation above, we have

1. $\dim \mathcal{H}^{p,q}_{\overline{\partial}}(M) < \infty$; and

2. because of this, the orthogonal projection $\mathcal{H}_{\overline{\partial}} : A^{p,q}(M) \to \mathcal{H}^{p,q}_{\overline{\partial}}(M)$ is well-defined. and there exists a unique operator, the Green's operator,

$$G_{\overline{\partial}} : A^{p,q}(M) \to A^{p,q}(M),$$

with $G_{\overline{\partial}}(\mathcal{H}^{p,q}_{\overline{\partial}}(M)) = 0$, $[G_{\overline{\partial}}, \overline{\partial}] = [G_{\overline{\partial}}, \overline{\partial}^*] = 0$ and

$$(\spadesuit) \qquad \mathrm{id} = \mathcal{H}_{\overline{\partial}} + \triangle_{\overline{\partial}} G_{\overline{\partial}}$$

on $A^{p,q}(M)$.

The equation ($\spadesuit$) in the form

$$\psi = \mathcal{H}_{\overline{\partial}}(\psi) + \overline{\partial}\left(\overline{\partial}^* G_{\overline{\partial}}\psi\right) + \overline{\partial}^*\left(\overline{\partial} G_{\overline{\partial}}\psi\right)$$

is called the Hodge decomposition on forms, since it directly implies the orthogonal direct-sum decomposition

$$A^{p,q}(M) = \mathcal{H}^{p,q}_{\overline{\partial}}(M) \oplus \overline{\partial} A^{p,q-1}(M) \oplus \overline{\partial}^* A^{p,q+1}(M).$$

Remark. On a compact Riemannian manifold M we may define the adjoint d^* of d, form the Laplacian $\triangle_d = dd^* + d^*d$, and arrive at the exact same formalism as for $\overline{\partial}$ on complex manifolds. The Hodge theorem is also true and the proof is the same as the one in the complex case.

0.2 Application of the Hodge Theorem

We first recall the Dolbeault theorem

Theorem (Dolbeault)**.** For complex manifold M, we have the isomorphism

$$H^q(M, \Omega^p) \cong H^{p,q}_{\overline{\partial}}(M).$$

Now, by the Hodge decomposition every $\overline{\partial}$-closed form $\psi \in Z^{p,q}_{\overline{\partial}}(M)$ can be written as

$$\psi = \mathcal{H}_{\overline{\partial}}(\psi) + \overline{\partial}\left(\overline{\partial}^* G\psi\right),$$

since $\overline{\partial} G\psi = G\overline{\partial}\psi = 0$. Therefore, we have the isomorphism

$$\mathcal{H}^{p,q}_{\overline{\partial}}(M) \cong H^{p,q}_{\overline{\partial}}(M).$$

Combining this with the Dolbeault isomorphism, we find

$$\mathcal{H}^{p,q}_{\overline{\partial}}(M) \cong H^q(M, \Omega^p).$$

1 Kähler Manifolds

1.1 The Kähler Condition

Let M be a compact complex manifold with Hermitian metric ds^2, and suppose that in some open set $U \subseteq M$, ds^2 is Euclidean; that is, there exist local holomorphic coordinates $z = (z_1, \ldots, z_n)$ such that

$$ds^2 = \sum_j dz_j \otimes d\overline{z}_j.$$

Write $z_j = x_j + iy_j$; one may directly verify that for a differential form

$$\varphi = \sum_{I,J} \varphi_{I\overline{J}} dz_I \wedge d\overline{z}_J$$

compactly supported in U,

$$\begin{aligned}
\triangle_{\overline{\partial}}(\varphi) &= -2 \sum_{I,J,j} \frac{\partial^2}{\partial z_j \partial \overline{z}_j} \varphi_{I\overline{J}}\, dz_I \wedge d\overline{z}_J \\
&= -\frac{1}{2} \sum_{I,J,j} \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right) \varphi_{I\overline{J}}\, dz_I \wedge d\overline{z}_J \\
&= \frac{1}{2} \triangle_d(\varphi)
\end{aligned}$$

i.e., the $\overline{\partial}$-Laplacian is equal to the ordinary d-Laplacian in U, up to a constant (cf. Section 6). Of course, very few compact complex manifolds have everywhere Euclidean metrics, but as it turns out in order to insure the identity

$$\triangle_{\overline{\partial}} = \frac{1}{2} \triangle_d$$

on a complex manifold, it is sufficient that the metric approximate the Euclidean metric to order 2 at each point. This is the Kähler condition, and we will spend the greater part of this section discussing the condition and its consequences.

We start by giving three alternate forms of the Kähler condition.

Definition-Proposition. Let M be a complex manifold with a Hermitian metric

$$ds^2 = \sum_j \varphi_j \otimes \overline{\varphi}_j.$$

We say that ds^2 is Kähler if one of the following equivalent statements holds:

(i) the associated $(1,1)$-form ω is d-closed, and

(ii) ds^2 osculates to order 2 to the Euclidean metric; that is, for every point $z \in M$ we can find a holomorphic coordinate system (z_j) in a neighborhood of z for which

$$ds^2 = \sum_{j,k} (\delta_{jk} + g_{jk})\, dz_j \otimes d\overline{z}_k,$$

where g_{jk} vanishes up to order 2 at z. We usually write

$$ds^2 = \sum_{j,k} (\delta_{jk} + [2])\, dz_j \otimes d\overline{z}_k.$$

Moreover, a manifold is called Kähler if it admits a Kähler metric.

Proof. One direction is clear. Conversely, we can always find coordinates (z_j) for which $h_{jk}(z) = \delta_{jk}$; i.e.,

$$\omega = \frac{i}{2} \sum_{j,k,\ell} (\delta_{jk} + a_{jk\ell} z_\ell + a_{jk\overline{\ell}} \overline{z}_\ell + [2])\, dz_j \wedge d\overline{z}_k;$$

note that

$$h_{jk} = \overline{h}_{kj} \implies a_{kj\overline{\ell}} = \overline{a}_{jk\ell}$$

and

$$d\omega = 0 \implies a_{jk\ell} = a_{\ell kj}.$$

We want to find a change of coordinates

$$z_j = w_j + \frac{1}{2} \sum_{k,\ell} b_{jk\ell} w_k w_\ell$$

such that

$$(\spadesuit) \qquad \omega = \frac{i}{2} \sum_{j,k} (\delta_{jk} + [2])\, dw_j \wedge d\overline{w}_k;$$

we normalize by requiring

$$b_{jk\ell} = b_{j\ell k}.$$

Then

$$dz_j = dw_j + \sum_{k,\ell} b_{jk\ell} w_k\, dw_\ell,$$

so that

$$-2i\omega = \sum_j \left(dw_j + \sum_{k,\ell} b_{jk\ell} w_k \, dw_\ell \right) \wedge \sum_m \left(d\overline{w}_m + \sum_{p,q} \overline{b}_{mpq} \overline{w}_p \, d\overline{w}_q \right)$$
$$+ \sum_{j,k,\ell} \left(a_{jk\ell} w_\ell + a_{jk\overline{\ell}} \overline{w}_\ell \right) dw_j \wedge dw_k + [2]$$
$$= \sum_{j,k,\ell} \left(\delta_{jk} + \sum_\ell \left(a_{jk\ell} w_\ell + a_{jk\overline{\ell}} \overline{w}_\ell + b_{k\ell j} w_\ell + \overline{b}_{j\ell k} \overline{w}_\ell \right) \right) dw_j \wedge d\overline{w}_k + [2].$$

If we set

$$b_{k\ell j} = -a_{jk\ell};$$

then

$$b_{k\ell j} = -a_{jk\ell} = -a_{\ell kj} = b_{kj\ell}$$

and

$$\overline{b}_{j\ell k} = -\overline{a}_{kj\ell} = -a_{kj\overline{\ell}},$$

so that the coordinate change does in fact satisfy the condition (♠). ■

Remark. Another way of expressing this condition that is useful in computation is to say that for each point $z \in M$ we can find a unitary coframe $\varphi_1, \ldots, \varphi_n$ for the metric in some neighborhood of z such that $d\varphi_j(z) = 0$.

Examples.

1. Any metric on a compact Riemann surface is Kähler, since $d\omega = 0$ for any 2-form ω.

2. If Λ is a lattice in $\mathbb{C}^n$, the complex torus $T = \mathbb{C}^n/\Lambda$ is Kähler with the Euclidean metric $ds^2 = \sum_j dz_j \otimes d\overline{z}_j$.

3. If M and N are Kähler, then $M \times N$ is Kähler with the product metric.

4. If $S \subseteq M$ is a submanifold, then by the analogue of implicit function theorem, the associated $(1,1)$-form of the induced metric on S is just the pullback of the associated $(1,1)$-form of the metric on M. Thus M is Kähler implies S is Kähler.

5. Recall that the Fubini-Study metric on $\mathbb{P}^n$ is given by its associated $(1,1)$-form

$$\omega = \frac{i}{2\pi} \partial\overline{\partial} \log \|Z\|^2 = \frac{i}{4\pi} d\left((\overline{\partial} - \partial) \log \|Z\|^2 \right),$$

so we see that ω is closed, and the Fubini-Study metric is Kähler.

Note: It is convenient to define an operator d^c by

$$d^c = \frac{i}{4\pi}(\overline{\partial} - \partial).$$

d and d^c are both real differential operators, and

$$dd^c = -d^c d = \frac{i}{2\pi}\partial\overline{\partial}.$$

We can consequently write

$$\omega = dd^c \log \|Z\|^2.$$

Remark. Note that by the last two examples, any compact manifold that can be embedded in projective space $\mathbb{P}^n$ is Kähler.

1.2 The Kähler Identities

Let M be a compact complex manifold with hermitian metric ds^2 and associated $(1,1)$-form ω. We define an additional operator

$$L : A^{p,q}(M) \longrightarrow A^{p+1,q+1}(M)$$

by $L(\eta) = \eta \wedge \omega$ and let

$$\Lambda = L^* : A^{p,q}(M) \longrightarrow A^{p-1,q-1}(M)$$

be its adjoint.

Now, for general M there are no non-obvious relationships among these various operators. If we assume that the metric on M is Kähler, however, we get a host of identities relating them, called the Kähler identities. Indeed, the Kähler condition is exactly that which insures a strong interplay between the real potential theory that associated to the Riemannian metric and the underlying complex structure.

Lemma. On $A^{p,q}(M)$, we have $\Lambda = (-1)^{p+q} * L *$.

Proof. For any $\psi \in A^{p,q}(M)$, $\eta \in A^{p+1,q+1}$,

$$\langle L\psi, \eta\rangle = \int_M \langle \psi \wedge \omega, \eta\rangle \,\Phi = \int_M \psi \wedge \omega \wedge (*\,\eta)$$

$$\langle \psi, (-1)^{p+q} * L * \eta\rangle = \int_M \langle \psi, (-1)^{p+q} * ((*\,\eta) \wedge \omega)\rangle \,\Phi = \int_M \psi \wedge * (-1)^{p+q} * ((*\,\eta) \wedge \omega).$$

Then the result follows from $\omega \wedge (*\,\eta) = (*\,\eta) \wedge \omega$ and $** = (-1)^{p+q}$ on $A^{n-p,n-q}$. ∎

Theorem (Kähler identities)**.** With the above notations, we have the following identities:

$$[\Lambda, d] = -4\pi d^{c*}, \ [L, d^*] = 4\pi d^c.$$

Proof. Clearly, these two identities are just equivalent. By decomposition into type, the first identity is equivalent to

$$[\Lambda, \partial] = i\overline{\partial}^* \quad \text{and} \quad [\Lambda, \overline{\partial}] = -i\partial^*.$$

Since Λ is a real operator, either of these implies the other. So it suffices to prove $[\Lambda, \partial] = i\overline{\partial}^*$. We may first make the computation on $\mathbb{C}^n$ with the Euclidean metric. For each $k = 1, \ldots, n$, let

$$e_k : A_c^{p,q}(\mathbb{C}^n) \to A_c^{p+1,q}(\mathbb{C}^n), \ e_k(\varphi) = dz_k \wedge \varphi,$$
$$\overline{e}_k : A_c^{p,q}(\mathbb{C}^n) \to A_c^{p,q+1}(\mathbb{C}^n), \ \overline{e}_k(\varphi) = d\overline{z}_k \wedge \varphi.$$

Let i_k and $\overline{i}_k$ be adjoints of e_k and $\overline{e}_k$, respectively. Note that e_k, $\overline{e}_k$, i_k and $\overline{i}_k$ are all linear over $C^\infty(\mathbb{C}^n)$, and

$$i_k(dz_J \wedge d\overline{z}_K) = 0, \quad \text{if } k \notin J, \ i_k(dz_k \wedge dz_J \wedge d\overline{z}_K) = 2\, dz_J \wedge d\overline{z}_K,$$
$$\overline{i}_k(dz_J \wedge d\overline{z}_K) = 0, \quad \text{if } k \notin K, \ \overline{i}_k(d\overline{z}_k \wedge dz_J \wedge d\overline{z}_K) = 2\, dz_J \wedge d\overline{z}_K.$$

Indeed, we have for any multi-indices L and M

$$\langle i_k(dz_J \wedge d\overline{z}_K), dz_L \wedge d\overline{z}_M\rangle = \langle dz_J \wedge d\overline{z}_K, dz_k \wedge dz_L \wedge d\overline{z}_M\rangle = 0,$$

so $i_k(dz_J \wedge d\overline{z}_K) = 0$, and

$$\begin{aligned}\langle i_k(dz_k \wedge dz_J \wedge d\overline{z}_K), dz_L \wedge d\overline{z}_M\rangle &= \langle dz_k \wedge dz_J \wedge d\overline{z}_K, dz_k \wedge dz_L \wedge d\overline{z}_M\rangle \\ &= 2\langle dz_J \wedge d\overline{z}_K, d\overline{z}_L \wedge d\overline{z}_M\rangle.\end{aligned}$$

Then for any monomial $dz_J \wedge d\overline{z}_K$,

$$i_k(e_k(dz_J \wedge d\overline{z}_K)) = \begin{cases} 0, & \text{if } k \in J, \\ 2\, dz_J \wedge d\overline{z}_K, & \text{if } k \notin J \end{cases}$$

while

$$e_k(i_k(dz_J \wedge d\overline{z}_K)) = \begin{cases} 2\, dz_J \wedge d\overline{z}_K, & \text{if } k \in J, \\ 0, & \text{if } k \notin J. \end{cases}$$

Therefore, we have $i_k e_k + e_k i_k = 2$. On the other hand, for $k \neq \ell$ we have

$$\begin{aligned}
i_k(e_\ell(dz_k \wedge dz_J \wedge d\overline{z}_K)) &= i_k(dz_\ell \wedge dz_k \wedge dz_J \wedge d\overline{z}_K) \\
&= i_k(-dz_k \wedge dz_\ell \wedge dz_J \wedge d\overline{z}_K) \\
&= -2\, dz_\ell \wedge dz_J \wedge d\overline{z}_K \\
&= -2e_\ell(dz_J \wedge d\overline{z}_K) \\
&= -e_\ell(i_k(dz_k \wedge dz_J \wedge d\overline{z}_K)),
\end{aligned}$$

while $i_k(e_\ell(dz_J \wedge d\overline{z}_K)) = e_\ell(i_k(dz_J \wedge d\overline{z}_K)) = 0$ in the case $k \notin J$, so we conclude

$$e_k i_\ell + i_\ell e_k = 2\delta_{k\ell}.$$

Similarly, $\overline{i}_k \overline{e}_\ell + \overline{e}_\ell \overline{i}_k = 2\delta_{k\ell}$. We also define operators ∂_k and $\overline{\partial}_k$ on $A^{p,q}_c(\mathbb{C}^n)$ by

$$\partial_k(\varphi_{I\overline{J}}\, dz_I \wedge d\overline{z}_J) = \frac{\partial \varphi_{I\overline{J}}}{\partial z_k}\, dz_I \wedge d\overline{z}_J \quad \text{and} \quad \overline{\partial}_k(\varphi_{I\overline{J}}\, dz_I \wedge d\overline{z}_J) = \frac{\partial \varphi_{I\overline{J}}}{\partial \overline{z}_k}\, dz_I \wedge d\overline{z}_J.$$

Note that ∂_k and $\overline{\partial}_k$ commute with e_ℓ, $\overline{e}_\ell$, i_ℓ, and $\overline{i}_\ell$ and with each other. One can see that the adjoint of ∂_k is $-\overline{\partial}_k$: for any $\varphi = \sum_{I,J} \varphi_{I\overline{J}}\, dz_I \wedge d\overline{z}_J$ and any compactly supported form $\psi\, dz_L \wedge d\overline{z}_M$, we have

$$\begin{aligned}
\langle -\overline{\partial}_k \varphi, \psi\, dz_L \wedge d\overline{z}_M \rangle &= \left\langle -\frac{\partial}{\partial \overline{z}_k}(\varphi_{L\overline{M}})\, dz_L \wedge d\overline{z}_M, \psi\, dz_L \wedge d\overline{z}_M \right\rangle \\
&= 2^{|L|+|M|} \int_{\mathbb{C}^n} -\frac{\partial}{\partial \overline{z}_k}(\varphi_{L\overline{M}}) \overline{\psi} \\
&= 2^{|L|+|M|} \int_{\mathbb{C}^n} \varphi_{L\overline{M}} \frac{\partial}{\partial \overline{z}_k}(\overline{\psi}) \\
&= 2^{|L|+|M|} \int_{\mathbb{C}^n} \varphi_{L\overline{M}} \overline{\frac{\partial}{\partial z_k}(\psi)} \\
&= \langle \varphi_{L\overline{M}}\, dz_L \wedge d\overline{z}_M, \partial_k(\psi\, dz_L \wedge d\overline{z}_M) \rangle \\
&= \langle \varphi, \partial_k(\psi\, dz_L \wedge d\overline{z}_M) \rangle.
\end{aligned}$$

Likewise, the adjoint of $\overline{\partial}_k$ is $-\partial_k$.

We can express all of our operators on $A^*_c(\mathbb{C}^n)$ in terms of these elementary operators: clearly

$$\partial = \sum_k \partial_k e_k = \sum_k e_k \partial_k, \quad \overline{\partial} = \sum_k \overline{\partial}_k \overline{e}_k = \sum_k \overline{e}_k \overline{\partial}_k,$$

and, taking adjoints,

$$\overline{\partial}^* = -\sum_k \partial_k \overline{i}_k, \quad \partial^* = -\sum_k \overline{\partial}_k i_k.$$

Moreover, L is defined as exterior product with the standard Kähler form defined on $\mathbb{C}^n$, so

$$L = \frac{i}{2} \sum_k e_k \overline{e_k}$$

and, taking the adjoint,

$$\Lambda = -\frac{i}{2} \sum_k \bar{i}_k i_k.$$

Therefore,

$$\begin{aligned}
\Lambda\partial &= -\frac{i}{2} \sum_{k,\ell} \bar{i}_k i_k \partial_\ell e_\ell \\
&= -\frac{i}{2} \sum_{k,\ell} \partial_\ell \bar{i}_k i_k e_\ell \\
&= -\frac{i}{2} \sum_{k,\ell} \partial_\ell \bar{i}_k (2\delta_{k\ell} - e_\ell i_k) \\
&= -i \sum_k \partial_k \bar{i}_k + \frac{i}{2} \sum_{k,\ell} \partial_\ell e_\ell \bar{i}_k i_k = i\overline{\partial}^* + \partial\Lambda.
\end{aligned}$$

Thus the identity is proved on $\mathbb{C}^n$.

To prove the result an a Kähler manifold M we use the condition of osculation to show that the identity holds at any point: for $z \in M$, we can choose a holomorphic coordinate system (z_j) in a neighborhood of z for which

$$ds^2 = \sum_{j,k} (\delta_{jk} + [2])\, dz_j \otimes d\overline{z}_k.$$

Then $\omega = \frac{i}{2} \sum_{j,k} (\delta_{jk} + [2])\, dz_j \wedge d\overline{z}_k$, and hence $L = \frac{i}{2} \sum_{j,k} (\delta_{jk} + [2])\, e_j \overline{e}_k$. Hence by the lemma above,

$$\Lambda = -\frac{i}{2} \sum_{j,k} (\delta_{jk} + [2])\, \bar{i}_k i_j$$

Since $[\Lambda, \overline{\partial}]$ on M will be the same as it on $\mathbb{C}^n$ with the Euclidean metric except for additional terms of order no less than 1, i.e. they vanish at z, we see that the identity holds at any $z \in M$, and hence everywhere. ∎

Corollary. We have $[L, \triangle_d] = 0$, or, equivalently, $[\Lambda, \triangle_d] = 0$.

Proof. Since ω is d-closed, we have $d(\omega \wedge \eta) = \omega \wedge d\eta$. So $[L, d] = 0$, and hence $[\Lambda, d^*] = 0$.

Then

$$
\begin{aligned}
\Lambda(dd^* + d^*d) &= (d\Lambda d^* - 4\pi d^{c*}d^*) + d^*\Lambda d \\
&= d\Lambda d^* + (4\pi d^* d^{c*} + d^*\Lambda d) \\
&= (dd^* + d^*d)\Lambda.
\end{aligned}
$$

■

Corollary. $\triangle_d = 2\triangle_{\overline{\partial}} = 2\triangle_\partial$. In particular, $\triangle_d$ preserves bi-degree, i.e. $[\triangle_d, \pi^{p,q}] = 0$.

Proof. Since $\Lambda\partial - \partial\Lambda = i\overline{\partial}^*$, we have

$$
i(\partial\overline{\partial}^* + \overline{\partial}^*\partial) = \partial(\Lambda\partial - \partial\Lambda) + (\Lambda\partial - \partial\Lambda)\partial = \partial\Lambda\partial - \partial\Lambda\partial = 0,
$$

which imples $\partial\overline{\partial}^* + \overline{\partial}^*\partial = 0$. Then

$$
\begin{aligned}
\triangle_d &= (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) \\
&= (\partial\partial^* + \partial^*\partial) + (\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}) = \triangle_\partial + \triangle_{\overline{\partial}},
\end{aligned}
$$

So it suffices to show $\triangle_\partial = \triangle_{\overline{\partial}}$. Note that

$$
-i\triangle_\partial = \partial(\Lambda\overline{\partial} - \overline{\partial}\Lambda) + (\Lambda\overline{\partial} - \overline{\partial}\Lambda)\partial = \partial\Lambda\overline{\partial} - \partial\overline{\partial}\Lambda + \Lambda\overline{\partial}\partial - \overline{\partial}\Lambda\partial
$$

Consequently,

$$
i\triangle_{\overline{\partial}} = \left(\overline{\partial}(\Lambda\partial - \partial\Lambda) + (\Lambda\partial - \partial\Lambda)\overline{\partial}\right) = i\triangle_\partial
$$

since $\overline{\partial}\partial = -\partial\overline{\partial}$. ■

1.3 The Hodge Decomposition

Set

$$
\begin{aligned}
H_d^{p,q}(M) &= \frac{Z_d^{p,q}(M)}{dA^*(M) \cap Z_d^{p,q}(M)}, \\
\mathcal{H}_d^{p,q}(M) &= \{\eta \in A^{p,q}(M) \mid \triangle_d \eta = 0\}, \\
\mathcal{H}_d^r(M) &= \{\eta \in A^r(M) \mid \triangle_d \eta = 0\}.
\end{aligned}
$$

Note the the first group is intrinsically defined by the complex structure, while the latter two depend on the particular metric.

Theorem (Hodge Decomposition)**.** For a compact Kähler manifold M, the complex cohomology satisfies

$$\begin{cases} H^r(M,\mathbb{C}) \cong \bigoplus_{p+q=r} H_d^{p,q}(M), \\ H_d^{p,q}(M) = \overline{H_d^{q,p}(M)}. \end{cases}$$

Proof. By the commutativity of $\triangle_d$ and $\pi^{p,q}$ and the fact that $\triangle_d$ is real, the harmonic forms satisfy

$$(\clubsuit) \qquad \begin{cases} \mathcal{H}_d^r(M) = \bigoplus_{p+q=r} \mathcal{H}_d^{p,q}(M), \\ \mathcal{H}_d^{p,q}(M) = \overline{\mathcal{H}_d^{q,p}(M)}. \end{cases}$$

On the other hand, for η a closed form of pure type (p,q), $\eta = \mathcal{H}_d(\eta) + dd^*G(\eta)$, where the harmonic part η also has pure type (p,q). Thus

$$H_d^{p,q}(M) \cong \mathcal{H}_d^{p,q}(M).$$

Combining this with ($\clubsuit$) and an application of the Hodge theorem for $\triangle_d$

$$H_{\mathrm{dR}}^*(M) \cong \mathcal{H}_d^*(M),$$

we get the decomposition of $H^r(M,\mathbb{C}) = H_{\mathrm{dR}}^r(M)$ by de Rham isomorphism. ∎

Corollary. $H^{p,q}(M) \cong H^q(M,\Omega^p)$. In particular, the holomorphic forms are harmonic for any Kähler metric on a compact manifold.

Proof. Since $\triangle_d = 2\triangle_{\overline{\partial}}$, we have $\mathcal{H}_d^{p,q}(M) = \mathcal{H}_{\overline{\partial}}^{p,q}(M)$. Consequently, we obtain

$$H^{p,q}(M) \cong H_{\overline{\partial}}^{p,q}(M) \cong H^q(M,\Omega^p).$$

By taking $q = 0$, we see all holomorphic p-forms are harmonic. ∎

Corollary.

$$H^q(\mathbb{P}^n,\Omega^p) = H_{\overline{\partial}}^{p,q}(\mathbb{P}^n) = \begin{cases} 0, & \text{if } p \neq q, \\ \mathbb{C}, & \text{if } p = q. \end{cases}$$

Proof. Since $H^{2k+1}(\mathbb{P}^n,\mathbb{Z}) = 0$, we have $H_{\overline{\partial}}^{p,q}(\mathbb{P}^n) = 0$ for $p+q$ odd; since $H^{2k}(\mathbb{P}^n,\mathbb{Z}) = \mathbb{Z}$, we have for $p \neq k$,

$$1 = b_{2k}(\mathbb{P}^n) \geq h^{p,2k-p}(\mathbb{P}^n) + h^{2k-p,p}(\mathbb{P}^n) = 2h^{p,2k-p}$$