Thm (Grothendieck-Riemann-Roch).

Let $f: X \to Y$ projective (proper) morphism between smooth quasi-projective varieties.

Let $td(X)$ be the todd class
$\left(td(X) = td(T_X) = \prod \frac{a_i}{1-e^{-a_i}}, \text{ when } c_t(T_X) = \prod_{i=1}^{r} (1+a_i t) \right)$

Let $\mathcal{F} \in K(X)$ the Grothendieck gp, the chern character $ch(\mathcal{F}) = \sum e^{a_i}$

$$f_*(ch(\mathcal{F}) . T(X)) = ch(f_!(\mathcal{F})) . T(Y).$$

Where $f_! \mathcal{F} = \sum (-1)^i R^i (f_* \mathcal{F})$

Outline of proof:

For a projective morphism, we can factor in to

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & \mathbb{P}^n \times Y \arrow[d, "G"] \\ & Y \end{tikzcd}$$

So we have 3 steps.
①: Show that GRR true for closed immersion
②: Show that GRR true for $\mathbb{P}^n \times Y \to Y$.
③: Show that if GRR true for both $f: X \to Y$, $g: Y \to Z$, then also true for $g \circ f: X \to Z$.

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Lemma:
Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ proper morphisms.
$x \in K(X)$, $y := f_! x \in K(Y)$.
①: If GRR true for (f, x) i.e.
$f_*(ch(x) Td(X)) = ch(f_!(x)) \cdot Td(Y)$,
and GRR true for (g, y),
then GRR true for $(g \circ f, x)$

②: If GRR true for (g, y), $(g \circ f, x)$,
and g_* is inj on Chow ring,
then GRR true for (f, x)

Pf ①: We have $(g \circ f)_*(ch(x) Td(X))$
$= g_* f_*(ch(X) Td(X))$
(GRR for x) $= g_*(ch(f_!(x)) \cdot Td(Y))$
(GRR for y) $= ch(g_! f_! x) \cdot Td(Z)$

Note that $g_!(f_!(x)) = \sum_i (-1)^i R^i g_*(f_!(x))$
$= \sum_i (-1)^i R^i g_*(\sum_j (-1)^j R^j f_* x)$

$= \sum_i \sum_j (-1)^{i+j} (R^i g_*)(R^j f_*)(x)$

$= \sum_k \sum_{i+j=k} (-1)^k R^i g_* R^j f_*(x)$

$= \sum_k (-1)^k R^k (g_* f_*)(x)$ (By the Leray spectral sequence).

$= \sum_k (-1)^k R^k (g \circ f)(x) = (g \circ f)_!(x)$ □

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② : Goal: $\overbrace{f_*(ch(x)Td(X))}^{\alpha} = \overbrace{ch(f_!(x))Td(Y)}^{\beta}$.

Since g_* is inj, it suffice to show that $g_*\alpha = g_*\beta$.

Where
$$\begin{aligned} g_*\alpha &= (g\circ f)_*(ch(x)Td(X)) \\ &= ch((g\circ f)_!(x))Td(Z) \quad (\text{GRR of } g\circ f). \\ &= ch(g_! f_!(x))Td(Z). \\ &= g_*(ch(f_!(x))Td(Y)) \quad (\text{GRR of } g) \\ &= g_*\beta \quad \square \end{aligned}$$

For the case of product space, first note that we have natural maps
$K(X) \longrightarrow K(X\times Y)$
$K(Y) \longrightarrow K(X\times Y)$
by
$$\begin{tikzcd} & X\times Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y \end{tikzcd}$$
and pullback.

Thus it induces
$$\begin{aligned} K(X)\otimes K(Y) &\longmapsto K(X\times Y) \\ \mathcal{F}\otimes\mathcal{G} &\longmapsto p_1^*\mathcal{F}\otimes p_2^*\mathcal{G}. \end{aligned}$$

Now we can give a new lemma:

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Lemma: Let $f: X \to Y$, $f': X' \to Y'$ proper morphisms.
$x \in K(X)$, $x' \in K(X')$
If GRR true for (f, x) and (f', x'),
then GRR holds for $(f \times f', x \otimes x')$,
where $f \times f': X \times X' \longrightarrow Y \times Y'$.

Pf: Goal: $(f \times f')_*(ch(x \otimes x')\, td(X \times X'))$

$$\overset{?}{=} ch((f \times f')_!(x \otimes x'))\, td(Y \times Y')$$

Note that $td(X \times X') = td(X) \otimes td(X')$
since Todd class is come from tangent sheaf.

Claim: ①: $(f \times f')_!(x \otimes x') = f_!(x) \otimes f'_!(x')$, $x \in K(X)$
②: $(f \times f')_*(\alpha \otimes \alpha') = f_*(\alpha) \otimes f'_*(\alpha')$, $\alpha \in A(X)$
③: $ch(x \otimes x') = ch(x) \otimes ch(x')$.

Assume all the claim, then we have
$(f \times f')_*(ch(x \otimes x')\, td(X \times X'))$
$= (f \times f')_*(ch(x)\, td(X) \otimes ch(x')\, td(X'))$ (③).

$= f_*(ch(x)\, td(X)) \otimes f'_*(ch(x')\, td(X'))$ (②)

$= ch(f_! x)\, td(Y) \otimes ch(f'_! x')\, td(Y')$ (GRR of f, f')

$= ch(f_!(x) \otimes f'_!(x'))\, td(Y \times Y')$ (③).

$= ch((f \times f')_!(x \otimes x'))\, td(Y \times Y')$ (①)

Pf of claims:

① : $LHS = \sum_k (-1)^k R^k (f\times f')_* (X \otimes X')$

$RHS = \sum_i (-1)^i R^i f_*(X) \otimes \sum_j (-1)^j R^j f'_*(X')$

$= \sum_k \sum_{i+j=k} (-1)^{i+j} R^i f_*(X) \otimes R^j f'_*(X')$

$= \sum_k (-1)^k R^k (f\times f)_! (X \otimes X')$

(the Künneth formula, since we can identify higher direct image by coho gp)

② : Recall: for an irreducible cycle,

$$f_*(\alpha) = \begin{cases} \deg f|_\alpha \cdot f(\alpha), & \dim f(\alpha) = \dim \alpha \\ 0, & \text{otherwise} \end{cases}$$

To prove it, we may assume α, α' are reduced irred subvar, thus

$$(f\times f')(\alpha \otimes \alpha') = (f\times f')(\alpha \times \alpha')$$

$$= \begin{cases} \deg(f\times f')|_{\alpha\times\alpha'} (f\times f')(\alpha\times\alpha'), & \dim(f\times f')(\alpha\times\alpha') = \dim \alpha\times\alpha' \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} (\deg f|_\alpha)(\deg f'|_{\alpha'})\, f(\alpha)\times f(\alpha'), & \dim f(\alpha) = \dim\alpha,\ \dim f(\alpha') = \dim\alpha' \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} f_*(\alpha) \otimes f_*(\alpha') & \text{''} \\ 0, & \text{otherwise} \end{cases} \qquad \square$$

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③: Recall: $C_t(E\otimes F)=\prod_{i,j}(1+(a_i+b_j)t)$

if $C_t(E)=\prod_i(1+a_it)$
$C_t(F)=\prod_j(1+b_jt)$

$ch(E)=\sum e^{a_i}$.

So we have $ch(X\otimes X')=\sum_{i,j}e^{a_i+b_j}$

$ch(X)\otimes ch(X')=\sum_i e^{a_i}\otimes\sum_j e^{b_j}$

So what we need to prove is $e^{a_i+b_j}=e^{a_i}\otimes e^{b_j}$.

$RHS=\sum\frac{1}{n!}(a_i^n)\otimes\sum\frac{1}{m!}(b_j)^m=\sum_{n,m}\frac{1}{n!m!}a_i^nb_j^m$

$LHS=\sum_k\frac{1}{k!}(a_i+b_j)^k=\sum_k\frac{1}{k!}\left(\sum_{m+n=k}C_n^k a_i^nb_j^m\right)$

$=\sum_{n,m}\frac{1}{n!m!}a_i^nb_j^m=RHS$ □

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Cor: (Hirzebruch Riemann-Roch)
Let X be a sm proj var of dimension n, $\mathcal{E}$: locally free sheaf on X, then

$$\deg(ch(\mathcal{E})td(X))_n = \chi(\mathcal{E})$$

Pf: applying GRR on $f: X \to \{pt\}$, we have.

$$f_*(ch(\mathcal{E})td(X)) = ch(f_!(\mathcal{E})).td(pt)$$

LHS $= \deg(ch(\mathcal{E})td(X))_n$ by defn of f_*.

RHS $= ch(f_!(\mathcal{E})) = ch(\Sigma(-1)^i R^i f_*(\mathcal{E}))$

$= \Sigma(-1)^i \dim_k R^i f_*(\mathcal{E})$

$= \Sigma(-1)^i \dim_k H^i(X, \mathcal{E})$ (Hart III.8.5)

$= \chi(\mathcal{E})$ $\square$

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Cor: (Standard R-R on curve and surface)

Let $\mathcal{E} = \mathcal{O}(D)$ be a line bundle,
then we have $\chi(\mathcal{O}(D)) = \deg(ch(\mathcal{O}(D))\, td(X))_n$.

Curve: $RHS = \deg((1+\deg D)(1+\frac{1}{2}\deg(-K)))_1$

$= \deg D - \frac{1}{2}\deg K = \deg D + 1 - g.$

Surface: $c_i(\mathcal{O}(D)) = 0 \ \forall\, i \geq 2$ since $\mathcal{O}(D)$: l. b.

So $RHS = \deg((1+D+\frac{1}{2}D^2)(1+\frac{1}{2}(-K)+\frac{1}{12}(K^2+c_2)))_2$

$= \frac{1}{2}D^2 - \frac{1}{2}DK + \frac{1}{12}(K^2+c_2)$

Where c_2: 2nd chern class of T_X.

Noether's formula:
$10 + 12P_a = K^2 + b_2 - 2b_1$
$\Rightarrow K^2 = 12(1+P_a) + 2b_1 - b_2 - 2.$

$\Rightarrow c_2 = \frac{1}{12}(b_2 - 2b_1 + 2)$ by Standard R-R.

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Exercise A.6.8.

Let $\mathcal{E}$: locally free sheaf of rank 2 on $\mathbb{P}^3$
with chern class c_1, c_2.
$A(\mathbb{P}^3) \cong \mathbb{Z}[h]/h^4$ h: plane in $\mathbb{P}^3$.
Identified c_1, c_2 be integer.
Show that $c_1c_2 \equiv 0 \pmod 2$

Pf: By HRR, we have
$$\chi(\mathcal{E}) \overset{!}{=} \deg(ch(\mathcal{E})\, td(\mathbb{P}^3))_3$$

By the exact sequence
$$0 \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1)^4 \to T_{\mathbb{P}^3} \to 0$$
(Thm II.8.13 and taking dual)

We have
$$\begin{aligned} c_t(T_{\mathbb{P}^3}) &= c_t(\mathcal{O}_{\mathbb{P}^3})\, c_t(\mathcal{O}_{\mathbb{P}^3}(1)^4) \\ &= (c_t(\mathcal{O}_{\mathbb{P}^3}(1))^4 = (1+ht)^4 \end{aligned}$$
$$\Rightarrow c_t(T_{\mathbb{P}^3}) = 1 + 4ht + 6h^2t^2 + 4h^3t^3$$
$$\begin{aligned} \Rightarrow td(T_{\mathbb{P}^3}) &= 1 + \tfrac{1}{2}(4h) + \tfrac{1}{12}((4h)^2 + 6h^2) + \tfrac{1}{24}(4h)(6h^2) \\ &= 1 + 2h + \tfrac{11}{6}h^2 + h^3 \end{aligned}$$

$$ch(\mathcal{E}) = 2 + c_1h + \tfrac{1}{2}(c_1^2 - 2c_2)h^2 + \tfrac{1}{6}(c_1^3 - 3c_1c_2)h^3$$

$$\Rightarrow \chi(\mathcal{E}) = \tfrac{1}{6}(c_1^3 - 3c_1c_2) + (c_1^2 - 2c_2) + \tfrac{11}{6}c_1 + 2 \in \mathbb{Z}$$

$$\Rightarrow \frac{1}{6}(c_1^3 + 11c_1 - 3c_1c_2) \in \mathbb{Z}.$$

If $c_1c_2 \not\equiv 0 \pmod 2$, then $(c_1^3 + 11c_1 - 3c_1c_2)$ is odd $\Rightarrow \frac{1}{6}(c_1^3 + 11c_1 - 3c_1c_2) \notin \mathbb{Z}$ ※ $\square$

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