Main theorem (Chow Moving Lemma).

Let k be an (algebraically closed) field.
V : smooth irreducible subvariety in $\mathbb{P}^n_k$
(not need projective).

Let Y, Z be 2 cycles on V, then
$\exists\ Z' \underset{rat}{\sim} Z$ s.t. $(Y . Z')$ is defined
(i.e. every component $T \subset (\mathrm{Supp}\, Y_i \cap \mathrm{Supp}\, Z'_j)$
has $\mathrm{codim}(T, V) = \mathrm{codim}(Y_i, V) + \mathrm{codim}(Z'_j, V)$)
(Here we allow intersection multiplicity > 1).
Moreover, Z' can be chosen from a cycle W on $V \times \mathbb{P}^1$, and deform Z to Z' on W, in fact, for almost all $Z_x := \{(v,x) \in W \subset V \times \mathbb{P}^1\}$, $(Y . Z_x)$ is defined.

Definition: Let $Z \subset \mathbb{P}^n$ be a closed subvar,
L : linear subspace of $\mathbb{P}^n$ s.t. $L \cap Z = \phi$.
Define $C_L(Z) := \{\overline{zx} \mid z \in Z, x \in L\}$,
then (cf. Hartshorne Ex I.7.7)
$C_L(Z)$ is a closed subvariety of $\mathbb{P}^n$
with dimension $(\dim L + \dim Z + 1)$.
(Note that $L \cap Z = \phi$ implies $\dim L + \dim Z < n$).

(Algebraic defn: $C_L(Z) = V(I(Z) \cap I(L))$)

Defn: Let $V \subset \mathbb{P}^n$ be a variety of dim r,
$Y_1, \cdots, Y_m$ be subvarieties of V
(not need closed).
$Z \subset V$ be a closed subvar.
We define the excess of Z
relative to $Y_1, \cdots, Y_m$ by

$$e(Z) = \begin{cases} \max\limits_i \left\{ \max\limits_{W \subset Y_i \cap Z} \{\dim W + r - \dim Y - \dim Z\} \right\}, & Y_i \cap Z \neq \phi \text{ for some } i \\ 0, & Y_i \cap Z = \phi, \forall i \end{cases}$$

So we have
$e(Z) = 0 \Leftrightarrow Y_i \cap Z$ properly, $\forall i$.

When $Z = \sum n_j Z_j$ is a cycle, we
define $e(Z) = \max\limits_j \{e(Z_j)\}$

Key lemma:
Use the above notation and setting, if we
have the additional conditions:
① V is a closed subvariety of $\mathbb{P}^n$.
② $Y_i \subset V_{sm}$ $\forall i$.

Then let Z be a cycle on V, $\exists$ a dense
open subset $U \subset G(n, n-r-1)$ (Grassmannian)
s.t. $\forall$ $[L] \in U$ closed point,
① $L \cap V = \phi$
② $e(C_L(Z) \cdot V - Z) \leq \max(e(Z) - 1, 0)$.

Lemma: Let $Z \subset V \subset \mathbb{P}^n$ closed subvars, $\dim Z = d$
$\dim V = r$, let $[L] \in \mathbb{G}(n, n-r-1)$
s.t. $L \cap V = \phi$, then
$C_L(Z) \cdot V$ is defined

Pf: $\dim C_L(Z) = d + (n-r-1) + 1 = n - r + d$.
so $C_L(Z) \cdot V$ is defined
$\Leftrightarrow$ every comp has codim $(r-d) + (n-r) = n-d$
$\Leftrightarrow$ every comp has dimension $= d$.

$\geq$ always hold, for $\leq$,
consider the projection from L
$f: V \longmapsto \mathbb{P}^m$ for some m.
We have $f(x) = f(y) \Leftrightarrow \overline{xy} \cap L \neq \phi$.

So $x \in C_L(Z) \cap V$
$\Leftrightarrow x \in V$ and $\exists\, y \in L,\ z \in Z$ s.t. $x \in \overline{yz}$
$\Leftrightarrow \exists\, z \in Z$ s.t. $f(x) = f(z)$
$\Leftrightarrow x \in f^{-1} f(Z)$.

Since f is finite, we conclude
that $\dim(C_L(Z) \cap V) = \dim Z = d$ $\square$.

Pf of Chow Moving Lemma (assuming Key lemma)

Fix an immersion $V \hookrightarrow \mathbb{P}^n$, let $\overline{V}$, $\overline{Z}$ be the closure of V, Z in $\mathbb{P}^n$
Write $Y = \sum n_i Y_i$, each Y_i are locally closed subvariety of $\overline{V}$.

Now let $e := e(\overline{Z})$, then by the key lemma, $\exists$ $(n-r-1)$ dim'l subspaces $L_1, \ldots, L_e$ s.t. if we define $X_0 = \overline{Z}$, and $X_i = C_{L_i}(X_{i-1}) \cdot \overline{V} - X_{i-1}$, then $e(X_i) \le e(X_{i-1})$, and "$=$" iff $e(X_{i-1}) = 0$.
In particular, $e(X_e) = 0$ i.e. $X_e \cap Y_i$ properly.

Note that we have $X_0 = \left(\sum_{i=1}^{e} (-1)^{i-1} C_{L_i}(X_{i-1}) \cdot \overline{V}\right) + (-1)^e X_e$

(Since $C_{L_i}(X_{i-1}) \cdot \overline{V} = X_{i-1} + X_i$).

Fact: $\exists$ $g \in \mathrm{Aut}(\mathbb{P}^n)$ s.t.
$$\begin{cases} (g \cdot C_{L_i}(X_{i-1})) \cdot \overline{V} \\ g \cdot C_{L_i}(X_{i-1}) \cdot Y_j \end{cases} \text{are defined.}$$

i.e. $\mathrm{ht}(I(g \cdot C_{L_i}(X_{i-1})) + I(Y_j)) \text{"="} \mathrm{ht}(I(g \cdot C_{L_i})) + \mathrm{ht}(I(Y_j))$
LHS $= \mathrm{ht}(g \cdot I(C_{L_i}(X_{i-1})) + I(Y_j))$, RHS $= \mathrm{ht}(I(C_{L_i}(X_{i-1}))) + \mathrm{ht}(I(Y_j))$

So for general g, $g \cdot I(C_{L_j}(X_{i-1})) \cap I(Y_j) = 0$, thus equality holds.

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Now Let Γ be a rational curve in $\mathrm{Aut}(\mathbb{P}^n)$, connecting 1 and g, then we have a cycle
$W := \{(v,x) \in V \times \mathbb{P}^1 \mid v \in (g_x \cdot \bigcup_i \mathrm{Supp}\, C_{L_i}(X_{i-1})) \cdot V\}$
(where $g_1 = id$).
then we deform Z to $Z' := \sum (-1)^i g(C_{L_i}(X_{i-1})) \cdot V + (-1)^e X_e$.
$\square$

To prove key lemma, first we introduce some definition:

Let $\mathbb{G}(n,q)$ be the Grassmannian of q dim'l subspace of $\mathbb{P}^n$.
Consider $Z \subset V \subset \mathbb{P}^n$ closed subvars. $\dim V = r$.
Δ: diagonal of $V \times V$.
$\Sigma_0 := \{(z,x,y) \in (Z \times V - \Delta) \times \mathbb{P}^n \mid z, x, y \text{ colinear in } \mathbb{P}^n\}$.
$\Sigma := \overline{\Sigma_0}$.
Fact: if $(z,z,y) \in \Sigma$, then $y \in T_z V$.

Let $J := \{(x,L) \in \mathbb{P}^n \times \mathbb{G}(n,n-r-1) \mid x \in L\}$.

Then in $Z \times V \times \mathbb{P}^n \times \mathbb{G}(n,n-r-1)$, we have the closed subset $T := (\Sigma \times \mathbb{G}(n,n-r-1)) \cap (Z \times V \times J)$ and define $W = P_{124}(T) \subset Z \times V \times \mathbb{G}(n,n-r-1)$.
Called the secant correspondence.

Thus, for $x \neq z$, $(z,x,L) \in W \Leftrightarrow L \cap \overline{zx} \neq \phi$,
if $(z,z,L) \in W \Rightarrow T_z V \cap L \neq \phi$. (4)

Now let $q_{12}: W \to Z \times V$ be the projection.

For $x \neq z$, $q_{12}^{-1}(z,x) \cong \{[L] \in \mathbb{G}(n, n-r-1) \mid L \cap \overline{xz} \neq \phi\}$.

Fact about Grassmannian:
Let $0 \le q \le n$, $s \le n-q$, Λ be a s dim-l subsp of $\mathbb{P}^n$, then
$\{L \in \mathbb{G}(n,q) \mid \Lambda \cap L \neq \phi\}$
is a closed subvar of $\mathbb{G}(n,q)$
with codimension $n-q-s$ in $\mathbb{G}(n,q)$.

Thus, $q_{12}^{-1}(z,x)$ has codimension r in $\mathbb{G}(n,n-r-1)$,
hence, ~~dim~~ $W \cap ((Z \times Y - \Delta) \times \mathbb{G}(n, n-r-1))$
has dimension $\dim Y + \dim Z + \dim \mathbb{G}(n, n-r-1) - r$,
Hence for general L, $\dim((Z \times Y - \Delta) \times [L] \cap W) = \dim Y + \dim Z - r$
by the semi-continuity thm. $(*)$

Lemma:
Let $Z \subset V \subset \mathbb{P}^n$ closed subvar, $[L] \in \mathbb{G}(n, n-r-1)$
$x \in \operatorname{Supp}(C_L(Z) \cdot V - Z)$, then one of the following holds:
① $\exists$ closed pt $z \in Z$, $z \neq x$ s.t. $\overline{xz} \cap L \neq \phi$
② $x \in \operatorname{Supp} Z \cap \Gamma(L)$

Where $\Gamma(L) = \{x \in V \mid L \cap T_x(V) \neq \phi\}$.
By dimension counting, $\Gamma(L) \supset \operatorname{Sing}(V)$.

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Pf of Key Lemma (assuming the above lemma)

By the above fact of Grassmannian ① is OK.

For ②, we may assume Z is a closed subvar and $m=1$ (since we work on infinite field)

Let W as above, then for $[L] \in \mathbb{G}(n, n-r-1)$, by the above lemma, if $x \in \mathrm{Supp}((C_L(Z) \cdot V - Z))$, then either ① $x \in Z$ and $L \cap T_x V \neq \phi$, ② $\exists\, z \neq x$ s.t. $(z, x, L) \in W$.

What we need to show is: if $\dim(Z \cap Y) > \dim Z + \dim Y - r$ then $\dim(\mathrm{Supp}(C_L(Z) \cdot V - Z) \cap Y) < \dim(Z \cap Y)$ for general L. Let $x \in LHS \subset Y$

By $(*)$, for general L, the set of points $x \in Y$ satisfies ② has dimension exactly $\dim Z + \dim Y + r$. So we remain to deal with points satisfies ①.

Now if $x \in Y$ and x satisfies ①, then $x \in Y \cap Z$. But by the general fact about Grassmannian, $\{L \in \mathbb{G}(n, n-r-1) \mid L \cap T_x V \neq \phi\}$ has codimension $n - (n-r-1) - \dim T_x V = r+1-r = 1$ (since $x \in Y \subset V_{sm}$).
So for general L, x not satisfies ①
$\Rightarrow$ for general L, $\{x \in Y \cap Z \mid x \text{ satisfies } ①\}$ is a proper closed subset of $Y \cap Z$ $\Rightarrow$ done $\square$

Pf of Lemma:

Let $j: Z\times V \hookrightarrow Z\times V\times G(n, n-r-1)$

$(z,x) \longrightarrow (z,x,L)$

Let $E := j^{-1}(W) = \{(z,x) \mid (z,x,L)\in W\}$

$H := P_2(E) = \{x\in V \mid \exists z\in Z \text{ s.t. } (z,x,L)\in W\}$

$\Rightarrow$ by ✡, $\forall x\in H$, ① or ② holds.

Now let $x\in Z'-Z$, where Z' is a component of $C_L(Z)\cdot V$ other than Z, then by defn of $C_L(Z)$, $\exists\, z\in Z$ s.t. $\overline{xz}\cap L\neq\phi \Rightarrow$ by ✡, $x\in H$

$\Rightarrow Z'\subset H$, so $H\supset \mathrm{Supp}(C_L(Z)\cdot V)-\mathrm{Supp}(Z)$

$\Rightarrow \forall x\in RHS$, x satisfies ① or ②.

So if $\mathrm{Supp}(C_L(Z)\cdot V-Z)\not\supset Z$ i.e. $\mathrm{multi}_Z C_L(Z)\cdot V=1$, then we are done.

If not in the case, but $Z\subset H$ is also OK.

So it remain to deal with $x\in Z-H$.

In the case we have $(z,x,L)\notin W$ $\forall\, z\in Z$.

Now if $x\notin \mathbb{P}(L)$, then $T_xV\cap L=\phi \Rightarrow x\in V_{sm}$.

Since now $H\cap Z$ proper closed in Z, for general $z\in V_{sm}\cap Z_{sm}$,

$\mathbb{P}(L)\cap Z$. $z\notin \mathbb{P}(L)$.

$\Rightarrow z\in C_L(Z)_{sm}$.

Now we have $\dim(T_z C_L(Z)\cap T_z V)\geq \dim \overset{n-r+d}{C_L(Z)}+\dim \overset{r}{V}-n=d$.

And since $\mathrm{multi}_Z C_L(Z)\cdot V\geq 2$, the strict inequality holds.

Since $L\cap T_zV=\phi$, we have $T_zC_L(Z)\not\supset L$ by dimension counting. But since $\mathrm{multi}_Z C_L(Z)\cdot V\geq 2$, we have

$T_zC_L(Z)\supset L$ ✱ □