§. Serre's GAGA principle (6/23, 7/2)

We first give some basic definitions, which are analogues of affineness and varieties.

Def. 1. A subset $U \subseteq \mathbb{C}^n$ is said to be analytic if $\forall x \in U$, $\exists f_1, \dots, f_k$: holomorphic functions in a neighborhood of x, say W, such that $U \cap W = \{ z \in W \mid f_i(z) = 0,\ \forall i = 1, \dots, k \}$

· We equip with such U a sheaf $\mathcal{H}_U$ as follow : let $\mathcal{H}$:= the sheaf of germs of holomorphic functions on $\mathbb{C}^n \subseteq \mathcal{C}(\mathbb{C}^n)$:= the sheaf of $\mathbb{C}$-valued functions on $\mathbb{C}^n$ (resp. X), (resp. $\mathcal{C}(X)$, where X is a top. space).

Now let ϵ_x be the restriction map, $\epsilon_x : \mathcal{C}(\mathbb{C}^n)_x \longrightarrow \mathcal{C}(U)_x$. This gives a sheaf $\mathcal{H}_U \subseteq \mathcal{C}(U)$.

$$\begin{array}{ccc} \mathcal{C}(\mathbb{C}^n)_x & \longrightarrow & \mathcal{C}(U)_x \\ \cup\!\!| & & \cup\!\!| \\ \mathcal{H}_x & \longmapsto & \mathcal{H}_{x,U} \end{array}$$

Also, we denote the kernel of $\mathcal{H}_x \xrightarrow{\epsilon_x} \mathcal{H}_{x,U}$ by $\mathcal{A}_{x,U}$.

Def. 2. Let $U \subseteq \mathbb{C}^r$, $V \subseteq \mathbb{C}^s$ be analytic subsets. A map $\phi : U \longrightarrow V$ is called holomorphic if ϕ is continuous and "$f \in \mathcal{H}_{\phi(x),V} \Rightarrow f\phi \in \mathcal{H}_{x,U}$". A bijection who and whose inverse are both holomorphic is called an (analytic) isomorphism.

3. A topological space X with a sheaf $\mathcal{H}_X \subseteq \mathcal{C}(X)$ satisfying the following axioms is called an analytic space : (H1) $\exists$ an open covering $\{V_i\}$ of X such that $(V_i, \mathcal{H}_X|_{V_i})$ is analytically isomorphic to an analytic subset of some $\mathbb{C}^n$, (H2) X is Hausdorff.

4. If X is an analytic space, a sheaf $\mathcal{F}$ on X is called analytic if it's a sheaf of $\mathcal{H}_X$-modules.

5. $Y \subseteq X$: closed analytic subspace, we can define the ideal sheaf of Y as follow : $\forall x \in X$, let $\mathcal{A}_{x,Y} := \{ f \in \mathcal{H}_{x,X} \mid f|_Y = 0$ on a neighborhood of $x \}$. This gives a short exact sequence :

$$0 \longrightarrow \mathcal{A}_Y \longrightarrow \mathcal{H}_X \longrightarrow \mathcal{H}_Y \longrightarrow 0$$

Note that polynomials and regular functions are in particular holomorphic, so we know that every Zariski locally closed subset (of affine space) is analytic, and regular maps between them are holomorphic.

Proposition 1. X : variety over $\mathbb{C}$. $\exists!$ structure of an analytic space such that for any affine chart $\phi : V \longrightarrow U$ (i.e. $V \subseteq X$: Zariski open, $U \subseteq$ some affine space : Zariski locally closed) V is open in usual topology and ϕ is an analytic isomorphism. This is denoted by X^h.

In other words, we regard every algebraic chart as an analytic chart.

So now for each $x \in X$, we have two local rings, $\mathcal{O}_x$ and $\mathcal{H}_x$ with a map $\theta : \begin{array}{ccc} \mathcal{O}_x & \longrightarrow & \mathcal{H}_x \\ \cup\!\!| & & \cup\!\!| \\ m_x & \longmapsto & m_x^h \end{array}$, which is a local homomorphism. By continuity, it extends to $\hat{\theta} : \hat{\mathcal{O}}_x \longrightarrow \hat{\mathcal{H}}_x$.

Proposition 2. $\hat{\theta}$ is bijective, and for $Y \subseteq X$: Zariski locally closed, $\theta(\mathcal{I}_{x,Y})$ generates $\mathcal{A}_{x,Y}$.

Pf. The question is local, so we may assume that $X = \mathbb{C}^n$

The first assertion is then trivial since $\hat{\mathcal{O}}_x = \mathbb{C}[\![z_1, \dots, z_n]\!] = \hat{\mathcal{H}}_x$

For the second, let $\mathfrak{a} := \langle \theta(\mathcal{I}_{x,Y}) \rangle \subseteq \mathcal{H}_x \Rightarrow V(\mathfrak{a}) = Y$. Now let $f \in \mathcal{A}_{x,Y} \subseteq \mathcal{H}_x$

Analytic nullstellensatz $\Rightarrow \exists\, r \geq 0$ such that $f^r \in \mathfrak{a} \Rightarrow f^r \in \mathfrak{a}\hat{\mathcal{H}}_x = \mathcal{I}_{x,Y}\, \hat{\mathcal{O}}_x$.

$\mathcal{I}_{x,Y}$ is an intersection of prime ideals (corresponding to irreducible components of Y passing through x), and hence so is $\mathcal{I}_{x,Y}\hat{\mathcal{O}}_x \Rightarrow f \in \mathcal{I}_{x,Y}\hat{\mathcal{O}}_x = \mathfrak{a}\hat{\mathcal{H}}_x \Rightarrow f \in \mathfrak{a}\hat{\mathcal{H}}_x \cap \mathcal{H}_x = \mathfrak{a}$.

(Here we use an algebraic fact : $\widehat{\bigcap_{i=1}^n p_i} = \bigcap_{i=1}^n \hat{p}_i$) □

Remark. This proposition gives that $\begin{array}{ccc} \hat{\mathcal{O}}_x & \xrightarrow{\sim} & \hat{\mathcal{H}}_x \\ \uparrow & \circlearrowright & \uparrow \\ \mathcal{O}_x & \xhookrightarrow{\theta} & \mathcal{H}_x \end{array}$: θ is injective and since both $\mathcal{O}_x \hookrightarrow \hat{\mathcal{O}}_x$ and $\mathcal{H}_x \hookrightarrow \hat{\mathcal{H}}_x$ are faithfully flat, so is $\mathcal{O}_x \xhookrightarrow{\theta} \mathcal{H}_x$. In Serre's paper, he defined a notion called "flat couples" and developed some theorems in Appendix, but we do not need it here.

Def 6. X : variety over $\mathbb{C}$, $\mathcal{F}$: sheaf on X, we define a sheaf $\mathcal{F}'$ on X^h by $\begin{array}{ccc} \mathcal{F}' & \longrightarrow & \mathcal{F} \\ \downarrow & \square & \downarrow \\ X^h & \xrightarrow{\text{"id"}} & X \end{array}$

Moreover, if $\mathcal{F}$ is an $\mathcal{O}_X$-module, we define $\mathcal{F}^h := \text{"id"}^* \mathcal{F} = \mathcal{F}' \otimes_{\mathcal{O}} \mathcal{H}$.

· First note that $\mathcal{O}_X^h = \mathcal{H}_X$ and that $\forall\, \mathcal{O}$-linear $\phi : \mathcal{F} \longrightarrow \mathcal{G}$, this operation gives a $\mathcal{H}$-linear map $\mathcal{F}^h \longrightarrow \mathcal{G}^h$, denoted by ϕ^h, i.e. $(\ )^h$ gives a covariant functor.

Notation. X : variety over $\mathbb{C}$. We denote the categories of $\mathcal{O}_X$-modules, $\mathcal{H}_X$-modules, coherent $\mathcal{O}_X$-modules and coherent $\mathcal{H}_X$-modules by (Mod_X), (Mod_{X^h}), (Coh_X), and (Coh_{X^h}) respectively. So now we have $\begin{array}{ccc} (\mathrm{Mod}_X) & \xrightarrow{(\ )^h} & (\mathrm{Mod}_{X^h}) \\ \cup\!\!| & & \cup\!\!| \\ (\mathrm{Coh}_X) & & (\mathrm{Coh}_{X^h}) \end{array}$

Def 7. X : topological space, $\mathcal{A} \in \mathrm{Ob}(\mathrm{Ring}_X)$, $\mathcal{M}$: $\mathcal{A}$-module, $x \in X$.

· $\mathcal{M}$ is called finitely generated over $\mathcal{A}$ at x if $\exists\, U$: neighborhood of x with an exact sequence on it : $(\mathcal{A}|_U)^{\oplus r} \longrightarrow \mathcal{M}|_U \longrightarrow 0$

- $\mathcal{M}$ is called relation finite over $\mathcal{A}$ at x if for any map $(\mathcal{A}|_U)^{\oplus r} \longrightarrow \mathcal{M}|_U$ where $x \in U \underset{\text{open}}{\subseteq} X$ and $r \in \mathbb{N}$, the kernel of it is $\mathcal{A}|_U$-finitely generated at x

- $\mathcal{M}$ is coherent over $\mathcal{A}$ at x if it's both $\mathcal{A}$-finitely generated and $\mathcal{A}$-relation finite at x.
  $\mathcal{M}$ is coherent over $\mathcal{A}$ if it's coherent over $\mathcal{A}$ at every $x \in X$.

Theorem (Oka) $\mathcal{H}_X$ is $\mathcal{H}_X$-coherent.

Now, our goal is to show that there's an exact equivalence $(Coh_X) \longrightarrow (Coh_{X^h})$ which makes the above diagram commutes.

- Let $\mathcal{F} \in Ob(Coh_X)$, $x \in X$. We can find a finite presentation $\mathcal{O}_X^p \longrightarrow \mathcal{O}_X^q \longrightarrow \mathcal{F} \longrightarrow 0$ near x
$\Rightarrow \mathcal{H}_X^p \longrightarrow \mathcal{H}_X^q \longrightarrow \mathcal{F}^h \longrightarrow 0 \Rightarrow \mathcal{F}^h$ is $\mathcal{H}_X$-coherent near x since $\mathcal{H}_X$ itself is
$\Rightarrow \mathcal{F}^h \in Ob(Coh_{X^h})$ i.e. $(\ )^h$ descends to a functor $(Coh_X) \longrightarrow (Coh_{X^h})$.

- Exactness: for $\mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3$: exact in (Coh_X) (or (Mod_X)), then so is $\mathcal{F}_1' \longrightarrow \mathcal{F}_2' \longrightarrow \mathcal{F}_3'$
$\mathcal{H}_X$ is $\mathcal{O}_X$-flat $\Rightarrow \mathcal{F}_1^h \longrightarrow \mathcal{F}_2^h \longrightarrow \mathcal{F}_3^h$ is exact in (Coh_{X^h}).

- X : variety over $\mathbb{C}$, $\mathcal{F} \in Ob(Mod_X)$, $U \subseteq X$: Zariski open, $s \in \Gamma(U, \mathcal{F})$ can be viewed as a section s' of $\mathcal{F}'$ over $U^h \subseteq X^h \Rightarrow \exists$ a map $\epsilon : \Gamma(U, \mathcal{F}) \longrightarrow \Gamma(U^h, \mathcal{F}^h)$.
$$s \longmapsto s' \otimes 1_{\mathcal{H}_X}$$
This induces eventually a map $\epsilon : H^q(X, \mathcal{F}) \longrightarrow H^q(X^h, \mathcal{F}^h)$ which is functorial.

Main Theorem. X : projective variety over $\mathbb{C}$. 1) For any $\mathcal{F} \in Ob(Coh_X)$, $q \geq 0$, the map $\epsilon : H^q(X, \mathcal{F}) \longrightarrow H^q(X^h, \mathcal{F}^h)$ is an isomorphism. 2) $(Coh_X) \xrightarrow{(\ )^h} (Coh_{X^h})$ is an equivalence.

Pf. 1) $X \overset{i}{\hookrightarrow} \mathbb{P}_{\mathbb{C}}^r$. By pushing forward, we may assume $X = \mathbb{P}_{\mathbb{C}}^r$.

Case 1. $\mathcal{F} = \mathcal{O}_X$

We know from Hartshorne that $H^0(X, \mathcal{O}_X) = \mathbb{C}$ and $H^q(X, \mathcal{O}_X) = 0 \ \forall q > 0$.

For the analytic case, by Dolbeault's theorem (see page 14 of the note Transcendental Methods) we also see $H^0(X, \mathcal{H}) = \mathbb{C}$ and $H^q(X^h, \mathcal{H}) \simeq H^{0,q}(X) = 0$.

Case 2. $\mathcal{F} = \mathcal{O}_X(n)$

Induct on r (dimension) and consider $0 \longrightarrow \mathcal{O}_X(n-1) \longrightarrow \mathcal{O}_X(n) \longrightarrow \mathcal{O}_E(n) \longrightarrow 0$ where E is a hyperplane.

This gives two long exact cohomology sequences:

$$\begin{array}{ccccccc} \cdots \to H^q(X, \mathcal{O}_X(n-1)) & \to & H^q(X, \mathcal{O}_X(n)) & \to & H^q(X, \mathcal{O}_E(n)) & \to & H^{q+1}(X, \mathcal{O}_X(n-1)) \\ \downarrow \epsilon^q_{n-1} & & \downarrow \epsilon^q_n & & \downarrow \wr & & \epsilon^{q+1}_{n-1} \downarrow \\ \cdots \to H^q(X^h, \mathcal{H}_X(n-1)) & \to & H^q(X^h, \mathcal{H}_X(n)) & \to & H^q(X^h, \mathcal{H}_E(n)) & \to & H^{q+1}(X^h, \mathcal{H}_X(n-1)) \end{array}$$

$\Rightarrow$ " ϵ^q_{n-1} is an isomorphism if and only if ϵ^q_n is, $\forall q$". So Case 1 $\Rightarrow$ Case 2

Case 3. General $\mathcal{F}$.

We use descending induction on q (When $q \gg 0$, use Grothendieck vanishing)

$\mathcal{F}$ is $\mathcal{O}_X$-coherent $\Rightarrow \exists\ 0 \to \mathcal{R} \to \mathcal{L} = \bigoplus_i \mathcal{O}(n_i) \to \mathcal{F} \to 0$

$$\Rightarrow \begin{array}{ccccccccc} \cdots H^q(X, \mathcal{R}) & \to & H^q(X, \mathcal{L}) & \to & H^q(X, \mathcal{F}) & \to & H^{q+1}(X, \mathcal{R}) & \to & H^{q+1}(X, \mathcal{L}) \\ \epsilon^q_{\mathcal{R}} \downarrow & & \downarrow \wr & & \epsilon^q_{\mathcal{F}} \downarrow & & \wr \downarrow & & \wr \downarrow \\ H^q(X^h, \mathcal{R}^h) & \to & H^q(X^h, \mathcal{L}^h) & \to & H^q(X^h, \mathcal{F}^h) & \to & H^{q+1}(X^h, \mathcal{R}^h) & \to & H^{q+1}(X^h, \mathcal{L}^h) \end{array}$$

Four Lemma $\Rightarrow \epsilon^q_{\mathcal{F}}$ is surjective. But $\mathcal{F}$ stands in a general position $\Rightarrow \epsilon^q_{\mathcal{R}}$ is also surjective. Again by four Lemma, $\epsilon^q_{\mathcal{F}}$: injective. This proves 1).

2) Step 1. $(\ )^h$ is fully faithful.

Let $\mathcal{A} := \mathcal{H}om(\mathcal{F}, \mathcal{G})$, $\mathcal{B} := \mathcal{H}om(\mathcal{F}^h, \mathcal{G}^h)$ (here $\mathcal{F}, \mathcal{G} \in Ob(Coh_X)$)

We first note that $f \in \mathcal{A}_x = Hom_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{G}_x)$ defines an element $f^h \in Hom_{\mathcal{H}_x}(\mathcal{F}_x \otimes \mathcal{H}_x, \mathcal{G}_x \otimes \mathcal{H}_x)$. Extending by linearity we obtain a map $\iota: \mathcal{A}^h \to \mathcal{B}$.

Moreover, at the stalk, this map is just the natural map $\iota_x: Hom_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{G}_x) \otimes_{\mathcal{O}_x} \mathcal{H}_x \to Hom_{\mathcal{H}_x}(\mathcal{F}_x \otimes \mathcal{H}_x, \mathcal{G}_x \otimes \mathcal{H}_x)$, which is an isomorphism since $\mathcal{H}_x$ is (faithfully) flat over $\mathcal{O}_x$ and $\mathcal{F}_x$ admits a finite presentation (by coherence)

Now consider $Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = H^0(X, \mathcal{A}) \xrightarrow{\sim} H^0(X^h, \mathcal{A}^h) \xrightarrow{\sim} H^0(X^h, \mathcal{B}) = Hom_{\mathcal{H}_X}(\mathcal{F}^h, \mathcal{G}^h)$

Step 2. To show a fully faithful exact functor is an equivalence, it suffices to show that it's "essentially surjective", in other words, "$\forall \mathcal{M} \in Ob(Coh_{X^h})$, $\exists \mathcal{F} \in Ob(Coh_X)$ such that $\mathcal{F}^h \simeq \mathcal{M}$". One may notice that this implies the uniqueness of $\mathcal{F}$.

Reduction to $X = \mathbb{P}^r_{\mathbb{C}}$: given $Y \xhookrightarrow{i} X$: a projective variety and $\mathcal{M}$: a $\mathcal{H}$-coherent sheaf on Y^h.

Were the assertion proven for X, $\exists\, \mathcal{G} \in Ob(Coh_X)$ such that $\mathcal{G}^h \simeq i_* \mathcal{M}$.

Let $\mathcal{I} := \mathcal{I}_Y \subseteq \mathcal{O}_X$. Since $\mathcal{G}$ is supported on Y, $\mathcal{I}\cdot\mathcal{G} = 0 \Rightarrow \exists\, \mathcal{F} \in Ob(Coh_Y)$ such that $\mathcal{G} = i_* \mathcal{F}$ (in fact, $\mathcal{F} = \mathcal{G}|_Y$) $\Rightarrow i_*(\mathcal{F}^h) = (i_*\mathcal{F})^h = \mathcal{G}^h = i_* \mathcal{M}$ $\Rightarrow \mathcal{F}^h \simeq \mathcal{M}$. This proves the assertion for Y.

<u>Step 3</u>. $X := \mathbb{P}^r_{\mathbb{C}}$, $\mathcal{M} \in Ob(Coh_{X^h})$. Induct on r.

We first prove two fact, which are analogues of <u>Theorem</u> II.5.17 and III.5.2 in Hartshorne.

Let $E \simeq \mathbb{P}^{r-1}_{\mathbb{C}}$ be a hyperplane in X, $\mathcal{A} \in Ob(Coh_{E^h}) \Rightarrow \exists\, \mathcal{F} \in Ob(Coh_E)$ such that $\mathcal{F}^h \simeq \mathcal{A} \Rightarrow \mathcal{F}(n)^h \simeq \mathcal{F}^h(n) \simeq \mathcal{A}(n)$.

By part 1) and algebraic Serre vanishing, we see $H^q(E^h, \mathcal{A}(n)) = 0 \ \forall q > 0,\ n \gg 0$.

<u>Step 4</u>. $\exists\, n_0 = n_0(\mathcal{M})$ such that $\forall n \geq n_0$, $x \in X$, $\mathcal{M}(n)_x$ is generated by $H^0(X^h, \mathcal{M}(n))$.

First note that X^h is quasi-compact and if $H^0(X^h, \mathcal{M}(n))$ generates $\mathcal{M}(n)_x$, it also generates $\mathcal{M}(n)_y$ for all y in a neighborhood of x. So we only need to prove that "$\forall x \in X$, $\exists\, n = n(x, \mathcal{M})$ such that $\mathcal{M}(n)_x$ is generated by $H^0(X^h, \mathcal{M}(n))$."

Choose E: hyperplane passing through $x \Rightarrow 0 \to \mathcal{H}(-1) \to \mathcal{H} \to \mathcal{H}_E \to 0$

Let $\mathcal{B} := \mathcal{M} \otimes \mathcal{H}_E$ and $\mathcal{C} := \ker(\mathcal{M}(-1) \to \mathcal{M}) \Rightarrow 0 \to \mathcal{C} \to \mathcal{M}(-1) \to \mathcal{M} \to \mathcal{B} \to 0$.

Twisting by n and let $\mathcal{P}_n := \ker(\mathcal{M}(n) \to \mathcal{B}(n))$, we split this into two short exact sequences: $0 \to \mathcal{C}(n) \to \mathcal{M}(n-1) \to \mathcal{P}_n \to 0$ and $0 \to \mathcal{P}_n \to \mathcal{M}(n) \to \mathcal{B}(n) \to 0$.

By definition, $\mathcal{B}, \mathcal{C} \in Ob(Coh_{E^h})$, so <u>Step 3</u> $\Rightarrow \exists\, n_0$ such that $H^2(X^h, \mathcal{C}(n)) = 0$ and $H^1(X^h, \mathcal{B}(n)) = 0 \ \forall n \geq n_0$. Hence, for sufficiently large n, we have the following inequality:

$h^1(X^h, \mathcal{M}(n-1)) \geq h^1(X^h, \mathcal{P}_n) \geq h^1(X^h, \mathcal{M}(n))$ (recall that all these cohomologies are finite dimensional $\mathbb{C}$-vector spaces by part 1).) In other words, the dimension of $H^1(X^h, \mathcal{M}(n))$ decreases as $n \geq n_0$ increases.

$\Rightarrow \exists\, n_1 (\geq n_0)$ such that $h^1(X^h, \mathcal{M}(n))$ is constant for $n \geq n_1$. Moreover, it equals to $h^1(X^h, \mathcal{P}_n) \Rightarrow H^1(X^h, \mathcal{P}_n) \twoheadrightarrow H^1(X^h, \mathcal{M}(n))$ and hence it's an isomorphism.

$\Rightarrow H^0(X^h, \mathcal{M}(n)) \twoheadrightarrow H^0(X^h, \mathcal{B}(n)) \ \forall n \geq n_1$.

$\mathcal{B} \in Ob(Coh_{E^h}) \Rightarrow \exists\, \mathcal{G} \in Ob(Coh_E)$ such that $\mathcal{G}^h \simeq \mathcal{B}$ by induction hypothesis.

$\Rightarrow \mathcal{G}(n)_x$ is generated by $H^0(X, \mathcal{G}(n)) \underset{1)}{=} H^0(X^h, \mathcal{G}^h(n)) = H^0(X^h, \mathcal{B}(n))$ for $n \gg 0$. Now $\mathcal{B}(n)_x = \mathcal{M}(n)_x \otimes_{\mathcal{H}_x} \mathcal{H}_{x,E} = \mathcal{M}(n)_x \otimes_{\mathcal{H}_x} \mathcal{H}_x / \mathcal{A}_{x,E} = \mathcal{M}(n)_x / \mathcal{A}_{x,E}\mathcal{M}(n)_x$ and

$\overline{\langle H^0(X^h, \mathcal{M}(n)) \rangle}$ generates $\mathcal{M}(n)_x / \mathcal{A}_{x,E}\mathcal{M}(n)_x \Rightarrow \mathcal{M}(n)_x = \langle H^0(X^h, \mathcal{M}(n)) \rangle + \mathcal{A}_{x,E}\mathcal{M}(n)_x$

$\Rightarrow \mathcal{M}(n)_x = \langle H^0(X^h, \mathcal{M}(n)) \rangle + m_x^h \mathcal{M}(n)_x$. Nakayama lemma $\Rightarrow \mathcal{M}(n)_x = \langle H^0(X^h, \mathcal{M}(n)) \rangle$.

Step 5. By Step 4, $\exists\ \mathcal{H}^p \longrightarrow \mathcal{M}(n) \longrightarrow 0$. Twisting by $-n$, we have a short exact sequence $0 \to \underset{\text{kernel}}{\mathcal{R}} \to (\mathcal{H}(-n))^p \to \mathcal{M} \to 0$.

$(\mathcal{H}(-n))^p = (\underset{\mathcal{L}_0}{\mathcal{O}(-n)^p})^h$. Similarly we can find $\mathcal{L}_1^h \to \mathcal{R} \to 0 \Rightarrow \mathcal{L}_1^h \xrightarrow{f} \mathcal{L}_0^h \to \mathcal{M} \to 0$

$(\ )^h$ is fully faithful $\Rightarrow \exists\ g : \mathcal{L}_1 \longrightarrow \mathcal{L}_0$ such that $g^h = f \Rightarrow \mathcal{M} \simeq (\operatorname{cok} g)^h$ □

Corollary. (Chow's theorem) Every closed analytic subspace of projective space is algebraic.

Pf. Let $X := \mathbb{P}^r_{\mathbb{C}}$. $Y \subseteq X^h$: closed analytic subspace.

Oka $\Rightarrow \mathcal{H}_Y = \mathcal{H}_X / \mathcal{A}_Y$ is coherent over $\mathcal{H}_X \Rightarrow \exists\ \mathcal{F} \in Ob(Coh_X)$ such that $\mathcal{F}^h \simeq \mathcal{H}_Y$

$Y = \operatorname{Supp} \mathcal{H}_Y = \operatorname{Supp} \mathcal{F}^h = \operatorname{Supp} \mathcal{F}$ which is Zariski closed. □

§. Appendix : Oka's coherence theorem & Analytic nullstellensatz.

First we need the following two theorems :

Thm. (Weierstrass preparation theorem) Let g be a holomorphic function defined near the origin of $\mathbb{C}^n$ s.t. $\frac{g(0, z_n)}{z_n^s}$ has a non-zero finite limit at $z_n = 0$.

One can write $g(z) = u(z) \cdot P(z', z_n)$ where u is an invertible holo. func. on a nbd. of $\{|z'| \le r'\} \times \{|z_n| \le r_n\}$ for some $r', r_n > 0$ and P is of the form :

$P(z', z_n) = z_n^s + a_1(z') z_n^{s-1} + \cdots + a_s(z')$ (Weierstrass polynomial) and a_k is holo. on a nbd. of $\{|z'| \le r'\} \subseteq \mathbb{C}^{n-1}$, $a_k(0) = 0$.

Pf. $\exists r_n > 0$ s.t. $g(0, \ldots, 0, z_n) \ne 0$ for $0 < |z_n| \le r_n$.

Since g is continuous and $\{|z_n| = r_n\}$ is compact, $\exists r' > 0$ and $\varepsilon > 0$ s.t. $g(z', z_n) \ne 0$ for $\begin{cases} |z'| \le r' \\ |z_n - r_n| \le \varepsilon \end{cases}$.

Now consider $S_k(z') := \frac{1}{2\pi i} \int_{|z_n| = r_n} \frac{1}{g(z', z_n)} \cdot \frac{\partial g}{\partial z_n} \cdot z_n^k \, dz_n$ $(k \in \mathbb{N})$ which is holo. in a nbd. of $\{|z'| \le r'\}$.

First note that $S_0(z')$ is the number of roots z_n of $g(z', z_n) = 0$ in $\{|z_n| < r_n\}$
$\Rightarrow S_0(z') \equiv s$, say the roots are $w_1(z'), \ldots, w_s(z')$.

Argument principle $\Rightarrow S_k(z') = \sum_{j=1}^{s} w_j(z')^k$. Let $C_k(z')$ be the elementary symmetric func. of degree k in $w_1(z'), \ldots, w_s(z')$. Since $C_k(z')$ is a polynomial in $S_1(z'), \ldots, S_k(z')$, $C_k(z')$: holo. in a nbd. of $\{|z'| \le r'\}$.

Define $P(z', z_n) := z_n^s - C_1(z') z_n^{s-1} + \cdots + (-1)^s C_s(z') = \prod_{j=1}^{s} (z_n - w_j(z'))$. Then for $|z'| \le r'$, $u := \frac{g}{P}$ is holo. in z_n on $|z_n| < r_n + \varepsilon$ because g and P has exactly the same zeros with the same multiplicities. Also $u(z', z_n)$ is holo. in z' for $|z_n - r_n| \le \varepsilon$ $\Rightarrow$ u is holo. in z on a nbd. of the polydisk $\{|z'| \le r'\} \times \{|z_n| \le r_n\}$.

$\Rightarrow g(z) = u(z) \cdot P(z', z_n)$ satisfies the required conditions. □

Thm. (Weierstrass division theorem) Every bounded holo. func. f on $\Delta := \{|z'| < r'\} \times \{|z_n| < r_n\}$ can be represented in the form $f(z) = g(z) \cdot q(z) + R(z', z_n)$ (g as above) where q, R are analytic in Δ, $R(z', z_n)$ is a poly. in z_n, of deg. $\le s-1$ and $\sup_{\Delta} |q| \le C \sup_{\Delta} |f|$, $\sup_{\Delta} |R| \le C \sup_{\Delta} |f|$ for some constant $C \ge 0$ (indep. of f).

The representation is unique.

Pf. By the preparation thm., it suffices to prove for the case that $g(z) = P(z', z_n)$ is a W-poly.

- Uniqueness: If $f = Pq_1 + R_1 = Pq_2 + R_2$, then $P(q_2 - q_1) + (R_2 - R_1) = 0$ $\Rightarrow$ s roots z_n of $P(z', \cdot)$ are zeros of $R_2 - R_1$, whose degree in z_n $\leq s-1$ $\Rightarrow R_2 \equiv R_1 \Rightarrow q_2 \equiv q_1$.
- Existence: Set $q(z', z_n) := \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{|w_n| = r_n - \varepsilon} \frac{f(z', w_n)}{P(z', w_n)(w_n - z_n)} dw_n \quad (z \in \Delta)$

When ε is sufficiently small, the integral doesn't depend on $\varepsilon \Rightarrow q$ is holo. on Δ. Define $R := f - Pq$, which is also holo. on Δ and

$$R = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{|w_n| = r_n - \varepsilon} \left(\frac{f(z', w_n)}{P(z', w_n)}\right) \underbrace{\left(\frac{P(z', w_n) - P(z', z_n)}{w_n - z_n}\right)} dw_n$$

$\underbrace{\quad} = \frac{(w_n^s - z_n^s) + \sum_{j=1}^{s} a_j(z')(w_n^{s-j} - z_n^{s-j})}{w_n - z_n}$ is a poly. in z_n of deg. $\leq s-1$ with coeff.s holo. funcs in z'.

$\Rightarrow f = Pq + R$ and $\sup_\Delta |R| \leq C_1 \sup_\Delta |f|$ for some const. C_1, depending on bounds for $a_j(z')$ and on $\mu := \min |P(z', z_n)|$ on $\{|z'| \leq r'\} \times \{|z_n| = r_n\}$.

Apply the max. principle to $q = \frac{f - R}{P}$ on each disk $\{z'\} \times \{|z_n| < r_n - \varepsilon\}$, we get $\sup_\Delta |q| \leq \frac{1 + C_1}{\mu} \sup_\Delta |f|$. □

Now let M be a n-dim'l complex analytic manifold. $\mathcal{O}_M :=$ the sheaf of germs of holo. func. on M.

Theorem (Oka) $\mathcal{O}_M$ is coherent as a $\mathcal{O}_M$-module.

Pf. Induct on $\dim_{\mathbb{C}} M = n$. For $n = 0$, $\mathcal{O}_{M,x} = \mathbb{C}$ and the result is trivial.

Let $n \geq 1$. We need to show that for $U \underset{open}{\subseteq} M$, $F_1, \ldots, F_q \in \mathcal{O}_M(U)$, the sheaf $Rel(F_1, \ldots, F_q)$ is locally finitely generated. May assume $U = \Delta' \times \Delta_n$. Changing coordinates and multiplying by invertible holo. func.s, we can assume that $F_1, \ldots, F_q$ are W-poly.s in z_n, with coeff.s in $\mathcal{O}(\Delta')$.

Lemma! If $x = (x', x_n) \in \Delta$, the $\mathcal{O}_{\Delta, x}$-module $Rel(F_1, \ldots, F_q)_x$ is generated by those of its elements, whose components are germs of analytic polynomials in $\mathcal{O}_{\Delta', x'}[z_n]$ with degrees in $z_n \leq \mu := \max\{\deg_{z_n} F_1, \ldots, \deg_{z_n} F_q\}$.

Lemma 2. Let $P, F \in \mathcal{O}_{n-1}[z_n]$. P: W-poly. If $P | F$ in $\mathcal{O}_n$, then $P|F$ in $\mathcal{O}_{n-1}[z_n]$

Pf. Assume $F(z', z_n) = P(z', z_n) \cdot h(z)$ for some $h \in \mathcal{O}_n$. Consider the standard division of F by P in $\mathcal{O}_{n-1}[z_n]$, $F = P\underset{\underset{\mathcal{O}_{n-1}[z_n]}{\wedge}}{Q} + R$ with $\deg_{z_n} R < \deg_{z_n} P$.

Then the uniqueness of Weierstrass division thm. yields $h(z) = Q$ and $R \equiv 0$. □

Lemma 3 Let $P(z', z_n)$ be a W-poly. Then : a) If $P = P_1 \cdots P_N$ with $P_j \in \mathcal{O}_{n-1}[z_n]$, then all P_j are W-poly.s up to invertible elements of $\mathcal{O}_{n-1}$ b) $P(z', z_n)$ is irr. in $\mathcal{O}_n \Leftrightarrow$ it's irr. in $\mathcal{O}_{n-1}[z_n]$.

Pf. a) Assume P_j are of degree s_j and $\sum_{j=1}^{N} s_j = \overset{\deg P}{\overset{\shortparallel}{s}}$. We may also assume that all of them are monic and $s_j > 0$ for all j.

$P(0, z_n) = z_n^s = P_1(0, z_n) \cdots P_N(0, z_n) \Rightarrow P_j(0, z_n) = z_n^{s_j}$ and hence P_j: W-poly.

b) "$\Rightarrow$" follows from a). "$\Leftarrow$": Assume that P is reducible in $\mathcal{O}_n$, say $P(z', z_n) = g_1(z) \cdot g_2(z)$ for non-invertible $g_1, g_2 \in \mathcal{O}_n$. Then $g_1(0, z_n)$ and $g_2(0, z_n)$ have vanishing orders $s_1, s_2 > 0$ with $s_1 + s_2 = \overset{\shortparallel}{s}$.

Weierstrass preparation thm. $\Rightarrow g_j = u_j P_j$, $\deg_{z_n} P_j = s_j$ $(j = 1, 2)$, where P_j: W-poly. and $u_j \in \mathcal{O}_n$: invertible $\Rightarrow P_1 P_2 = uP$ for some invertible $u \in \mathcal{O}_n$.

Lemma 2 $\Rightarrow P | P_1 P_2$ in $\mathcal{O}_{n-1}[z_n]$. Since P_1, P_2: monic and $s = s_1 + s_2$, $P = P_1 P_2$ i.e. P is reducible in $\mathcal{O}_{n-1}[z_n]$. □

Pf (of Lemma 1) WLOG, we can assume that $\deg F_q = \mu$.

Write $F_{q,x} = f' \cdot f''$ where $f', f'' \in \mathcal{O}_{\Delta', x'}[z_n]$, f': W-poly. in $z_n - x_n$, $f''(x) \neq 0$

Let $\mu' := \deg_{z_n} f'$, $\mu'' := \deg_{z_n} f''$.

Given $(g^1, \ldots, g^q) \in Rel(F_1, \ldots, F_q)_x$, the Weierstrass division thm. yields that $g^j = F_{q,x} t^j + r^j$ $(j = 1, \ldots, q-1)$ where $t^j \in \mathcal{O}_{\Delta, x}$ and $r^j \in \mathcal{O}_{\Delta', x'}[z_n]$: of deg $< \mu'$.

Define $r^q := g^q + \sum_{j=1}^{q-1} t^j F_{j,x} \Rightarrow (g^1, \ldots, g^q) = \sum_{j=1}^{q} t^j (0, \ldots, \underset{j\text{-th}}{F_q}, \ldots, 0, -F_j)_x + (r^1, \ldots, r^q)$. Since these tuples are in $Rel(F_1, \ldots, F_q)_x$, so is $(r^1, \ldots, r^q)$, that is,

$\underbrace{\sum_{j=1}^{q-1} F_{j,x} \cdot r^j}_{\text{a poly. in } z_n,\ \deg < \mu + \mu'} + f' \cdot f'' \cdot r^q = 0 \overset{\text{Lemma 3}}{\Longrightarrow} f'' r^q$ is a poly. in z_n of deg $< \mu$.

Now $(r^1, \ldots, r^q) = \frac{1}{f''}(f'' r^1, \ldots, f'' r^q)$ and each $f'' r^j$ is of deg. $< \mu' + \mu'' = \mu$. □

Pf (of Oka Theorem) Let $(g^1, \dots, g^q)$ be one of the poly.s in $Rel(F_1, \dots, F_q)$ described in Lemma 3. Write $g^j = \sum_{k=0}^{\mu} u^{jk} z_n^k$ $(u^{jk} \in \mathcal{O}_{\Delta', x'})$

$(g^1, \dots, g^q) \in Rel(F_1, \dots, F_q)_x$ gives $2\mu+1$ linear conditions for (u^{jk}) with coeff.s in $\mathcal{O}(\Delta')$. Induction hypothesis $\Rightarrow$ $\mathcal{O}_{\Delta'}$ is coherent.

Consider the following diagram:

$$\begin{array}{ccccccc} 0 \to & Rel(F_1, \dots, F_q) & \longrightarrow & \mathcal{O}^{\oplus q}_{\Delta' \times \Delta_n} & \xrightarrow{(F_1, \dots, F_q)} & \mathcal{O}_{\Delta' \times \Delta_n} \\ & \cup\!\mid & & \cup\!\mid & \circlearrowright & \cup\!\mid \\ 0 \to & \text{"}Rel'(F_1, \dots, F_q)\text{"} & \longrightarrow & \mathcal{O}_{\Delta'}[z_n]^{\oplus q}_{\le \mu} & \xrightarrow{(F_1, \dots, F_q)} & \mathcal{O}_{\Delta'}[z_n] \\ & & & \mid\!\wr & & \end{array}$$

Locally finitely generated $\leftarrow$ $\mathcal{O}_{\Delta'}^{\oplus K}$ (for some K) $\Rightarrow$ it's coherent. $\Rightarrow$ $\exists$ a nbd. Ω' of 0 s.t. the relation modules are generated over $\mathcal{O}_{\Delta', x'}$ for all x' near 0.

Lemma 1 $\Rightarrow$ generators of $Rel(F_1, \dots, F_q)$ can be chosen to be in the bottom row. and hence $Rel(F_1, \dots, F_q)$ is locally f.g. (Explicitly, if "$Rel'(F_1, \dots, F_q)$" is locally generated by finitely many $(q \times u)$-tuples $U_1, \dots, U_N$, then $Rel(F_1, \dots, F_q)$ is generated by the germs of $G_l(z) := \left(\sum_{k=0}^{\mu} U_l^{jk}(z') z_n^k\right)_{1 \le j \le q}$ at every point $x \in \Omega' \times \Delta_n$). $\square$

We now turn to the analytic nullstellensatz.

Theorem (Analytic nullstellensatz) $\forall$ ideal $\mathcal{I} \subseteq \mathcal{O}_n$, $\mathcal{I}_{V(\mathcal{I}), 0} = \sqrt{\mathcal{I}}$.

Pf We first reduce to the case of prime ideals.

"$\subseteq$": $\sqrt{\mathcal{I}} = \bigcap_{\mathcal{I} \subseteq \mathcal{P}} \mathcal{P}$. For such prime ideal $\mathcal{P}$, $(V(\mathcal{I}), 0) \supseteq (V(\mathcal{P}), 0)$

$\Rightarrow \mathcal{I}_{V(\mathcal{I}), 0} \subseteq \mathcal{I}_{V(\mathcal{P}), 0} = \mathcal{P} \Rightarrow \mathcal{I}_{V(\mathcal{I}), 0} \subseteq \bigcap_{\mathcal{I} \subseteq \mathcal{P}} \mathcal{P} = \sqrt{\mathcal{I}}$.

"$\supseteq$": If $f^k \in \mathcal{I}$, then $f^k = 0$ on B and $f \in \mathcal{I}_{B, 0}$.

So it suffices to show that: "if $\mathcal{I} \in \operatorname{Spec} \mathcal{O}_n$ and $A := V(\mathcal{I})$, then $\mathcal{I}_{A, 0} = \mathcal{I}$."

Also, the inclusion "$\supseteq$" is obvious as above. We now need the following:

Lemma 1 $\exists\, d \in \mathbb{Z}$, a basis $(e_1, \dots, e_n)$ of $\mathbb{C}^n$ and associated coordinates $(z_1, \dots, z_n)$ such that $\mathcal{I} \cap \mathbb{C}\{z_1, \dots, z_d\} = 0$ and $\forall\, k = d+1, \dots, n$, $\exists$ a W-poly. $P_k \in \mathcal{I} \cap \mathbb{C}\{z_1, \dots, z_k\}$ of the form $P_k(z', z_k) = z_k^{s_k} + \sum_{j=1}^{s_k} a_{j,k}(z') z_k^{s_k - j}$ $(a_{j,k}(z') \in \mathcal{O}_{k-1})$ where $a_{jk}(z') = O(|z'|^j)$

Moreover, $(e_1, \dots, e_n)$ can be chosen arbitrarily close to any preassigned basis $(e_1^\circ, \dots, e_n^\circ)$

Pf. Induct on n. Set $\mathcal{I}_k := \mathcal{I} \cap \mathbb{C}\{z_1, \dots, z_k\}$. If $\mathcal{I} = \mathcal{I}_n = 0$, there's nothing to prove.

Otherwise, find $g_n^{*^0} \in \mathcal{I}$ and a vector e_n s.t. $\mathbb{C} \to \mathbb{C}$, $w \mapsto g_n(we_n)$ has minimum vanishing order s_n.

This excludes at most $\{ g_n^{(s_n)}(v) = 0 \}$, so e_n can be taken arbitrarily close to e_n^0. Let $(\tilde{z}_1, \dots, \tilde{z}_{n-1}, z_n)$ be the associated coordinate of the basis $(e_1^0, \dots, e_{n-1}^0, e_n)$

By the preparation thm., we may assume that g_n is a W-poly. $P_n(\tilde{z}, z_n) = z_n^{s_n} + \sum_{j=1}^{s_n} a_{j,n}(\tilde{z}) z_n^{s_n - j}$ $(a_{j,n} \in \mathcal{O}_{n-1})$, $a_{j,n}(\tilde{z}) = O(|\tilde{z}|^j)$. Now if $\mathcal{I}_{n-1} = 0$ then $d = n-1$ and the construction is finished. Otherwise we apply the induction hypothesis to $\mathcal{I}_{n-1} \lhd \mathcal{O}_{n-1}$. □

This gives that $\exists$ an injection $\mathcal{O}_d = \mathbb{C}\{z_1, \dots, z_d\} \hookrightarrow \mathcal{O}_n/\mathcal{I}$.

<u>Lemma 2</u>. $\mathcal{O}_n/\mathcal{I}$ is a finite integral extension of $\mathcal{O}_d$.

Pf. Let $f \in \mathcal{O}_n$. Dividing it by P_n gives $f = P_n q_n + R_n$ with $R_n \in \mathcal{O}_{n-1}[z_n]$ and $\deg_{z_n} R_n < s_n$. Further divisions yield $R_{k+1} = P_k q_k + R_k$ with $R_k \in \mathcal{O}_k[z_{k+1}, \dots, z_n]$ where $\deg_{z_j} R_k < s_j$ for $j > k$.

$\Rightarrow f = R_d + \sum_{k=d+1}^{n} P_k q_k \Rightarrow \bar{f}$ in $\mathcal{O}_n/\mathcal{I}$ is equal to $R_d \in \mathcal{O}_d[z_{d+1}, \dots, z_n]$

Also, $\mathcal{O}_n/\mathcal{I}$ is f.g. as an $\mathcal{O}_d$-module by the family of monomials $z_{d+1}^{\alpha_{d+1}} \cdots z_n^{\alpha_n}$ with $\alpha_j < s_j$. □

Now, for $f \in \mathcal{I}_{A,0}$, <u>Lemma 2</u> implies that $\bar{f}$ in $\mathcal{O}_n/\mathcal{I}$ satisfies an irreducible equation $\bar{f}^r + b_1(z')\bar{f}^{r-1} + \dots + b_r(z') = 0$ in $\mathcal{O}_n/\mathcal{I}$. (Suppose it's of the lowest degree r). Then if we can show that $b_r(z') \in \mathcal{I} \Rightarrow r = 1$ i.e. $f \in \mathcal{I}$.

We'll prove a series of lemmas:

<u>Lemma 3</u> (local parametrization theorem) $\mathcal{I} \in \operatorname{Spec} \mathcal{O}_n$, $A := V(\mathcal{I})$, $(z', z'') := (z_1, \dots, z_d; z_{d+1}, \dots, z_n)$ is chosen as in <u>Lemma 1</u>. $\mathcal{M}_A$, $\mathcal{M}_d :=$ the quotient fields of $\mathcal{O}_n/\mathcal{I}$ and $\mathcal{O}_d$ resp. $q := [\mathcal{M}_A : \mathcal{M}_d]$

Let $\delta(z') \in \mathcal{O}_d$ be the discriminant of the irr. poly. of $u(z') = \sum_{d<k} c_k z_k$: a primitive element

Consider Δ', Δ'' : poly-disks of sufficiently small radii r', r'' and $r' \le \frac{r''}{C}$ with C large.

The projection $\pi : A \cap (\Delta' \times \Delta'') \longrightarrow \Delta'$, $(z', z'') \longmapsto z'$ is a ramified covering with q sheets whose ramification locus is contained in $S := \{ z' \in \Delta' \mid \delta(z') = 0 \}$. Namely:

a) $A_S := A \cap ((\Delta' \setminus S) \times \Delta'')$ is a smooth d-dim'l manifold.

b) $\pi : A_S \longrightarrow \Delta' \setminus S$ is a covering

c) $\pi^{-1}(z')$ has exactly q elements if $z' \in \Delta' \setminus S$ and at most q elements if $z' \in S$.

Pf $\bar{u} \in \mathcal{O}_n/\mathcal{I}$ is integral over $\mathcal{O}_d$ $\Rightarrow$ $\exists$ a monic irr. poly. $W_u(z';T) = T^q + \sum_{j=1}^{q} a_j(z')T^{q-j}$

with $a_j(z') \in \mathcal{O}_d$. (It must be a W-poly. by the preparation thm!)

We can similarly find irr. W-poly.s $W_k \in \mathcal{O}_d[T]$ such that $W_k(z'; \tilde{z}_k) = 0$ with

degree $= \deg \tilde{z}_k \le q$ $(d+1 \le k \le n)$

Lemma 4 $\forall g \in \mathcal{M}_A$, if g is integral over $\mathcal{O}_d$, then $\delta g \in \mathcal{O}_d[\bar{u}]$.

Pf Let $\sigma_1, \dots, \sigma_q$ be the embeddings of $\mathcal{M}_A$ into $\overline{\mathcal{M}}_A$ over $\mathcal{M}_d$.

$\Rightarrow \delta(z') = \prod_{j<k} (\sigma_k \bar{u} - \sigma_j \bar{u})^2 \neq 0$. Write $g \in \mathcal{M}_A = \mathcal{M}_d[\bar{u}]$ as $g = \sum_{j=0}^{q-1} b_j \bar{u}^j$ $(b_j \in \mathcal{M}_d)$

where $b_0, \dots, b_{q-1}$ are the solutions of the linear system $\sigma_k g = \sum b_j (\sigma_k \bar{u})^j$.

The determinant is $\delta^{\frac{1}{2}}$ $\Rightarrow$ $\delta b_j \in \mathcal{M}_d$ are poly.s in $\sigma_k g$ and $\sigma_k \bar{u}$ $\Rightarrow$ δb_j : integral over

$\mathcal{O}_d$, which is an UFD ($\Rightarrow$ integrally closed) $\Rightarrow$ $\delta b_j \in \mathcal{O}_d$ $\Rightarrow$ $\delta g \in \mathcal{O}_d[\bar{u}]$. □

In particular, $\exists$ $B_{d+1}, \dots, B_n \in \mathcal{O}_d[T]$ with degrees $\le q-1$ s.t. $\delta(z') \cdot z_k = B_k(z', u(z''))$ in $\mathcal{O}_n/\mathcal{I}$.

$\Rightarrow$ $\delta(z')^q \cdot W_k\left(z'; \frac{B_k(z';T)}{\delta(z')}\right) \in \langle W_u(z';T) \rangle_{\mathcal{O}_d[T]}$ since LHS is a poly. in $\mathcal{O}_d[T]$, admitting

$T = \bar{u}$ as a root in $\mathcal{O}_n/\mathcal{I}$.

Now consider $\mathcal{G} := \langle W_u(z'; u(z'')), \delta(z')z_k - B_k(z', u(z'')) \rangle \subseteq \mathcal{I}$, $m := \max\{q, (n-d)(q-1)\}$

Lemma 5. $\forall f \in \mathcal{O}_n$, $\exists!$ poly. $R \in \mathcal{O}_d[T]$ with $\deg_T R \le q-1$ such that $\delta(z')^m f(z) - R(z', u(z'')) \in \mathcal{G}$

Moreover, if $f \in \mathcal{I}$, then $R = 0$ (i.e. $\delta^m \mathcal{I} \subseteq \mathcal{G}$).

Pf. First by above, $\delta(z')^q \cdot W_k(z'; z_k) \in \mathcal{G}$. By division as in Lemma 2, we obtain that

$f = R_d + \sum_{k=d+1}^{n} W_k g_k$ $(R_d \in \mathcal{O}_d[z_{d+1}, \dots, z_n])$ with $\deg_{z_k} R_d < \deg W_k \le q$.

$\Rightarrow$ $\delta^m f - \delta^m R_d \in \mathcal{G}$, so we can replace f by R_d : a poly. in $\mathcal{O}_d[z_{d+1}, \dots, z_n]$,

of total degree $\le (n-d)(q-1) \le m$.

Also, $\exists G \in \mathcal{O}_d[T]$ s.t. $\delta(z')^m f(z) - G(z', u(z'')) \in \langle \delta(z')z_k - B_k(z', u(z'')) \rangle$. A division

$G = \underbrace{W_u}_{\deg_T = q} Q + R$ gives the desired R.

• If $f \in \mathcal{I}$ and $\delta^m(z') f(z) - R(z', u(z'')) \in \mathcal{G}$ for $\deg_T R < q$, then $R(z', \bar{u}) = 0$ since $\bar{u} \in \mathcal{O}_n/\mathcal{I}$

is of deg q $\Rightarrow$ $R = 0$. Uniqueness is similar. □

We come back to the proof of Lemma 3.

After a linear coordinate change in $z_{d+1}, \dots, z_n$, we may assume $u(z'') = z_{d+1}$

$\Rightarrow W_u = W_{d+1}$, $B_{d+1}(z'; T) = \delta(z') T$

Lemma 5 $\Rightarrow \delta^m \mathcal{J} \subseteq \mathcal{G} \subseteq \mathcal{J}$. We claim that we can find a poly-disk $\Delta = \Delta' \times \Delta''$ of sufficiently small radii r', r'' s.t. $V(\mathcal{J}) \subseteq V(\mathcal{G}) \subseteq V(\delta^m \mathcal{J})$ in Δ.

Claim. The germ $(A, 0)$ is contained in a cone $\{|z''| \le C|z'|\}$ for some constant C and the map π is proper if r'' is small enough and $r' \le \frac{r''}{C}$.

First, if $w \in \mathbb{C}$ is a root of $w^d + a_1 w^{d-1} + \cdots + a_d = 0$ ($a_j \in \mathbb{C}$), then $|w| \le 2 \cdot \max |a_j|^{\frac{1}{j}}$. Since otherwise $|w| > 2|a_j|^{\frac{1}{j}}$ for all j and $-1 = \frac{a_1}{w} + \frac{a_2}{w^2} + \cdots + \frac{a_d}{w^d}$

$\Rightarrow 1 \le 2^{-1} + \cdots + 2^{-d}$ ※.

Now the poly.s $P_k(z_1, \dots, z_{k-1}; z_k)$ (in Lemma 1) vanish on $(A, 0)$, so every pt. $z \in A$ sufficiently close to 0 satisfies $|z_k| \le C_k(|z_1| + \cdots + |z_{k-1}|)$ $(d+1 \le k \le n)$

$\Rightarrow |z''| \le C|z'|$.

By definition, $V(\delta) \cap \Delta = S \times \Delta'' \Rightarrow A \cap \Delta \subseteq V(\mathcal{G}) \cap \Delta \subseteq (A \cap \Delta) \cup (S \times \Delta'')$.

In particular $A_S = A \cap ((\Delta' \setminus S) \times \Delta'')$ coincides with $V(\mathcal{G}) \cap ((\Delta' \setminus S) \times \Delta'')$.

The latter is the set of points $z \in \Delta$ parametrized by $\begin{cases} \delta(z') \ne 0, \ z_k = \frac{B_k(z', z_{d+1})}{\delta(z')} \\ W_{d+1}(z', z_{d+1}) = 0 \end{cases}$ (*)

for all $d+2 \le k \le n$.

As $\delta(z')$ is the resultant of W_{d+1} and $\frac{\partial W_{d+1}}{\partial T}$, we have $\frac{\partial W_{d+1}}{\partial T}(z', z_{d+1}) \ne 0$ on A_S

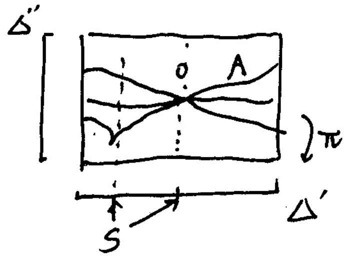
IFT $\Rightarrow z_{d+1}$ is locally a holo. func. of z' on A_S, and the same is true for $z_k (k \ge d+2)$

$\Rightarrow A_S$: smooth mfd. and $\pi : A_S \longrightarrow \Delta' \setminus S$ is a proper local diffeo. for $r' \le \frac{r''}{C}$: small.

By (*), $\pi^{-1}(z')$ have at most q points (corresponding to some of the q roots of $W_{d+1}(z', w) = 0$). Since $\Delta' \setminus S$ is connected, the analytic continuation $\Rightarrow$ either $A_S = \emptyset$ or π is a covering of constant sheet number $q' \le q$.

Consider the equation $W_{d+1}(z', w) = 0$ with $z' \in \Delta' \setminus S$. (*) $\Rightarrow z_{d+1}$ is a root, and $z_k = \frac{B_k(z', z_{d+1})}{\delta(z')}$ $(k \ge d+2)$ satisfies $W_k(z', z_k) = 0$ by Lemma 4. In particular, $|z_k| = O(|z'|^{\frac{1}{q}})$ (as in the Claim above). For z' with $|z'|$ small enough, the q points $(z', z'') = z$ defined in this way lie in $\Delta \Rightarrow q' = q$.

This proves some parts of Lemma 3, including the first part of c).



Now, on $\Delta' \setminus S$, for each z', there exists points $z = (z', z'')$ on A. This gives that $b_r(z') = 0$ on $\Delta' \setminus S$ and hence on whole Δ'. View $b_r(z')$ as an element in $\mathcal{O}_n/\mathcal{I}$.

$\Rightarrow \overline{b_r} = 0$. This proves the assertion of the analytic nullstellensatz. $\square$