Transcendental Methods

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1 Some Vanishing Theorems and Corollaries

1.1 Hodge theory on vector bundles

Let M be a compact Kähler manifold.

Recall that for any holomorphic vector bundle $E \to M$, the $\overline{\partial}$-operator

$$\overline{\partial} : A^{p,q}(E) \to A^{p,q+1}(E)$$

is defined for global C^∞ E-valued differential forms, and satisfies $\overline{\partial}^2 = 0$. We let $Z^{p,q}_{\overline{\partial}}(E)$ and $\mathscr{Z}^{p,q}_{\overline{\partial}}(E)$ denote the space and the sheaf of $\overline{\partial}$-closed E-valued differential forms of type (p,q), respectively, and we define the Dolbeault cohomology groups $H^{p,q}_{\overline{\partial}}(E)$ of E to be

$$H^{p,q}_{\overline{\partial}}(E) = \frac{Z^{p,q}(E)}{\overline{\partial} A^{p,q-1}(E)}.$$

The exact sheaf sequences

$$0 \longrightarrow \Omega^p(E) \longrightarrow \mathscr{C}^{p,0}(E) \xrightarrow{\overline{\partial}} \mathscr{Z}^{p,1}_{\overline{\partial}}(E) \longrightarrow 0$$
$$0 \longrightarrow \mathscr{Z}^{p,q}_{\overline{\partial}}(E) \longrightarrow \mathscr{C}^{p,q}(E) \xrightarrow{\overline{\partial}} \mathscr{Z}^{p,q+1}_{\overline{\partial}}(E) \longrightarrow 0$$

gives us isomorphisms

$$H^i(M, \mathscr{Z}^{p,1}_{\overline{\partial}}(E)) \xrightarrow{\sim} H^{i+1}(M, \Omega^p(E)),$$
$$H^i(M, \mathscr{Z}^{p,q+1}_{\overline{\partial}}(E)) \xrightarrow{\sim} H^{i+1}(M, \mathscr{Z}^{p,q}_{\overline{\partial}}(E)), \qquad i \geq 1,$$

since the sheaves $\mathscr{C}^{p,q}(E)$ admit partitions of unity and hence have no Čech cohomology. Thus,

$$\begin{aligned} H^q(M, \Omega^p(E)) &\cong H^{q-1}(M, \mathscr{Z}_{\overline{\partial}}^{p,1}) \cong \cdots \cong H^1(M, \mathscr{Z}_{\overline{\partial}}^{p,q-1}) \\ &\cong \frac{H^0(M, \mathscr{Z}_{\overline{\partial}}^{p,q})}{\overline{\partial} H^0(M, \mathscr{C}_{\overline{\partial}}^{p,q-1})} = H_{\overline{\partial}}^{p,q}(E). \end{aligned}$$

Suppose we have metrics given on M and E; we have then induced metrics on all tangential tensor bundles of M tensored with E or E^*. In particular, if $\{\varphi_j\}$ is a local coframe for the metric on $T_M^{*\prime}$ and $\{e_\alpha\}$ a unitary frame for E, any section η of $A^{p,q}(E)$ can be written locally as

$$\eta(z) = \frac{1}{p!q!} \sum_{I,J,\alpha} \eta_{I,J,\alpha}(z)\, \varphi_I \wedge \overline{\varphi}_J \otimes e_\alpha;$$

for $\eta, \psi \in A^{p,q}(E)$,

$$\langle \eta(z), \psi(z) \rangle = \frac{2^{p+q-n}}{p!q!} \sum_{I,J,\alpha} \eta_{I,J,\alpha}(z) \cdot \overline{\psi_{I,J,\alpha}(z)}.$$

Again, we define an inner product on $A^{p,q}(E)$ by setting

$$\langle \eta, \psi \rangle = \int_M \langle \eta(z), \psi(z) \rangle \, \Phi,$$

where Φ is the volume form on M.

We have a “wedge product”

$$\wedge : A^{p,q}(E) \otimes A^{p',q'}(E^*) \longrightarrow A^{p+p',q+q'}(M)$$

defined by

$$(\eta \otimes s) \wedge (\eta' \otimes s') = \langle\!\langle s, s' \rangle\!\rangle \cdot \eta \wedge \eta',$$

where $\langle\!\langle\ ,\ \rangle\!\rangle$ is the bilinear pairing $A^0(E) \otimes A^0(E') \to A^0(M)$; we define an operator

$$*_E : A^{p,q}(E) \longrightarrow A^{n-p,n-q}(E^*)$$

by requiring, for $\eta, \psi \in A^{p,q}(E)$,

$$\langle \eta, \psi \rangle = \int_M \eta \wedge *_E \psi.$$

Explicitly, if $\{e_\alpha\}$ and $\{e_\alpha^*\}$ are dual unitary frames for E and E^*, then for $\eta \in A^{p,q}(E)$ written as $\eta = \sum_\alpha \eta_\alpha \otimes e_\alpha$, we have $*_E \eta = \sum_\alpha (*\, \eta_\alpha) \otimes e_\alpha^*$, where $*$ is the usual star operator on $A^{p,q}(M)$.

We take

$$\overline{\partial}^* : A^{p,q}(E) \longrightarrow A^{p,q-1}(E)$$

to be given by $\overline{\partial}^* = - *_{E^*} \overline{\partial} *_E$; as before, $\overline{\partial}^*$ is the adjoint of $\overline{\partial}$. Finally, the $\overline{\partial}$-Laplacian on E is defined by

$$\triangle_E = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : A^{p,q}(E) \longrightarrow A^{p,q}(E).$$

An E-valued form φ is called harmonic if $\triangle_E \varphi = 0$. We let

$$\mathcal{H}^{p,q}(E) = \ker \triangle_E$$

be the harmonic space.

Theorem (Hodge)**.** Using the notation above, we have

1. $\dim \mathcal{H}^{p,q}(E) < \infty$.

2. If $\mathcal{H}$ denotes the orthogonal projection $\mathcal{H} : A^{p,q}(E) \to \mathcal{H}^{p,q}(E)$, there exists an operator

$$G : A^{p,q}(E) \longrightarrow A^{p,q}(E),$$

with $G(\mathcal{H}^{p,q}(E)) = 0$, $[G, \overline{\partial}] = [G, \overline{\partial}^*] = 0$ and

$$\text{id} = \mathcal{H} + \triangle_E G \qquad (\spadesuit_E)$$

on $A^{p,q}(E)$.

3. Consequently, there is an isomorphism

$$\mathcal{H}^{p,q}(E) \longrightarrow H^{p,q}_{\overline{\partial}}(E).$$

4. The $*$-operator gives an isomorphism

$$H^q(M, \Omega^p(E)) \cong H^{n-q}(M, \Omega^{n-p}(E^*))^*.$$

For $p = 0$, this last result reads

$$H^q(M, \mathcal{O}(E)) \cong H^{n-q}(M, \mathcal{O}(E^* \otimes K_M))^*.$$

This isomorphism is called Kodaira-Serre duality.

Now if M is Kähler with associated $(1,1)$-form ω, we define the operator

$$L : A^{p,q}(E) \longrightarrow A^{p+1,q+1}(E)$$

by setting, for $\eta \in A^{p,q}(M)$ and $s \in A^0(E)$,

$$L(\eta \otimes s) = \omega \wedge \eta \otimes s;$$

let $\Lambda = L^*$ be the adjoint of L. If $\nabla = \nabla' + \overline{\partial}$ is the metric connection on E, then we have the basic identity

$$[\Lambda, \overline{\partial}] = -i\nabla'^*.$$

This identity follows from the analogous identity $[\Lambda, \overline{\partial}] = -i\partial^*$ on scalar forms $A^{p,q}(M)$, which we have already proved. To see this, pick a local frame $\{e_\alpha\}$ for E; if $A = A^{1,0} + A^{0,1}$ is the connection matrix for ∇ in terms of $\{e_\alpha\}$, we can write, for $\eta \in A^{p,q}(E)$,

$$\eta = \sum_\alpha \eta_\alpha \otimes e_\alpha, \quad \overline{\partial}\eta = \sum_\alpha \overline{\partial}\eta_\alpha \otimes e_\alpha + \sum_{\alpha,\beta} (\eta_\alpha \wedge A^{0,1}_{\alpha\beta}) \otimes e_\beta,$$

and $\Lambda\eta = \sum_\alpha \Lambda(\eta_\alpha) \otimes e_\alpha$, so

$$[\Lambda, \overline{\partial}]\eta = \sum_\alpha \big([\Lambda, \overline{\partial}]\eta_\alpha\big) \otimes e_\alpha + [\Lambda, A^{0,1}]\eta = -i\sum_\alpha \partial^*\eta_\alpha \otimes e_\alpha + [\Lambda, A^{0,1}]\eta.$$

Similarly,

$$\nabla'\eta = \sum_\alpha \partial\eta_\alpha \otimes e_\alpha + \sum_{\alpha,\beta} (\eta_\alpha \wedge A^{1,0}_{\alpha\beta}) \otimes e_\beta,$$

i.e.,

$$\nabla'^*\eta = \sum_\alpha \partial^*\eta_\alpha \otimes e_\alpha + (A^{1,0})^*\eta.$$

Now, we choose at each $z \in M$ a frame for E in a neighborhood of z for which $A(z)$ vanishes. Indeed, a given trivialization can be changed by a local $\phi : U \to GL(r, \mathbb{C})$, whose Taylor expansion is of the form

$$\phi(z_1, \ldots, z_n) = \mathrm{id} - \sum_j z_j A_j(0) + \text{ higher order terms.}$$

Here, $z_1, \ldots, z_n$ are local coordinates with z as the origin and $A = \sum A_j dz_j$. Then we see that

$$[\Lambda, \overline{\partial}] + i\nabla'^* = [\Lambda, A^{0,1}] + i(A^{1,0})^* = 0.$$

1.2 The Kodaira Vanishing Theorem

Definition. A line bundle $L \to M$ is positive if there exists a metric on L with curvature form Θ such that $(i/2\pi)\Theta$ is a positive $(1,1)$-form, i.e., for any $z \in M$ and $0 \neq v \in T'_z M$,

$$-i\left\langle \frac{i}{2\pi}\Theta(z), v \wedge \overline{v} \right\rangle = \frac{1}{2\pi}\langle \Theta(z), v \wedge \overline{v} \rangle > 0;$$

L is negative if L^* is positive. A divisor D on M is positive if the line bundle $[D]$ is.

Proposition. If ω is any real, closed $(1,1)$-form with

$$[\omega] = c_1(L) \in H^2_{\mathrm{dR}}(M),$$

then there exists a metric connection on L with curvature form $\Theta = -2\pi i \omega$. Thus L is positive iff its Chern class may be represented by a positive form in $H^2_{\mathrm{dR}}(M)$.

Proof. Let $|\cdot|^2$ be a metric on L with curvature form Θ. If $\varphi : L_U \to U \times \mathbb{C}$ is a trivialization of L over an open set U, s a section of L over U and s_U the corresponding holomorphic function, then

$$|s|^2 = h(z) \cdot |s_U|^2$$

for some positive function h. The curvature form and Chern class are given by

$$\Theta = -\partial\overline{\partial} \log h(z), \quad c_1(L) = \left[\frac{i}{2\pi}\Theta\right] \in H^2_{\mathrm{dR}}(M).$$

Now let $\widetilde{|\cdot|}^2$ be another metric on L with curvature form $\widetilde{\Theta}$. Then $\widetilde{|s|}^2/|s|^2 = e^\rho$ for some real C^∞ function ρ on M, and from the local formula

$$\widetilde{h}(z) = e^{\rho(z)} h(z)$$

it follows that $\Theta = \partial\overline{\partial}\rho + \widetilde{\Theta}$.

Working in the other direction, let $\widetilde{\Theta} = -2\pi i \omega$. If we can solve the equation

$$\Theta = \partial\overline{\partial}\rho + \widetilde{\Theta}$$

for a real C^∞ function ρ, then the metric $e^\rho |s|^2$ on L will have curvature form φ.

Let $\eta = \Theta - \widetilde{\Theta}$ and let G_d denote the Green's operator associated to the Laplacian $\triangle_d$, and similarly for G_∂ and $G_{\overline{\partial}}$. From the basic identity

$$\frac{1}{2}\triangle_d = \triangle_\partial = \triangle_{\overline{\partial}},$$

it follows first that $2G_d = G_\partial = G_{\overline{\partial}}$, and then that all the operators d, ∂, $\overline{\partial}$, d^*, ∂^*, and $\overline{\partial}^*$ commute with the Green's operators.

Now, since η is d-exact, its harmonic projection under any of the above Laplacians is zero. By the decomposition for $\overline{\partial}$,

$$\eta = \mathcal{H}_{\overline{\partial}}(\eta) + \overline{\partial}^*\overline{\partial} G_{\overline{\partial}}\eta + \overline{\partial}\overline{\partial}^* G_{\overline{\partial}}\eta = \overline{\partial}\overline{\partial}^* G_{\overline{\partial}}\eta.$$

But $\overline{\partial}^* G_{\overline{\partial}}\eta$ has pure type $(1,0)$ and so

$$\partial(\overline{\partial}^* G_{\overline{\partial}}\eta) = -\overline{\partial}^* G_{\overline{\partial}}(\partial\eta) = 0.$$

Since the harmonic space for ∂ is the same as the harmonic space for $\overline{\partial}$ and hence is orthogonal to the range of $\overline{\partial}^*$, we deduce by the decomposition for ∂ that

$$\overline{\partial}^* G_{\overline{\partial}}\eta = \mathcal{H}_\partial(\overline{\partial}^* G_{\overline{\partial}}\eta) + \partial^*\partial G_\partial(\overline{\partial}^* G_{\overline{\partial}}\eta) + \partial\partial^* G_\partial(\overline{\partial}^* G_{\overline{\partial}}\eta) = \partial\partial^* G_\partial(\overline{\partial}^* G_{\overline{\partial}}\eta).$$

Since $i\eta$ is real, $\rho = -\partial^*\overline{\partial}^* G_{\overline{\partial}}^2\eta$ is also real, and we are done. ∎

Let J be the tautological line bundle of $\mathbb{P}^n$, which is the dual of the hyperplane bundle $[H]$; we can put a metric on J by setting $|(Z_0,\ldots,Z_n)|^2 = \sum_i |Z_i|^2$. If Z is any nonzero section of J, then the curvature form in J is given by

$$\Theta^* = \overline{\partial}\partial \log \|Z\|^2 = 2\pi i dd^c \log \|Z\|^2.$$

The curvature form Θ for the dual metric in $[H]$ is then $-\Theta^*$, and consequently

$$\frac{i}{2\pi}\Theta = dd^c \log \|Z\|^2,$$

i.e., $(i/2\pi)\Theta$ is just the associated $(1,1)$-form ω of the Fubini-Study metric on $\mathbb{P}^n$, which we have seen is positive.

Note that since the restriction to a submanifold $V \subseteq M$ of a positive form is again positive, $L|_V \to V$ will be positive if $L \to M$ is. In particular, the hyperplane bundle on any complex submanifold of $\mathbb{P}^n$ is positive.

Theorem (Kodaira-Nakano Vanishing Theorem)**.** If $L \to M$ is a positive line bundle, then

$$H^q(M, \Omega^p(L)) = 0 \quad \text{for } p+q > n.$$

Proof. By hypothesis we can find a metric in L whose curvature form Θ is $-2\pi i\omega$, where ω is the associated $(1,1)$-form of a Kähler metric. Now by harmonic theory

$$H^q(M, \Omega^p(L)) \cong H^{p,q}_{\overline{\partial}}(L) \cong \mathcal{H}^{p,q}(L).$$

Let $\eta \in \mathcal{H}^{p,q}(L)$ be a harmonic form. Then

$$\Theta = \nabla^2 = \overline{\partial}\nabla' + \nabla'\overline{\partial},$$

so from $\overline{\partial}\eta = 0$, $\overline{\partial}^*\eta = 0$, and $\Theta\eta = \overline{\partial}\nabla'\eta$,

$$\begin{aligned} i\langle \Lambda\Theta\eta, \eta\rangle &= i\langle \Lambda\overline{\partial}\nabla'\eta, \eta\rangle \\ &= i\left\langle \left(\overline{\partial}\Lambda - i\nabla'^*\right)\nabla'\eta, \eta\right\rangle \\ &= \langle \nabla'^*\nabla'\eta, \eta\rangle = \|\nabla'\eta\|^2 \geq 0, \end{aligned}$$

since $\langle \overline{\partial}\Lambda\nabla'\eta, \eta\rangle = \langle \Lambda\nabla'\eta, \overline{\partial}^*\eta\rangle = 0$. Similarly,

$$\begin{aligned} i\langle \Theta\Lambda\eta, \eta\rangle &= i\langle \nabla'\overline{\partial}\Lambda\eta, \eta\rangle \\ &= i\left\langle \nabla'\left(\Lambda\overline{\partial} + i\nabla'^*\right)\eta, \eta\right\rangle \\ &= -\langle \nabla'\nabla'^*\eta, \eta\rangle = -\|\nabla'^*\eta\|^2 \leq 0. \end{aligned}$$

Combining, $i\left\langle [\Lambda, \Theta]\eta, \eta\right\rangle \geq 0$. But $\Theta = -2\pi i L$, and so

$$0 \leq i\langle [\Lambda, \Theta]\eta, \eta\rangle = 2\pi\left\langle [\Lambda, L]\eta, \eta\right\rangle = 2\pi(n-p-q)\|\eta\|^2.$$

Thus $p + q > n$ implies $\eta = 0$. ∎

Dualizing the Kodaira vanishing theorem, we obtain:

$$H^q(M, \Omega^p(L)) = 0 \text{ for } p+q < n \text{ in case } L \to M \text{ is a negative line bundle.}$$

Example. As an immediate consequence of the vanishing theorem, we see that

$$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(kH)) = 0 \qquad \text{for } 1 \leq q \leq n-1, \quad \text{all } k.$$

This follow directly from the dualized version of the vanishing theorem in case k is negative; if k is nonnegative,

$$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(kH)) = H^q(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}(kH - K_{\mathbb{P}^n})) = H^q(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}((k+n+1)H)) = 0$$

by the original version of the theorem.

1.3 The Serre Vanishing Theorem

Our second vanishing theorem for the cohomology of holomorphic vector bundles is less precise but broader in scope than the Kodaira Vanishing Theorem.

Theorem (Serre Vanishing Theorem)**.** Let M be a compact, complex manifold and $L \to M$ a positive line bundle. Then for any holomorphic vector bundle E, there exists μ_0 such that

$$H^q(M, \mathcal{O}(L^\mu \otimes E)) = 0 \quad \text{for } q > 0,\ \mu \geq \mu_0.$$

Proof. First, by Kodaira-Serre duality,

$$H^q(M, \mathcal{O}(L^\mu \otimes E)) \cong H^{n-q}(M, \mathcal{O}(L^{-\mu} \otimes E^* \otimes K_M)),$$

so it will be sufficient to prove that for any E, there exists μ_0 such that

$$H^{0,q}_{\bar\partial}(M, L^{-\mu} \otimes E) \cong H^q(M, \mathcal{O}(L^{-\mu} \otimes E)) = 0$$

for $\mu \geq \mu_0$, $q < n$.

Choose a metric in L such that $\omega = (i/2\pi)\Theta_L$ is positive; let the metric on M be the one given by ω. Now we have seen that if E, E' are two hermitian vector bundles and if we give $E \otimes E'$ the induced metric, then

$$\nabla_{E\otimes E'} = \nabla_E \otimes 1 + 1 \otimes \nabla_{E'}$$

and so

$$\Theta_{E\otimes E'} = \Theta_E \otimes 1 + 1 \otimes \Theta_{E'}.$$

In particular, for L and E as above with any metric on E,

$$\Theta_{L^{-\mu}\otimes E} = 2\pi i \mu\, \omega \otimes 1_E + \Theta_E.$$

Let $\eta \in \mathcal{H}^{0,q}(L^{-\mu} \otimes E)$ be harmonic. Writing Θ for $\Theta_{L^{-\mu}\otimes E}$, ∇ for $\nabla_{L^{-\mu}\otimes E}$, we have $i\langle[\Lambda, \Theta]\eta, \eta\rangle \geq 0$ by the proof of Kodaira theorem. But now

$$\Theta = \Theta_{L^{-\mu}\otimes E} = \Theta_E + 2\pi i \mu\, \omega,$$

and so

$$\begin{aligned} i\langle[\Lambda, \Theta]\eta, \eta\rangle &= i\langle[\Lambda, \Theta_E]\eta, \eta\rangle - 2\pi\mu\langle[\Lambda, L]\eta, \eta\rangle \\ &= i\langle[\Lambda, \Theta_E]\eta, \eta\rangle - 2\pi\mu(n-q)\|\eta\|^2. \end{aligned}$$

Now $[\Lambda, \Theta_E]$ is bounded on $A^{0,q}(L^{-\mu} \otimes E)$, so we can write

$$|\langle [\Lambda, \Theta_E]\eta, \eta\rangle| \leq \|[\Lambda, \Theta_E]\eta\| \cdot \|\eta\| \leq C\|\eta\|^2,$$

and consequently for $q < n$,

$$\mu > \frac{C}{2\pi} \implies \eta = 0,$$

i.e., $\mathcal{H}^{0,q}(L^{-\mu} \otimes E) = 0$ for $\mu > C/(2\pi)$, $q < n$. ∎

1.4 The Lefschetz Theorem on Hyperplane Sections

Using the Kodaira vanishing theorem, we can give a proof of the famous Lefschetz theorem relating the homology of a projective variety to that of its hyperplane sections.

Theorem (Lefschetz Hyperplane Theorem)**.** Let M be an n-dimensional compact, complex manifold and $V \subset M$ a smooth hypersurface with $L = [V]$ positive. Then the map

$$H^r(M, \mathbb{Q}) \longrightarrow H^r(V, \mathbb{Q})$$

induced by the inclusion $i : V \hookrightarrow M$ is an isomorphism for $r \leq n-2$ and injective for $r = n-1$.

Proof. It will suffice to prove the result over $\mathbb{C}$. By the Hodge decomposition and Dolbeault,

$$H^r(M, \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(M) \cong \bigoplus_{p+q=r} H^{p,q}_{\bar{\partial}}(M) \cong \bigoplus_{p+q=r} H^q(M, \Omega^p_M).$$

The same holding for V, it is sufficient to prove that the map

$$H^q(M, \Omega^p_M) \longrightarrow H^q(V, \Omega^p_V)$$

is an isomorphism for $p + q \leq n-2$, and injective for $p + q = n-1$.

To see this, we factor the restriction map $\Omega^p_M \to \Omega^p_V$ by

$$\Omega^p_M \xrightarrow{\text{res}} \Omega^p_M|_V \xrightarrow{i} \Omega^p_V,$$

where $\Omega^p_M|_V$ is the sheaf of sections of $\left(\bigwedge^p T^{*\prime}_M\right)\Big|_V$. The kernel of the restriction map is clearly just the sheaf of holomorphic p-forms on M vanishing along V, so we have an exact sequence of sheaves on M

$$(\heartsuit) \qquad 0 \longrightarrow \Omega^p_M(-V) \longrightarrow \Omega^p_M \xrightarrow{\text{res}} \Omega^p_M|_V \longrightarrow 0.$$

For $z \in V$, the sequence

$$0 \longrightarrow N^*_{V,z} \longrightarrow T^{*\prime}_z M \longrightarrow T^{*\prime}_z V \longrightarrow 0,$$

yields, by linear algebra,

$$0 \longrightarrow N^*_{V,z} \otimes \bigwedge^{p-1} T^{*\prime}_z V \longrightarrow \bigwedge^{p} T^{*\prime}_z M \longrightarrow \bigwedge^{p} T^{*\prime}_z V \longrightarrow 0,$$

and consequently an exact sequence of sheaves on V

$$0 \longrightarrow \Omega^{p-1}_V(N^*_V) \longrightarrow \Omega^p_M|_V \xrightarrow{i} \Omega^p_V \longrightarrow 0.$$

By adjunction formula I, $N^*_V = [-V]|_V$; we can thus rewrite this last sequence as

$$(\diamondsuit) \qquad 0 \longrightarrow \Omega^{p-1}_V(-V) \longrightarrow \Omega^p_M|_V \longrightarrow \Omega^p_V \longrightarrow 0.$$

Now $[-V]$ is negative on M, and likewise $[-V]|_V$ is negative on V. The Kodaira vanishing theorem gives

$$H^q(M, \Omega^p_M(-V)) = H^q(V, \Omega^{p-1}_V(-V)) = 0, \quad p + q < n.$$

By the exact cohomology sequences associated to the sheaf sequences $(\heartsuit)$ and $(\diamondsuit)$,

$$H^q(M, \Omega^p_M) \xrightarrow{\sim} H^q(M, \Omega^p_M|_V) = H^q(V, \Omega^p_M|_V) \xrightarrow{\sim} H^q(V, \Omega^p_V)$$

for $p + q \leq n - 2$, and with both maps injective for $p + q = n - 1$. ■

Example. When $n = 2$—i.e., M is a (connected and compact) complex surface—and $V \subset M$ is a Riemann surface embedded as a positive divisor, then the Lefschetz theorem gives

$$H_0(V, \mathbb{Z}) \cong H_0(M, \mathbb{Z}) = \mathbb{Z}, \quad H_1(V, \mathbb{Z}) \longrightarrow H_1(M, \mathbb{Z}) \longrightarrow 0,$$

i.e., all of the first homology of the 4-manifold M lies on the irreducible embedded Riemann surface V.