which implies $h^{p,2k-p}(\mathbb{P}^n) = 0$ and hence $h^{p,p}_{\bar\partial}(\mathbb{P}^n) = 1$. Thus $H^{p,p}_{\bar\partial}(\mathbb{P}^n) \cong \mathbb{C}$. ■

Note in particular that there are no nonzero global holomorphic forms on $\mathbb{P}^n$.

1.4 The Lefschetz Decomposition

sl_2 is the Lie algebra of SL_2; it is realized as the vector space of 2×2 complex matrices with trace 0, and with the bracket $[A, B] = AB - BA$. We take as standard generators

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with the relations

$$[x, y] = h, \; [h, x] = 2x, \; [h, y] = -2y.$$

The irreducible (finite-dimensional) sl_2-modules are indexed by nonnegative integers n; for each such n the corresponding sl_2-module $V(n)$ has dimension $n + 1$. Explicitly,

$$V(n) \cong V_n \oplus V_{n-2} \oplus \cdots \oplus V_{-n+2} \oplus V_{-n}.$$

The eigenvalues of h acting on $V(n)$ are $n, n-2, \ldots, -n+2, -n$, each appearing with multiplicity 1.

For any (finite-dimensional) sl_2-module V (with $\rho : sl_2 \to gl(V)$), not necessarily irreducible, we define the Lefschetz decomposition of V as follows: let $PV = \ker \rho(x)$; then

$$V = PV \oplus yPV \oplus y^2 PV \oplus \cdots,$$

and this decomposition is compatible with the decomposition of V into eigenspaces V_m for h since $yPV(n) = V_{n-2k}$. We also see that the maps

$$V_m \xrightarrow{y^m} V_{-m} \quad \text{and} \quad V_{-m} \xrightarrow{x^m} V_m$$

are isomorphisms. Finally, in general,

$$\ker x \cap V_k = \ker \left(y^{k+1} : V_k \to V_{-k-2} \right).$$

We return now to our compact complex manifold M with Kähler metric

$$ds^2 = \sum_j \varphi_j \otimes \overline{\varphi}_j.$$

First, we want to compute the commutator $[L, \Lambda]$ on $\mathbb{C}^n$. Recall that

$$L = \frac{i}{2}\sum_k e_k\overline{e}_k \quad \text{and} \quad \Lambda = -\frac{i}{2}\sum_k \overline{i}_k i_k.$$

By our commutation relations

$$e_k i_\ell + i_\ell e_k = \overline{e}_k\overline{i}_\ell + \overline{i}_\ell\overline{e}_k = 2\delta_{k\ell} \quad \text{and} \quad [e_k, \overline{i}_\ell] = [\overline{e}_k, i_\ell] = 0,$$

we have then

$$\begin{aligned}
[L,\Lambda] &= \frac{1}{4}\left(\sum_{k,\ell} e_k\overline{e}_k\overline{i}_\ell i_\ell - \sum_{k,\ell}\overline{i}_\ell i_\ell e_k\overline{e}_k\right)\\
&= \frac{1}{4}\sum_{k,\ell}\left(2\delta_{k\ell}(e_k i_\ell - \overline{i}_\ell\overline{e}_k) - (e_k\overline{i}_\ell\overline{e}_k i_\ell - \overline{i}_\ell e_k i_\ell\overline{e}_k)\right)\\
&= \frac{1}{2}\sum_k (e_k i_k - \overline{i}_k\overline{e}_k) = \frac{1}{2}\sum_k (2 - i_k e_k - \overline{i}_k\overline{e}_k) = n - \frac{1}{2}\sum_k (i_k e_k + \overline{i}_k\overline{e}_k).
\end{aligned}$$

Note that $i_k e_k(dz_J \wedge d\overline{z}_K)$ is zero if $k \in J$, and $2\, dz_J \wedge d\overline{z}_K$ otherwise; $\overline{i}_k\overline{e}_k(dz_J \wedge d\overline{z}_K)$ is zero if $k \in K$, and $2\, dz_J \wedge d\overline{z}_K$ if not. Thus

$$\begin{aligned}
\sum_k (i_k e_k + \overline{i}_k\overline{e}_k)(dz_J \wedge dz_K) &= 2\sum_{k\notin J} dz_J \wedge d\overline{z}_K + 2\sum_{k\notin K} dz_J \wedge d\overline{z}_K\\
&= 2(2n - |J| - |K|)\, dz_J \wedge d\overline{z}_K.
\end{aligned}$$

and so on $A_c^{p,q}(\mathbb{C}^n)$, $[L,\Lambda] = p + q - n$. Since L and Λ are both algebraic operators, this identity will hold on any Kähler manifold.

Theorem (Hard Lefschetz Theorem)**.** The map $L^k : H^{n-k}(M) \to H^{n+k}(M)$ is an isomorphism, and if we define the primitive cohomology

$$P^r(M) = \begin{cases} \ker\left(L^{n-r+1}\right) = \ker\Lambda \cap H^r(M), & \text{if } r \le n,\\ 0, & \text{else,}\end{cases}$$

then we have

$$H^m(M) = \bigoplus_{k\ge 0} L^k P^{m-2k}(M),$$

called the Lefschetz decomposition.

Proof. Set

$$h = \sum_{r=0}^{2n}(n-r)\pi^r;$$

since $L : A^r(M) \to A^{r+2}(M)$ and $\Lambda : A^r(M) \to A^{r-2}(M)$, we obtain

$$(\heartsuit) \quad [\Lambda, L] = h, \ [h, L] = -2L, \ [h, \Lambda] = 2\Lambda.$$

The operators L, Λ, and h all commute with $\triangle_d$, and so act on the harmonic space $\mathcal{H}^*_d(M) \cong H^*(M)$ with relations $(\heartsuit)$. We may therefore give a representation of sl_2 on $H^*(M)$ by sending

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto \Lambda, \ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto L, \ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto h;$$

the eigenspace for h with eigenvalue $(n - r)$ will be $H^r(M)$. Applying the results on finite-dimensional representations of sl_2 to this representation we get the results. ■

Note that the Lefschetz decomposition is compatible with the Hodge decomposition, i.e., if we set $P^{p,q}(M) = \ker \Lambda \cap H^{p,q}(M)$, then

$$P^r(M) = \bigoplus_{p+q=r} P^{p,q}(M).$$

2 Divisors and Line Bundles

Let M be a compact complex manifold of dimension n. We know that the Picard group $\mathrm{Pic}(M) = H^1(M, \mathcal{O}^*)$ is the set of line bundles on M. The exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$$

gives a boundary map in cohomology

$$\mathrm{Pic}(M) = H^1(M, \mathcal{O}^*) \xrightarrow{\delta} H^2(M, \mathbb{Z}).$$

Definition. For a line bundle L, we define the first Chern class $c_1(L)$ of L (or simply Chern class) to be $\delta(L) \in H^2(M, \mathbb{Z})$; for D a divisor on M, we define the Chern class of D to be $c_1([D])$.

We sometimes write $c_1(L) \in H^2_{\mathrm{dR}}(M)$ for the image of $c_1(L)$ under the natural map $H^2(M, \mathbb{Z}) \to H^2_{\mathrm{dR}}(M)$.

Let $E \to M$ be a vector bundle of rank k and ∇ a connection on E, the curvature operator is ∇^2. Locally, if φ_α is trivialization of E over U_α, then ∇ is represented by $d + A$, where A is a $k \times k$ matrix of 1-forms, and ∇^2 is represented by $\Theta_\alpha = dA - A \wedge A$, which is a $k \times k$ matrix of 2-forms. If φ_β is another trivialization, we have

$$\Theta_\alpha = g_{\alpha\beta}\, \Theta_\beta\, g_{\alpha\beta}^{-1},$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_k$ is the transition function relative to φ_α and φ_β. In particular, if E is a line bundle, since $GL_1 = \mathbb{C}^\times$ is commutative, $\Theta = \Theta_\alpha = \Theta_\beta$ is a closed, globally defined differential form of degree 2, called the curvature form of E.

Definition. For any analytic subvariety $V \subseteq M$ of dimension k, we define the fundamental class $[V] \in H_{2k}(M, \mathbb{R})$ to be the linear functional

$$\begin{aligned} H^{2k}_{\mathrm{dR}}(M) &\longrightarrow \mathbb{R} \\ \varphi \quad &\mapsto \int_V \varphi \end{aligned}$$

we denote its Poincaré dual by η_V. In particular, we take the fundamental class of a divisor $D = \sum_j a_j V_j$ on M to be $\sum_j a_j [V_j]$, we denote its Poincaré dual by $\eta_D = \sum_j a_j \eta_{V_j}$.

Proposition. For any line bundle L with curvature form Θ,

1. $c_1(L) = \left[\frac{i}{2\pi}\Theta\right] \in H^2_{\mathrm{dR}}(M)$.
2. If $L = [D]$ for some $D \in \mathrm{Div}(M)$, $c_1(L) = \eta_D \in H^2_{\mathrm{dR}}(M)$.

Proof. Let φ_α be trivializations and let $g_{\alpha\beta}$ be transition functions relative to a cover $\mathcal{U} = \{U_\alpha\}$ of M. We may assume the open sets U_α are simply connected and set

$$h_{\alpha\beta} = \frac{1}{2\pi i}\log g_{\alpha\beta}.$$

By the definition of δ, if we set

$$z_{\alpha\beta\gamma} = h_{\beta\gamma} + h_{\gamma\alpha} + h_{\alpha\beta} = \frac{1}{2\pi i}(\log g_{\beta\gamma} + \log g_{\gamma\alpha} + \log g_{\alpha\beta}),$$

then $\{z_{\alpha\beta\gamma}\} \in Z^2(\mathcal{U}, \mathbb{Z})$ is a cocycle representing $c_1(L)$.

Now choose any connection ∇ on L. In terms of the frame $e_\alpha(z) = \varphi_\alpha^{-1}(z,1)$ on U_α, ∇ is given by its connection matrix, which in this case is a 1-form θ_α. In $U_\alpha \cap U_\beta$,

$$\theta_\alpha = g_{\alpha\beta}\theta_\beta g_{\alpha\beta}^{-1} + dg_{\alpha\beta}\, g_{\alpha\beta}^{-1},$$

i.e., $\theta_\beta - \theta_\alpha = -g_{\alpha\beta}^{-1}dg_{\alpha\beta} = -d(\log g_{\alpha\beta})$, and the curvature matrix is the global 2-form

$$\Theta = d\theta_\alpha - \theta_\alpha \wedge \theta_\alpha = d\theta_\alpha = d\theta_\beta.$$

Since Θ is given as a closed 2-form and $c_1(L)$ is given as a Čech cocycle, we must now look at the explicit form of the de Rham isomorphism. Let $\mathscr{Z}^r_d$ be the sheaf of d-closed r-forms, then we have exact sequences of sheaves

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathscr{C}^0 \longrightarrow \mathscr{Z}^1_d \longrightarrow 0,$$
$$0 \longrightarrow \mathscr{Z}^1_d \longrightarrow \mathscr{C}^1 \longrightarrow \mathscr{Z}^2_d \longrightarrow 0,$$

giving us boundary isomorphisms

$$H^1(\mathscr{Z}^1_d) \xrightarrow{\delta_0} H^2(\mathbb{C}) \quad \text{and} \quad \frac{H^0(\mathscr{Z}^2_d)}{dH^0(\mathscr{C}^1)} \xrightarrow{\delta_1} H^1(\mathscr{Z}^1_d).$$

Write Θ locally as $\{d\theta_\alpha\}$, we see from the definition of δ_0 that $\delta_0(\Theta) = \{\theta_\beta - \theta_\alpha\} \in Z^1(\mathscr{Z}^1_d)$. Now $\theta_\beta - \theta_\alpha = -d(\log g_{\alpha\beta})$, so

$$\delta_0(\delta_1(\Theta)) = \delta_0(\{\theta_\beta - \theta_\alpha\}) = \{-(\log g_{\beta\gamma} + \log g_{\gamma\alpha} + \log g_{\alpha\beta})\} = -2\pi i\, c_1(L).$$

To prove 2 we have to show that, for Θ a curvature matrix for the bundle $[D]$ and for every real, closed form $\psi \in A^{2n-2}(M)$,

$$\frac{i}{2\pi}\int_M \Theta \wedge \psi = \sum_j a_j \int_{V_j} \psi.$$

We may assume that $D = V$ is an irreducible subvariety.

First, we compute the curvature form of a metric connection on $[V]$. To do this, let e be a local nowhere vanishing holomorphic section of $[V]$ and write $|e(z)|^2 = h(z)$. Then for any section $s = \lambda \cdot e$, the connection matrix $A = \theta$ for the metric connection ∇ in terms of the frame e must satisfy $\theta = \theta^{1,0}$ and

$$\begin{aligned} d(|s|^2) &= \langle \nabla s, s\rangle + \langle s, \nabla s\rangle \\ &= \langle (d\lambda + \theta\lambda)e, \lambda e\rangle + \langle \lambda e, (d\lambda + \theta\lambda)e\rangle \\ &= h \cdot \overline{\lambda} \cdot d\lambda + h \cdot \lambda \cdot d\overline{\lambda} + h \cdot |\lambda|^2(\theta + \overline{\theta}). \end{aligned}$$

On the other hand,

$$d(|s|^2) = d(\lambda \cdot \overline{\lambda} \cdot h) = h \cdot \overline{\lambda} \cdot d\lambda + h \cdot \lambda \cdot d\overline{\lambda} + |\lambda|^2 \cdot dh.$$

So we have $\theta + \overline{\theta} = dh/h$, i.e., $\theta = \partial \log h = \partial \log |e|^2$, and

$$\Theta = d\theta - \theta \wedge \theta = d\theta = \overline{\partial}\partial \log |e|^2 = 2\pi i \, dd^c \log |e|^2.$$

Note that this holds for any nowhere vanishing holomorphic section e.

Now let V be given by local equation f_α and let s be a global section $\{f_\alpha\}$ of $[V]$ vanishing exactly on V. Set

$$D(\varepsilon) = \{z \in M \mid |s(z)| < \varepsilon\} \subseteq M.$$

For small ε, $D(\varepsilon)$ is just a tubular neighborhood around V in M, and

$$\int_M \Theta \wedge \psi = 2\pi i \lim_{\varepsilon \to 0} \int_{M - D(\varepsilon)} dd^c \log |s|^2 \wedge \psi = -2\pi i \lim_{\varepsilon \to 0} \int_{\partial D(\varepsilon)} d^c \log |s|^2 \wedge \psi$$

by Stokes' theorem and $d\psi = 0$. In $U_\alpha \cap D(\varepsilon)$, write

$$|s|^2 = |f_\alpha|^2 \cdot h_\alpha = f_\alpha \cdot \overline{f}_\alpha \cdot h_\alpha$$

with $h_\alpha > 0$; we have

$$d^c \log |s|^2 = d^c \log(f_\alpha \cdot \overline{f}_\alpha \cdot h_\alpha) = \frac{i}{4\pi}\left(\overline{\partial} \log \overline{f}_\alpha - \partial \log f_\alpha\right) + d^c \log h_\alpha.$$

Since $d^c \log h_\alpha = d^c h_\alpha / h_\alpha$ is bounded and $\mathrm{Vol}(\partial D(\varepsilon)) \to 0$ as $\varepsilon \to 0$, we deduce that

$$\lim_{\varepsilon\to 0} \int_{\partial D(\varepsilon)} d^c \log h_\alpha \wedge \psi = 0.$$

Moreover, $\overline{\partial} \log \overline{f}_\alpha = \overline{\partial \log f_\alpha}$ and, since ψ is real, this implies

$$\int_{\partial D(\varepsilon)} \overline{\partial} \log \overline{f}_\alpha \wedge \psi = \overline{\int_{\partial D(\varepsilon)} \partial \log f_\alpha \wedge \psi}.$$

Thus in U_α,

$$-2\pi i \lim_{\varepsilon\to 0} \int_{\partial D(\varepsilon)} d^c \log |s|^2 \wedge \psi = -i \lim_{\varepsilon\to 0} \mathrm{Im} \left(\int_{\partial D(\varepsilon)} \partial \log f_\alpha \wedge \psi \right).$$

Now in the neighborhood of any smooth point $z_0 \in V \cap U_\alpha$, we can find a holomorphic coordinate system $w = (w_1, \ldots, w_n)$ with $w_1 = f_\alpha$. Write $\psi = \psi(w) dw' \wedge d\overline{w}' + \varphi$, where $w' = (w_2, \ldots, w_n)$ and all terms of φ contain either dw_1 or $d\overline{w}_1$; then in any polydisc Δ around z_0,

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_{\partial D(\varepsilon)\cap\Delta} \partial \log f_\alpha \wedge \psi &= \lim_{\varepsilon\to 0} \int_{|w_1|=\varepsilon} \frac{dw_1}{w_1} \cdot \psi(w)\, dw' \wedge d\overline{w}' \\ &= 2\pi i \int_{w'} \psi(0, w')\, dw' \wedge d\overline{w}' \\ &= 2\pi i \int_{V\cap\Delta} \psi,\end{aligned}$$

and so

$$\int_M \Theta \wedge \psi = -i\ \mathrm{Im} \left(2\pi i \int_V \psi \right) = -2\pi i \int_V \psi. \qquad \blacksquare$$

Examples.

1. Let M be a compact complex manifold, $V \subset M$ a smooth analytic hypersurface. The normal bundle N_V on V is the quotient line bundle

$$N_V = \frac{T'_M|_V}{T'_V}.$$

We define the conormal bundle $N_V^* \subseteq {T_M^*}'\big|_V$ to be the dual of N_V.

Adjunction Formula I

$$N_V^* = [-V]|_V.$$

Proof. Suppose V is given locally by functions $f_\alpha \in \mathcal{O}(U_\alpha)$; the line bundle $[V]$ on M is then given by transition functions $\{g_{\alpha\beta} = f_\alpha / f_\beta\}$. Now since $f_\alpha = 0$ on $V \cap U_\alpha$, the differential df_α is a section of the conormal bundle N_V^* of V; since V is smooth, df_α is nowhere vanishing on V. On $U_\alpha \cap U_\beta \cap V$, moreover, we have

$$df_\alpha = d(g_{\alpha\beta} f_\beta) = dg_{\alpha\beta} \cdot f_\beta + g_{\alpha\beta} \cdot df_\beta = g_{\alpha\beta} \cdot df_\beta,$$

i.e., the sections $df_\alpha \in \Gamma(U_\alpha, \mathcal{O}(N_V^*))$ together give a nowhere vanishing global section of $N_V^* \otimes [V]|_V$. Thus $N_V^* \otimes [V]|_V$ is the trivial line bundle, as desired. ■

One of the most important line bundles for a general n-dimensional complex manifold M is the canonical bundle

$$K_M = \bigwedge^n {T_M^*}'.$$

Holomorphic sections of K_M are holomorphic n-forms, i.e. $\mathcal{O}(K_M) = \Omega_M^n$. In general, we can compute the canonical bundle K_V of a smooth analytic hypersurface V in a manifold M in terms of K_M.

Adjunction Formula II

$$K_V = (K_M \otimes [V])|_V \,.$$

Proof. We have an exact sequence of vector bundles on V

$$0 \longrightarrow N_V^* \longrightarrow {T_M^*}'\big|_V \longrightarrow {T_V^*}' \longrightarrow 0.$$

By simple linear algebra,

$$\left(\bigwedge^n {T_M^*}'\right)\Big|_V \cong \bigwedge^{n-1} {T_V^*}' \otimes N_V^*,$$

i.e., $K_V = K_M|_V \otimes N_V$. Then the formula follows from the adjunction formula I above. ■

We can give the corresponding map on sections

$$\Omega_M^n(V) \xrightarrow{\text{P.R.}} \Omega_V^{n-1}$$

as follows: Considering a section ω of $\Omega_M^n(V)$ as a meromorphic n-form with a simple pole along V and holomorphic elsewhere, we write

$$\omega = \frac{g(z)\, dz_1 \wedge \cdots \wedge dz_n}{f(z)},$$

where $z = (z_1, \ldots, z_n)$ are local coordinates on M and V is given locally by $f(z)$. Under the isomorphism, then, ω corresponds to the form ω' such that $\omega = (df/f) \wedge \omega'$. Explicitly,

$df = \sum_j (\partial f/\partial z_j)\, dz_j$, and so we can take

$$\omega' = (-1)^{j-1} \frac{g(z)\, dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n}{\partial f/\partial z_j}$$

for any j such that $\partial f/\partial z_j \neq 0$. The map

$$\frac{g(z)\, dz_1 \wedge \cdots \wedge dz_n}{f(z)} \mapsto (-1)^{j-1} \left. \frac{g(z)\, dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n}{\partial f/\partial z_j} \right|_{f=0}$$

is called the Poincaré residue map, denoted P.R.

Note that the kernel of the Poincaré residue map consists simply of the holomorphic n-forms on M. The exact sheaf sequence

$$0 \longrightarrow \Omega_M^n \longrightarrow \Omega_M^n(V) \xrightarrow{\text{P.R.}} \Omega_V^{n-1} \longrightarrow 0$$

gives us the exact sequence

$$H^0(M, \Omega_M^n(V)) \xrightarrow{\text{P.R.}} H^0(V, \Omega_V^{n-1}) \xrightarrow{\delta} H^1(M, \Omega_M^n),$$

so the Poincaré residue map is surjective on global sections if

$$H^1(M, \Omega_M^n) = H^{n,1}(M) = 0.$$

2. By the exact cohomology sequence

$$0 = H^1(\mathbb{P}^n, \mathcal{O}) \xrightarrow{\exp} H^1(\mathbb{P}^n, \mathcal{O}^*) \xrightarrow{c_1} H^2(\mathbb{P}^n, \mathbb{Z}) \longrightarrow H^2(\mathbb{P}^n, \mathcal{O}) = 0,$$

we see that

$$\operatorname{Pic}(\mathbb{P}^n) \cong H^2(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}.$$

In other words, every divisor on $\mathbb{P}^n$ is linearly equivalent to a multiple of the hyperplane divisor $H = \mathbb{P}^{n-1} \subset \mathbb{P}^n$. The line bundle $[H]$ associated to a hyperplane in $\mathbb{P}^n$ is called the hyperplane bundle.

Let $\mathbb{P}^n \times \mathbb{C}^{n+1}$ be the trivial bundle of rank $n+1$ on $\mathbb{P}^n$, with all fibers identified to $\mathbb{C}^{n+1}$. We define the tautological line bundle to be the subbundle J of $\mathbb{P}^n \times \mathbb{C}^{n+1}$ whose fiber at each point $Z \in \mathbb{P}^n$ is the line in $\mathbb{C}^{n+1}$ represented by Z, i.e.,

$$J_Z = \{\lambda(Z_0, \ldots, Z_n) \mid \lambda \in \mathbb{C}\} \subseteq (\mathbb{P}^n \times \mathbb{C}^{n+1})_Z.$$

In fact, $J = [-H]$. To see this, consider the section e_0 of J over $U_0 = D(Z_0) \subset \mathbb{P}^n$ given by

$$e_0(Z) = \left(1, \frac{Z_1}{Z_0}, \ldots, \frac{Z_n}{Z_0}\right).$$

e_0 is clearly holomorphic and nonzero in U_0 and extends to a global meromorphic section of J with a pole of order 1 along the hyperplane $V(Z_0) \subset \mathbb{P}^n$. Thus $J = [(e_0)] = [-H]$.

If $M \subseteq \mathbb{P}^n$ is a submanifold of projective space, we usually call the restriction $[H]|_M$ simply the hyperplane bundle on M; by functoriality, it is the line bundle associated to a generic hyperplane section $\mathbb{P}^{n-1} \cap M$ of M.

3. We compute the canonical bundle of $\mathbb{P}^n$: let $Z_0, \ldots, Z_n$ be homogeneous coordinates on $\mathbb{P}^n$, $w_i = Z_i/Z_0$ Euclidean coordinates on $U_0 = D(Z_0)$, and consider the meromorphic n-form

$$\omega = \frac{dw_1}{w_1} \wedge \frac{dw_2}{w_2} \wedge \cdots \wedge \frac{dw_n}{w_n}.$$

ω is clearly nonzero in U_0 with a single pole along each hyperplane $V(Z_i)$, $i = 1, \ldots, n$. Now if $w'_i = Z_i/Z_j$, $i = 0, \ldots, \widehat{j}, \ldots, n$ are Euclidean coordinates on $U_j = D(Z_j)$, then

$$w_i = \frac{w'_i}{w'_0},\ i \neq j; \quad w_j = \frac{1}{w'_0},$$

which gives

$$\frac{dw_i}{w_i} = \frac{dw'_i}{w'_i} - \frac{dw'_0}{w'_0},\ i \neq j; \quad \frac{dw_j}{w_j} = -\frac{dw'_0}{w'_0}$$

and so in terms of w'_i,

$$\omega = (-1)^j \frac{dw'_0}{w'_0} \wedge \cdots \wedge \widehat{\frac{dw'_j}{w'_j}} \wedge \cdots \wedge \frac{dw'_n}{w'_n}.$$

Thus we see that ω has likewise a single pole along the hyperplane $D(Z_0)$, and consequently

$$K_{\mathbb{P}^n} = [(\omega)] = [-(n+1)H].$$