Cohomology classes

Let X be a smooth (complete) variety of const. dim. d / $k=\bar{k}$,

$j: Y \to X$ a closed subvariety of dim s

Consider $\alpha: H_c^{2d-2s}(X, \mu_n^{d-s}) \xrightarrow{j^*} H_c^{2d-2s}(Y, \mu_n^{d-s}) \cong H_c^{2d-2s}(Y_{reg}, \mu_n^{d-s}) \xrightarrow{\int_{Y_{reg}}} \Lambda$

The image of α under the duality $\mathrm{Hom}(H_c^{2d-2s}(X, \mu_n^{d-s}), \Lambda) \cong H^{2s}(X, \mu_n^{s})$

is called the cohomology class associated with $Y \subseteq X$, denoted by $cl_X(Y)$,

i.e., $\int_X (cl(Y) \cup v) = \int_Y (j^*(v))$, $\forall v \in H_c^{2d-2s}(X, \mu_n^{d-s})$

Extended linearly, $cl_X: Z^s(X) \to H^{2s}(X, \mu_n^s)$

Now we want to calculate $cl_{X \times X}(D)$ for the diagonal $D \subseteq X \times X$

We ~~identify~~ $\mu_{nX} \cong \Lambda_X$. Let $p, q: X \times X \to X$ be the two projections,

$j: X \to X \times X$, $j(X) = D$.

Assume $H^v(X, \Lambda)$ are all free Λ-module, we choose a basis

$\{e_i^v\}_i$ for $H^v(X, \Lambda)$, and $\{f_i^v\}_i$ the dual basis in $H^{2d-v}(X, \Lambda)$

i.e. $\int_X (e_i^v \cup f_j^v) = \delta_{ij}$.

Note ~~that~~ by Kunneth formula, $H^{\cdot}(X) \otimes H^{\cdot}(X) \cong H^{\cdot}(X \times X)$

through $\alpha \otimes \beta \mapsto p^*(\alpha) \cup q^*(\beta)$

Prop. $cl_{X \times X}(D) = \sum_{i,v} p^*(f_i^v) \cup q^*(e_i^v)$

Pf. Let $\alpha = \sum_{i,v} p^*(f_i^v) \cup q^*(e_i^v)$, we verify $\int_{X \times X} (\alpha \cup u) = \int_X (j^*(u))$,

$\forall u \in H^{2d}(X \times X, \Lambda)$. For $u = p^*(e_r^\mu) \cup q^*(e_s^\mu)$,

$$\int_{X \times X} (\alpha \cup u) = \sum_{i,v} \int_{X \times X} (p^*(f_i^v) \cup q^*(e_i^v) \cup p^*(e_r^\mu) \cup q^*(f_s^\mu))$$

$$= \sum_i \int_{X \times X} (p^*(e_r^\mu \cup f_i^\mu) \cup q^*(e_i^\mu \cup f_s^\mu)) = \sum_i \delta_{ri} \delta_{is} = \delta_{rs}$$

(Since if $v \neq \mu$, one of $H^{2d \pm (v-\mu)}(X, \Lambda)$ is zero)

While $\int_X (j^*(u)) = \int_X ((p \circ j)^* e_r^\mu \cup (q \circ j)^* f_s^\mu)$

$$= \int_X (e_r^\mu \cup f_s^\mu) = \delta_{rs}$$

Now suppose $h: X \to X$, $\Gamma_h : X \to X \times X$, $h^*(e_i^\nu) = \sum_j c_{ji}^\nu e_j^\nu$

$x \to (x, h(x))$

Then $S_{X\times X}(cl_{X\times X}(\Gamma_h) \smile cl_{X\times X}(D))$

$= S_X(\Gamma_h^*(cl_{X\times X}(\Gamma))) = \sum_\nu (-1)^\nu \sum S_X(h^*(e_i^\nu) \smile f_i^\nu)$

$= (-1)^\nu \sum_{i,j} c_{ji}^\nu \delta_{ij} = \sum_\nu (-1)^\nu \operatorname{Trace}(h^* | H^\nu(X, \Lambda))$

(Note that $p^*(f_i^\nu) \smile q^*(e_i^\nu) = (-1)^{\nu(2d-\nu)} p^*(e_i^\nu) \smile q^*(f_i^\nu)$)

Thm. $(\Gamma \cdot D) = \sum (-1)^\nu \operatorname{Trace}(h^* | H^\nu(X, \mathbb{Q}_\ell))$

$= \#$-fixed pts of Γ, counted with multiplicity.

Suppose X is a projective nonsingular variety over $\mathbb{F}_q$.

Take $h = Fr^r$ the Frobenius morphism (sending $p \in X$ with coordinate (a_i) to $Fr(p)$ with coordinate (a^{q^r})), then Γ and D intersect transversally in $X \times X$

(Suppose p is fixed pt, then $(dFr^r)_p = 0$ has no eigenvalue 1, so $T_{(p,p)}\Gamma$, which is the graph of $(dFr^r)_p : T_pX \to T_pX$ is transversal to $T_{(p,p)}D$, the graph of $(d\,\mathrm{Id})_p$)

Thus $N_r = \#\{\text{fixed pt of } Fr^r\} = \sum (-1)^\nu \operatorname{Tr}(Fr^{r*} | H^\nu(X, \mathbb{Q}_\ell))$ is the number of points with coordinates lies in $\mathbb{F}_{q^r}$

Define $Z(t) = \exp(\sum_{r\geq 1} N_r \cdot \frac{t^r}{r})$ the zeta function of X

Prop. $Z(t) = P_1(t) P_3(t) \cdots P_{2d-1}(t) / P_0(t) P_2(t) \cdots P_{2d}(t)$

with $P_\nu(t) = \det(1 - Fr^* t | H^\nu(X, \mathbb{Q}_\ell))\frac{t^r}{r})$

Pf. $Z(t) = \exp(\sum_{r=1}^{\infty} \sum_{\nu=0} (-1)^\nu \operatorname{Tr}((Fr^*)^r | H^\nu(X, \mathbb{Q}_\ell)) \frac{t^r}{r}$

$= \prod_{\nu=0}^{2d} \exp(\sum_{r=1} \operatorname{Tr}((Fr^*)^r | H^\nu(X, \mathbb{Q}_\ell)) \frac{t^r}{r})^{(-1)^\nu}$

$= \prod_{\nu=0}^{2d} P_\nu(t)^{(-1)^\nu}$

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Rationality

Lemma. Let $k < K$ fields, then $k[[t]] \cap K(t) = k(t)$ $(\subseteq K[[t]])$

Pf. let $f(t) = \sum_{i=0}^{\infty} a_i t^i \in k[[t]] \cap K(t)$

$$H_n^{(k)} \triangleq \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+k-1} \\ a_{n+1} & & & \\ & & \ddots & \\ a_{n+k-1} & & & a_{n+2k-2} \end{vmatrix}$$

If for some r, d and all $j \geq 0$, $H_{d+j}^{(r+1)} = 0$ and $H_{d+j}^{(r)} \neq 0$

then $f(t) \in k(t)$ (Since we can find $\lambda_i \in k$, $i = 0, \cdots, r$, s.t.

$\lambda_0 a_n + \lambda_1 a_{n+1} + \cdots \lambda_r a_{n+r} = 0 \quad \forall n \geq d$)

Using $H_n^{(k)} H_{n+2}^{(k)} = (H_{n+1}^{(k)})^2 - H_n^{(k+1)} H_{n+2}^{(k-1)}$

(In general, if A is an $k \times k$ matrix, then

$\det A_{11} \det A_{kk} = \det A_{1k} \det A_{k1} - \det A \det B$.

where A_{ij} is the submatrix of A removing ith row & jth column

B is the submatrix of A removing 1st & kth row & column

Pf. Notice that left multiplying A by $\begin{pmatrix} 1 & \\ C & \\ & 1 \end{pmatrix}$ does not affect the truth of the formula, thus we may assume $B = \begin{pmatrix} Id & 0 \\ 0 & 0 \end{pmatrix}$, and the rest is trivial)

$H_{d+j}^{(r+1)} = 0 \ \forall j \geq 0 \Rightarrow H_{d+j}^{(r)}$ all zero or all nonzero $\forall j \geq 1$,

exist such r, d, since $f(t) \in K(t)$

Cor. $Z(t)$ is a rational function over $\mathbb{Q}$ (Since $Z(t) \in \mathbb{Q}[[t]] \cap \mathbb{Q}_\ell(t)$)

* This does not implies $P_\nu(t) \in \mathbb{Q}(t) \ \forall \nu$

Rm. Grothendieck Trace Formula

Let X_0 smooth / k finite, $\mathcal{G}_0 = (\mathcal{G}_{0,\nu})$ constructible ℓ-adic sheaf on X,

$Fr_{X_0} : X_0 \to X_0$, $Fr_{\mathcal{G}_0} : Fr_{X_0}^*(\mathcal{G}_0) \to \mathcal{G}_0$

X, $\mathcal{G}$, Fr_X, $Fr_{\mathcal{G}}$ the base change of X_0, $\mathcal{G}_0$, Fr_{X_0}, $Fr_{\mathcal{G}_0}$ to $\bar{k}$,

then $\displaystyle\sum_{a \in Fix(Fr_X^d)} Tr(Fr_{\mathcal{G}}^d{}_{a} : \mathcal{G}_a \to \mathcal{G}_a) = \sum_{\nu} (-1)^\nu Tr((Fr_X^*)^d \mid H_c(X, \mathcal{G}))$