Theorem on Formal Function

In this talk, we will introduce the definition of formal scheme and some essential propositions to help us go through the proof of theorem on formal function. This part is follow Section II.9 in Hartshorne's Algebraic Geometry.

Inverse Systems & Inverse Limits

- *Inverse system of abelian group* $(A_n, \varphi_{n'n})$ is a collection of abelian groups $\{A_n\}$ together with homorphism $\varphi_{n'n} : A_{n'} \to A_n$ for each $n' \geq n$ such that for all $n'' \geq n' \geq n$, we have $\varphi_{n''n} = \varphi_{n'n} \circ \varphi_{n''n}$.

$$\begin{array}{ccccc} A_{n''} & & \xrightarrow{\varphi_{n''n}} & & A_n \\ & \searrow^{\varphi_{n''n}} & & \nearrow_{\varphi_{n'n}} & \\ & & A_{n'} & & \end{array}$$

- *Inverse limit* $A = \varprojlim A_n := \{\{a_n\} \in \prod A_n \mid \varphi_{n'n}(a_{n'}) = a_n, \forall n' \geq n\}$.

- *Universal property*: Given group B and homomorphisms $\psi_n : B \to A_n, \forall n$ such that $\forall n' \geq n, \psi_n = \varphi_{n'n} \circ \psi_{n'}$, then $\exists! \psi : B \to A$ such that $\psi_n = p_n \circ \psi, \forall n$, where $p_n : A \to A_n$ is nth projection map $\prod A_k \to A_n$.

$$\begin{array}{ccccc} B & & \xrightarrow{\psi_n} & & A_n \\ & \searrow^{\psi_{n'}} & & \nearrow_{\varphi_{n'n}} & \\ & & A_{n'} & & \end{array} \quad \rightsquigarrow \quad \begin{array}{ccccc} B & & \xrightarrow{\psi} & & A \\ & \searrow^{\psi_n} & & \swarrow_{p_n} & \\ & & A_n & & \end{array}$$

- *Homomorphism of inverse system*: $(A_n) \to (B_n)$ is a collection of group homomorphism $f_n : A_n \to B_n$ such that $\forall n' \geq n$, the following diagram commute:

$$\begin{array}{ccc} A_{n'} & \xrightarrow{f_{n'}} & B_{n'} \\ {\scriptstyle \varphi_{n'n}} \downarrow & & \downarrow {\scriptstyle \psi_{n'n}} \\ A_n & \xrightarrow{f_n} & B_n \end{array}$$

- *Exactness*: $0 \to (A_n) \to (B_n) \to (C_n) \to 0$ is said to be exact if for each n, corresponding sequence of group is exact.

Proposition. *If*

$$0 \to (A_n) \to (B_n) \to (C_n) \to 0$$

is exact, then so is

$$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n.$$

□

9 Formal Scheme

Definition. We say an inverse system (A_n) satisfies *Mittag-Leffler condition* (ML), if for each n, the decreasing family $\{\varphi_{n'n}(A_{n'}) \subseteq A_n \mid n' \geq n\}$ of subgroups of A_n is stationary.

If (A_n) satisfies (ML), then for each n, we can define $A'_n \subseteq A_n$ to be the *stable image* $\varphi_{n'n}(A_n)$ for n' large enough. Then all maps of (A'_n) is surjective, and $\varprojlim A'_n = \varprojlim A_n$.

Proposition 9.1. *Let* $0 \longrightarrow (A_n) \xrightarrow{f} (B_n) \xrightarrow{g} (C_n) \longrightarrow 0$ *be exact sequence of inverse system of abelian groups.*

(a) if (B_n) *satisfies (ML), then so does* (C_n).

(b) if (A_n) *satisfies (ML), then* $0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to 0$ *is exact.*

Proof. (a) For each $n \geq n'$, the image of $B_{n'}$ in B_n maps surjectively to the image of $c_{n'}$ in c_n, so (ML) for (B_n) implies (ML) for (C_n) immediately.

(b) Just need to show

$$\varprojlim B_n \to \varprojlim C_n$$

is surjective. So fix $\{c_n\} \in \varprojlim C_n$. For each k, let $E_k := g^{-1}(c_k) \subseteq B_n$. Then (E_n) form an inverse system of sets. Also, since for each k, we have $0 \longrightarrow A_k \xrightarrow{f_k} B_k \xrightarrow{g_k} C_k \longrightarrow 0$ is exact, so

$$E_k = b_k \operatorname{Ker} g_k = b_k \operatorname{Im} f_k,$$

where b_k is an element in B_k such that $g_k(b_k) = c_k$. (Such b_k must exists since B_k maps surjective to C_k.) Hence, E_k is bijective to A_k. So (A_n) satisfies (ML) implies (E_n) satisfies (ML).

Now, consider the stable image of (E_n), we find that $\varprojlim E_n$ is nonempty, say $\{e_n\} \varprojlim E_n$, then $\{e_n\}$ is an element in $\varprojlim B_n$ which maps to $\{c_n\}$.

$\square$

Proposition 9.2. *X: topological space, $\mathfrak{C}$: category of sheaves of abelian groups on X. $(\mathscr{F}_n)$: inverse system of sheaves on X. Then $\mathscr{F} := \varprojlim \mathscr{F}_n$ exists in $\mathfrak{C}$ and for any U: open subset of X, $\Gamma(U, \mathscr{F}) = \varprojlim \Gamma(U, \mathscr{F}_n)$.*

Proof. Consider pre-sheaf

$$U \mapsto \varprojlim \Gamma(U, \mathscr{F}_n).$$

One can check this is a sheaf, denote it by $\mathscr{F}$. Now, we are going to check that $\mathscr{F}$ is satisfies the universal property. If there is any other sheaf $\mathscr{G}$ and compatible maps $\psi_n : \mathscr{G} \to \mathscr{F}_n$, $\forall n$. Then on each open set U, universal property of inverse limit of abelian groups ($\varprojlim \Gamma(U, \mathscr{F}_n)$) gives us an unique maps

$$\Gamma(U, \mathscr{G}) \to \Gamma(U, \mathscr{F}).$$

This gives a sheaf map $\mathscr{G} \to \mathscr{F}$. Thus, $\mathscr{F}$ is what we want. $\square$

Remark. We should notice that the statement of 9.1(b) may false in $\mathfrak{C}$. See counterexample in [Amnon Neeman, A counterexample to a 1961 "theorem" in homological algebra].

Definition. A: commutative ring with identity, and $I \lhd A$. Then (A/I^n) is an inverse system. We defined *I-adic completion of A* to be

$$\hat{A} := \varprojlim A/I^n.$$

By universal property, we get a homomorphism $A \to \hat{A}$. Also, for a A-module M,

$$\hat{M} := \varprojlim M/I^n M$$

is I-adic completion of M, and it is a $\hat{A}$-module.

Theorem 9.3. *A: noetherian ring, and $I \lhd A$. Let $\hat{}$ be I-adic completion. Then*

(a) $\hat{I} = \varprojlim I/I^n \lhd A$, and for each n, $\hat{I}^n = I^n \hat{A}$, $\hat{A}/\hat{I}^n \cong A/I^n$.

(b) M: finitely generated A-module. Then $\hat{M} \cong M \otimes_A \hat{A}$.

(c) $M \mapsto \hat{M}$ is an exact functor on the category of finitely generated A-modules.

(d) $\hat{A}$ is neotherian.

Proof.

(a) [Atiyah-Macdonald, Proposition 10.15]

(b) [Atiyah-Macdonald, Proposition 10.13]

(c) [Atiyah-Macdonald, Proposition 10.14]

(d) [Atiyah-Macdonald, Proposition 10.26]

□

Definition. X: neotherian scheme, Y: closed subscheme of X defined by a sheaf of ideals $\mathscr{I}$. Then we defined *formal completion of X along Y*, denote by $(\hat{X}, \mathscr{O}_{\hat{X}})$ is a topological space Y with sheaf of rings $\mathscr{O}_{\hat{X}} := \varprojlim \mathscr{O}_X/\mathscr{I}^n$.

Definition.

- A *noetherian formal scheme* $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ has finite open cover $\{\mathfrak{U}_i\}$ such that for each i, $(\mathfrak{U}_i, \mathscr{O}_{\mathfrak{X}}|_{\mathfrak{U}_i}) \cong$ completion of some noetherian scheme X_i along Y_i.
- A sheaf $\mathfrak{F}$ of $\mathscr{O}_{\mathfrak{X}}$- module is said to be *coherent* if there exists finite open cover $\{\mathfrak{U}_i\}$ such that $\mathfrak{U}_i \cong \hat{X}_i$ and $\exists \mathscr{F}_i$ on X_i such that $\mathfrak{F}|_{\mathfrak{U}_i} \cong \hat{\mathscr{F}}_i$ via given isomorphism $\mathfrak{U}_i \cong \hat{X}_i$.

Remark. $Y = \{P\}$. A $\hat{\mathscr{O}}_P$- module M is coherent on X if and only if M is finitely generated module.

Proof. (⇒) done!

(⇐) Obtain $\hat{X}$ by completing the scheme Spec $\hat{\mathscr{O}}_P$ at its closed point, and any finitely generated $\hat{\mathscr{O}}_P$-module M is correspond to a coherent sheaf on Spec $\hat{\mathscr{O}}_P$.

□

Next, we will give some motivation of theorem on formal function and go through the proof. After that, we are going to see how can we apply the theorem on formal function to prove the Zariski's main theorem and the Stein factorization. This part is follow Section III.11 in Hartshorne's Algebraic Geometry.

11 The Theorem on Formal Function

Now, Let's recall the following fact:

Fact 1 (Proposition III.9.3 & Remark 9.3.1 in Hartshorne's book)

Let $f : X \to Y$ is separated morphism of finite type of noetherian scheme. Let $\mathscr{F}$: quasi-coherent sheaf on X, and $u : Y' \to Y$ be morphism of noetherian schemes.

$$\begin{CD} X' @>v>> X \\ @VgVV @VVfV \\ Y' @>u>> Y \end{CD}$$

Then $\forall i \geq 0$, we have a map

$$u^* R^i f_*(\mathscr{F}) \to R^i g_*(v^* \mathscr{F}).$$

Moreover, if u is flat, then this map gives an isomorphism $u^* R^i f_*(\mathscr{F}) \cong R^i g_*(v^* \mathscr{F})$.

Now, consider $f : X \to Y$ be a projective morphism of noetherian schemes, $\mathscr{F}$: coherent sheaf on X. Fix $y \in Y$. For each $X_n = X \times_Y \operatorname{Spec} \mathscr{O}_y/\mathfrak{m}_y^n$. Then we have the following diagram:

$$\begin{CD} X_n @>v>> X \\ @Vf'VV @VVfV \\ \operatorname{Spec} \mathscr{O}_y/\mathfrak{m}_y^n @>>> Y \end{CD}$$

Let $\mathscr{F}_n := v^* \mathscr{F}$. By Fact 1, for each n, we have

$$R^i f_*(\mathscr{F}) \otimes \mathscr{O}_y/\mathfrak{m}_y^n \to R^i f'_*(\mathscr{F}_n).$$

Since $\operatorname{Spec} \mathscr{O}_y/\mathfrak{m}_y^n$ has only one point, concentrated at this point, then the right hand side is just $H^i(X_n, \mathscr{F}_n)$, and the left hand side is $R^i f_*(\mathscr{F})/\mathfrak{m}_y^n R^i f_*(\mathscr{F})$. (Since $M \otimes A/\mathfrak{m}^n = M/\mathfrak{m}^n M$.)

Notice that as n varies, both side form inverse systems, so we can take inverse limits and get

$$R^i f_*(\mathscr{F})_y^{\widehat{}} \to \varprojlim H^i(X_n, \mathscr{F}_n).$$

So we are going to introduce the following theorem:

Theorem 11.1 (Theorem on Formal Functions)**.** *Let $f : X \to Y$ be a projective morphism of noetherian schemes, $\mathscr{F}$: coherent sheaf on X. Let $y \in Y$. Then the natural map:*

$$R^i f_*(\mathscr{F})^{\widehat{}}_y \to \varprojlim H^i(X_n, \mathscr{F}_n).$$

is an isomorphism, $\forall i \geq 0$.

Proof.

1. Embed $X \hookrightarrow \mathbb{P}^N_Y$, and consider $\mathscr{F}$ as coherent sheaf on $\mathbb{P}^N_Y$. So we reduce to the case $X = \mathbb{P}^N_Y$.

2. Reduce to the case that Y is affine and restate the result as A-module:
Let $A = \mathscr{O}_y$. Make flat base extension $\operatorname{Spec} A \to Y$. Again by the Fact 1, we reduce the case to that Y is affine. In fact, we may assume $Y = \operatorname{Spec} A$ with A is noetherian local ring and $y \in Y$ is a closed point. So we have

$$R^i f_*(\mathscr{F})^{\widehat{}} = H^i(X, \mathscr{F})^{\widehat{}}.$$

Thus, we just need to show

$$H^i(X, \mathscr{F})^{\widehat{}} \xrightarrow{\sim} \varprojlim H^i(X_n, \mathscr{F}_n)$$

is an isomorphism as A-module.

3. Deal with the case $\mathscr{F} = \oplus_{i=1}^r \mathscr{O}(q_i)$:
Suppose $\mathscr{F} = \mathscr{O}(q)$ on $X = \mathbb{P}^N_Y$ for some $q \in \mathbb{Z}$. Then $\mathscr{F}_n = \mathscr{O}(q)$ on $X_n = \mathbb{P}^N_{A/\mathfrak{m}^n}$. So for each n, we have

$$H^i(X_n, \mathscr{F}_n) = H^i(\mathbb{P}^N_{A/\mathfrak{m}^n}, \mathscr{O}(q)) \cong H^i(\mathbb{P}^N_A, \mathscr{O}(q)) \otimes_A A/\mathfrak{m}^n = H^i(X, \mathscr{F}) \otimes_A A/\mathfrak{m}^n.$$

Then take inverse limit on both sides to get the desired result, and thus, results holds for finite direct sum of $\mathscr{O}(q_i)$.

4. For arbitrary coherent sheaf $\mathscr{F}$ on X:
By descending induction on i. Notice that X can be cover by $N + 1$ affine open sets, using this open cover to compute the Čech cohomology, and we will find that there is no $C^i(\mathfrak{U}, \mathscr{F})$, whenever $i > N$. Thus, if $i > N$, then both sides are 0. So we may assume the theorem holds for $i + 1$, and for all coherent sheaf. For any

coherent sheaf $\mathscr{F}$, we can write $\mathscr{F}$ as a quotient of $\mathscr{E} = \oplus_{i=1}^{r} \mathscr{O}(q_i)$. Let $\mathscr{R}$ be the kernel, then

$$0 \to \mathscr{R} \to \mathscr{E} \to \mathscr{F} \to 0 \qquad (\heartsuit)$$

is exact. Unfortunately, tensoring $\mathscr{O}_{X_n}$ is not exact. (Since $\mathscr{O}_y/\mathfrak{m}_y^n$ is not a free $\mathscr{O}_Y$-module.) So for each n, we only have

$$\mathscr{R}_n \to \mathscr{E}_n \to \mathscr{F}_n \to 0$$

is exact. Now, let $\mathscr{T}_n := \operatorname{Im}(\mathscr{R}_n \to \mathscr{E}_n)$ and $\mathscr{S}_n := \operatorname{Ker}(\mathscr{R}_n \to \mathscr{E}_n)$. Then we get:

$$0 \to \mathscr{S}_n \to \mathscr{R}_n \to \mathscr{T}_n \to 0 \qquad (\spadesuit)$$

and

$$0 \to \mathscr{T}_n \to \mathscr{E}_n \to \mathscr{F}_n \to 0 \qquad (\clubsuit)$$

Now, consider the following diagram:

$$\begin{array}{ccccccccc}
H^i(X,\mathscr{R})\hat{} & \longrightarrow & H^i(X,\mathscr{E})\hat{} & \longrightarrow & H^i(X,\mathscr{F})\hat{} & \longrightarrow & H^{i+1}(X,\mathscr{R})\hat{} & \longrightarrow & H^{i+1}(X,\mathscr{E})\hat{} \\
\downarrow{\scriptstyle \alpha_1} & & & & & & \downarrow{\scriptstyle \alpha_4}\,\cong & & \\
\varprojlim H^i(X_n,\mathscr{R}_n) & & \downarrow{\scriptstyle \alpha_2}\,\cong & & \downarrow{\scriptstyle \alpha_3} & & \varprojlim H^{i+1}(X_n,\mathscr{R}_n) & & \downarrow{\scriptstyle \alpha_5}\,\cong \\
\downarrow{\scriptstyle \beta_1} & & & & & & \downarrow{\scriptstyle \beta_2} & & \\
\varprojlim H^i(X_n,\mathscr{T}_n) & \to & \varprojlim H^i(X_n,\mathscr{E}_n) & \to & \varprojlim H^i(X_n,\mathscr{F}_n) & \to & \varprojlim H^{i+1}(X_n,\mathscr{T}_n) & \to & \varprojlim H^{i+1}(X_n,\mathscr{E}_n)
\end{array}$$

The top row comes from the cohomology sequence of $(\heartsuit)$ by completion. Since they are all finitely generated A-modules, completion is an exact functor. The bottom row comes from the cohomology sequence of $(\clubsuit)$ by taking inverse limits. These groups are all finitely generated $A/\mathfrak{m}^n$-modules, and so satisfy d.c.c. for submodules. Therefore the inverse systems all satisfy the (ML), and so the bottom row is exact. The vertical arrows $a_1, \ldots, a_5$ are the maps of the theorem. Finally, β_1 and β_2 are maps induced from the sequence $(\clubsuit)$.

By 3, α_2 and α_5 are isomorphisms. By induction hypothesis, α_4 is isomorphism.
Claim β_1, β_2 are isomorphisms.

(pf of claim) Take cohomology of $(\spadesuit)$, and again, since they are all finitely generated A-module, we can passing inverse limit and preserve the exactness. So it suffices to show that for each $i \geq 0$,

$$\varprojlim H^i(X_n, \mathscr{S}_n) = 0.$$

To show this, we just need to show that $\forall n, \exists n' > n$ such that $\mathscr{S}_{n'} \to \mathscr{S}_n$ is zero map. Notice that X is quasi-compact, so we can check it locally. We may assume $X = \operatorname{Spec} B$. Denote R, E, S_n be the B-module corresponding to $\mathscr{R}, \mathscr{E}, \mathscr{S}_n$. Let $\mathfrak{a} := \mathfrak{m}B$. Notice that R is a submodule of E, and

$$S_n = \operatorname{Ker}(R/\mathfrak{a}^n R \to E/\mathfrak{a}^n E).$$

So

$$S_n = (R \cap \mathfrak{a}^n E)/\mathfrak{a}^n R.$$

By Krull's theorem, $\forall n, \exists n' > n$ such that $R \cap \mathfrak{a}^{n'} \subseteq \mathfrak{a}^n R$, i.e. $S_{n'} \to S_n$ is zero. This prove the claim.

Now, by 5-lemma, α_3 is surjective. Since it will be true for any coherent sheaf, the map in theorem is surjective. This implies α_1 is also surjective. So by 5-lemma again, α_3 is injective. Thus, α_3 is isomorphism. This prove the theorem.

□

Remark. This is also true for the case f is proper.

Remark. $H^i(\hat{X}, \hat{\mathscr{F}})$ will equal to two quantities in the theorem.

Corollary 11.2. *Let $f : X \to Y$ be a projective morphism of neotherian schemes, $r = \max\{\dim X_y \mid y \in Y\}$. Then $R^i f_*(\mathscr{F}) = 0$, $\forall i > r$, $\forall \mathscr{F}$: coherent sheaf on X.*

Proof. For any $y \in Y$, $\operatorname{tp}(X_n) = \operatorname{tp}(X_y)$. By Grothendieck vanshing, $H^i(X_n, \mathscr{F}_n) = 0$, $\forall i > r$. By Theorem 11.1, $\widehat{R^i f_*(\mathscr{F})_y} = 0$, $\forall y \in Y$, $\forall i > r$. Also, since $R^i f_*(\mathscr{F})$ is coherent, so $R^i f_*(\mathscr{F})$ is zero at all stalk. (Since for M: finitely generated A-module, A: noetherian local ring, $\hat{M} = M \otimes_A \hat{A}$, and notice $\hat{A}$ is faithfully flat A-module.) □

Remark (Exercise III.11.1 in Hartshorne's book). Corollary 11.2 is false without the projective hypothesis. Let $X = \mathbb{A}^n_k$, $P = (0, \ldots 0)$, $U = X - P$, and $f : U \hookrightarrow X$ be the inclusion. Notice that the fibres of f all have dimension 0. (So $r = 0$.) But we can show that $R^{n-1} f_* \mathscr{O}_U \neq 0$. Since $R^{n-1} f_* \mathscr{O}_U$ is associated to $V \mapsto H^{n-1}(f^{-1}(V), \mathscr{O}_U|_{f^{-1}(V)})$, we just compute the $n-1$th Čech cohomology to show it is not zero. Take $U_i = \operatorname{Spec} k[x_1, \ldots, x_n, x_i^{-1}]$ be an open cover of U. Then the Čech complex will be

$$\cdots \longrightarrow \bigoplus_{i=1}^{n} k[x_1, \dots, x_n, x_1^{-1}, \dots, \hat{x_i}^{-1}, \dots x_n^{-1}] \longrightarrow k[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}] \longrightarrow 0$$

$$\bigoplus_{i=1}^{n} f_i \longmapsto \sum_{i=1}^{n} (-1)^{i-1} f_i$$

Thus, to show $n-1$th cohomology is not zero, just need to show the map above is not surjective. But $x_1^{-1}x_2^{-1}\dots x_n^{-1}$ is clearly not in the image. Hence, $R^{n-1}f_*\mathscr{O}_U \neq 0$.

Corollary 11.3. *Let $f : X \to Y$ be a projective morphism of neotherian schemes. Assume $f_*\mathscr{O}_X = \mathscr{O}_Y$. Then $f^{-1}(y)$ is connected, for every $y \in Y$.*

Proof. Suppose $f^y = X' \cup X''$, where X' and X'' are disjoint closed subsets. For each n, we have

$$H^0(X_n, \mathscr{O}_{X_n}) = H^0(X_n', \mathscr{O}_{X_n}) \oplus H^0(X_n'', \mathscr{O}_{X_n}).$$

Also,

$$\hat{\mathscr{O}}_y = (\mathscr{O}_Y)_y^{\widehat{}} = (f_*\mathscr{O}_X)_y^{\widehat{}} = \varprojlim H^0(X_n, \mathscr{O}_{X_n}) = \varprojlim H^0(X_n', \mathscr{O}_{X_n}) \oplus \varprojlim H^0(X_n'', \mathscr{O}_{X_n}).$$

But a local ring can not be a direct sum of two other rings. □

Fact 2 A local ring can not be a direct sum of teo other rings.

Proof. Let e', e'' be identity of A', A'' respectively. Then $e' + e'' = 1$ in $\hat{\mathscr{O}}_y = A' \oplus A''$ and $e'e'' = 0$. If one of e', e'' is unit, then $e' = 0$ or $e'' = 0$, so both e', e'' are non-unit. This implies $e', e'' \in \mathfrak{m}$. Thus $e' + e''$ can not be 1. $\rightarrow\leftarrow$ □

Corollary 11.4 (Zariski's Main Theorem)**.** *Let $f : X \to Y$ be a birational projective morphism of noetherian integral schemes. Assume Y is normal. Then $\forall y \in Y$, $f^{-1}(y)$ is connected.*

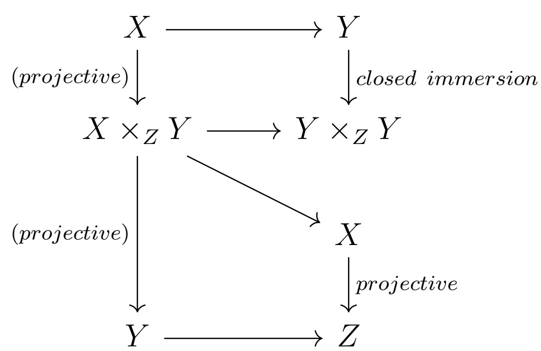
Proof. Only need to check $f_*\mathscr{O}_X = \mathscr{O}_Y$. This can be check locally on Y, so we assume that $Y = \operatorname{Spec} A$. Then $f_*\mathscr{O}_X$ is coherent $\mathscr{O}_Y$-algebra, so $B := \Gamma(Y, f_*\mathscr{O}_X)$ is finitely generated A-module. But A and B are integral domain with the same quotient field. (Since f is birational $\implies K(X) = K(Y)$) Also, A is integrally closed ($\because$ Y is normal.) Thus, $A = B$. Hence, $f_*\mathscr{O}_X = \mathscr{O}_Y$. Then by Corollary 11.3, this completes the proof. □

Corollary 11.5 (Stein Factorization)**.** *Let $f : X \to Y$ be a projective morphism of noetherian schemes. Then one can factor f into $g \circ f'$, where $f' : X \to Y'$ is projective morphism with connected fibres, and $g : Y' \to Y$ is a finite morphism.*

To proof this we need the following lemma:

Lemma. *Let $f : X \to Y$, $g : Y \to Z$ be morphisms of schemes. If $g \circ f$ is projective, and g is separated, then f is projective.*

Proof. Notice that g is separated means $Y \to Y \times_Z Y$ is a closed immersion, and thus, is projective. Since projectivity is stable under the base change, $X \to X \times_Z Y$ is projective. Also, $g \circ f : X \to Z$ is projective, then so is $X \times_Z Y \to Y$.

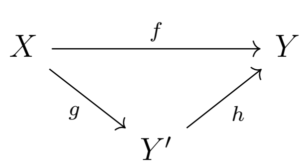


Finally, notice that the map $X \to X \times_Z Y \to Y$ is just f, but the composition of projective morphisms is projective, so f is projective. □

Proof. (of Corollary 11.5) Let $Y' = \mathbf{Spec}\, f_*\mathscr{O}_X$. Then notice that $f_*\mathscr{O}_X$ is coherent $\mathscr{O}_Y$-algebra. The natural map $g : Y' \to Y$ is affine, since for any open affine subset V of Y, by the definition of **Spec**, $g^{-1}(V) = \operatorname{Spec} \mathscr{O}_X(f^{-1}(V))$. Also, g is proper since f is. A proper affine morphism is projective, so g is finite. It is clearly that f factor through g, so we get $f' : X \to Y'$. Since g is separated, by Lemma, f' is projective. Notice that $f'_*\mathscr{O}_X = \mathscr{O}_{Y'}$, so the fibre is connected. □

Corollary 11.6 (Exercise III.11.2 in Hartshorne's book)**.** *A projective morphism with finite fibres is a finite morphism.*

Proof. Use Stein factorization then we have a projective morphism g with connected fibres and a finite morphism h such that the diagram commute:



Our goal is to show that g is an isomorphism so that f and h just deffer by an isomorphism, and thus, f is a finite morphism. Since we can replace Y' by $\operatorname{Im} g$, so we may assume g is surjective. Notice that $f^{-1}(h(y'))$ is finite and $g^{-1}(y') \subseteq f^{-1}(h(y'))$, but $g^{-1}(y')$ should be connected, and hence, $g^{-1}(y')$ is a single point. Also, g is projective and thus, is proper, so g give a homeomorphism on the under lying spaces. In the proof of Stein factorization, we see that $g_*\mathscr{O}_X = \mathscr{O}_{Y'}$, so their structure sheaf are the same. Thus, g is an isomorphism. This prove the statement.

□