Poincaré Duality

Goal. Let $\Lambda = \mathbb{Z}/n\mathbb{Z}$, X smooth (connected) projective scheme of const dim d over $k = \bar{k}$, n invertible in k.

Expect an perfect pairing:

$$\operatorname{Ext}^p(\mathcal{F}, \mu_{nX}^{d}) \times H^{2d-p}(X, \mathcal{F}) \to \Lambda$$

If X is not complete, $H^{2d-p}(X, \mathcal{F})$ should be replaced with $H_c^{2d-p}(X, \mathcal{F})$

$\mathcal{F}$ constructible sheaf of Λ-module

(If $\mathcal{F}$ is locally const., then $\operatorname{Ext}^p(\mathcal{F}, \mu_{nX}^{d}) \cong H^p(X, \mathcal{H}om(\mathcal{F}, \mu_n^{d}))$)

Trace Mapping

We want to construct $S = S_{X/k} : H^{2d}(X, \mu_{nX}^{d}) \to \mathbb{Z}/n\mathbb{Z}$,

more generally, for every smooth compactifiable $f: X \to S$ with const fiber dim d, construct $S_{X/S} : R^{2d} f_!(\mu_{nX}^{d}) \to \Lambda_S$, satisfying

1) $S_{X/S}$ compatible with base change:

Given
$$\begin{array}{ccc} X_T & \xrightarrow{f'} & T \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & S \end{array}, \quad \begin{array}{ccc} g^* R^{2d} f_!(\mu_{nX}^{d}) & \cong & R^{2d} f'_!(\mu_{nX_T}^{d}) \\ \downarrow g^*(S_{X/S}) & & \downarrow S_{X_T/T} \\ g^*(\Lambda_S) & \cong & \Lambda_T \end{array}$$

2) $S_{X/S}$ is transitive: Let $g: X \to T$, $h: T \to S$, $f = h \circ g$, define

$$S_{T/S} \,\square\, S_{X/T} : R^{2d(X/S)} f_!(\underline{\mu_{nX}^{d(X/S)}}) \xrightarrow{\cong}$$

(Leray Spec. Seq.) $\xrightarrow{\cong} R^{2d(T/S)} h_! (R^{2d(X/T)} g_! (\underline{\mu_{nX}^{d(X/T)} \otimes g^*(\mu_{nT}^{d(T/S)})}))$

(Projection Formula) $\xrightarrow{\cong} R^{2d(T/S)} h_! (R^{2d(X/T)} g_! (\mu_{nX}^{d(X/T)}) \otimes \mu_{nT}^{d(T/S)})$

$\xrightarrow{\text{"}S_{X/T}\text{"}} R^{2d(T/S)} h_!(\mu_{nT}^{d(T/S)}) \xrightarrow{S_{T/S}} \Lambda_S$

Expect $S_{X/S} \cong S_{T/S} \,\square\, S_{X/T}$

3) If f is etale, then $S_{X/S} : f_! f^*(\Lambda_S) \to \Lambda_S$ is the adjunction mapping

4) If $f: X \to \operatorname{spec} k$ is a smooth curve, then

$S_{X/S} : H^2(X, \mu_{nX}) \to \Lambda$ is given by the Kummer seq.

If X/S has connected and nonempty fiber, then $S_{X/S}$ is an isom.

(Note that $H_c^{2d}(X, \mu_{nX}^{d}) \cong H^{2d}(\bar{X}, \mu_{nX}^{d})$ by the exact seq:

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0$$

$j: X \to \bar{X}$, $i: Y = \bar{X} \setminus X \to \bar{X}$, $\mathcal{F} = \mu_{n\bar{X}}$)

Construction 1) If $X = \mathbb{P}^1_S$, and $\mathcal{L}$ the line bundle corresponding to the zero section, define $(\mathbb{Z}/n\mathbb{Z})_S \xrightarrow{\sim} R^2 f_*(\mu_{nX})$ as $S_{X/S}^{-1}$ $\quad 1 \mapsto cl_{X/S}(\mathcal{L})$

$(cl_{X/S} : \mathrm{Pic}(X) \xrightarrow{cl} H^2(X, \mu_{nX}) \to \Gamma(S, R^2 f_*(\mu_{nX})))$

For $X = \mathbb{A}^1_S$, let $S_{X/S} : R^2 f_!(\mu_{n\mathbb{A}^1_S}) \to R^2 f_*(\mu_{n\mathbb{P}^1_S}) \xrightarrow{S_{\mathbb{P}^1_S/S}} (\mathbb{Z}/n\mathbb{Z})_S$

2) For $f : X \to S$ having factorisation $X \xrightarrow{g} \mathbb{A}^d_S \to S$, where g is etale, let $S_{X/S} = S_{\mathbb{A}^1_S/S} \,\square\, S_{\mathbb{A}^2_S/\mathbb{A}^1_S} \,\square \cdots \square\, S_{\mathbb{A}^d_S/\mathbb{A}^{d-1}_S} \,\square\, S_{X/\mathbb{A}^d_S}$

3) For general smooth $f : X \to S$, there's finite Zariski open cover $\phi_i : U_i \to X$ with the factorisation as in 2), and the compatibility is given by $\bigoplus_{i,j} \phi_{ij!}(\mathcal{F}|U_{ij}) \rightrightarrows \bigoplus_k \phi_{k!}(\mathcal{F}|U_i) \to \mathcal{F} \to 0$

$R^{2d} f_!$ right exact $\bigoplus_{i,j} R^{2d}(f \circ \phi_{ij})(\mathcal{F}|U_{ij}) \to \bigoplus_k R^{2d}(f \circ \phi_i)(\mathcal{F}|U_i) \to R^{2d} f_!(\mathcal{F}) \to 0$

Perfect Pairing:

Consider the cup produce $Ext^p(\mathcal{F}, \mu_X^d) \times H_c^{2d-p}(X, \mathcal{F}) \to H^{2d}(X, \mu_X^d) \cong \Lambda$

we show that $Ext^p(-, \mu_X^p) \xrightarrow{\Delta} Hom(H_c^{2d-p}(X, -), \Lambda)$ is an isomorphism of δ-functor. $(F^\nu = G^\nu = 0 \ \forall \nu < 0)$

Lemma. If $F^\nu, G^\nu : \mathcal{U} \to Ab$ are two contravariant ∂-functors, and $F^\nu \xrightarrow{\Delta} G^\nu$ is a natural transformation, moreover $\mathfrak{M}$ is a class of object in $\mathcal{U}$ s.t.

1) $\forall A \in \mathcal{U}$, $a \in F^\nu(A)$ (or $G^\nu(A)$), $\exists M \in \mathfrak{M}$ and $M \to A$ epi. s.t. the image of a in $F^\nu(A) \to F^\nu(M)$ is zero (G^ν resp.ly)

2) $\forall M \in \mathfrak{M}$, $F^0(M) \to G^0(M)$ is an isom.

Then $F^\nu(A) \to G^\nu(A)$ is an isom, $\forall A \in \mathcal{U}$.

We take $\mathfrak{M}$ to be the family of sheaves $\bigoplus_\alpha j_{\alpha!}(\Lambda_\alpha)$, where $j_\alpha : U_\alpha \to X$ is a family of etale mapping. If $\mathcal{F} = j_!(\Lambda_U)$, then for $p=0$,

$$\Delta : Ext^0(\mathcal{F}, \mu_X^d) = \underset{\underset{\Lambda}{\text{||}}}{\Gamma(U, \mu_U^d)} \to Hom_\Lambda(\underset{\underset{\Lambda}{\text{||}}}{H_c^{2d}(U, \Lambda_U)}, \Lambda)$$

is an isom.

For any constructible $\mathcal{F}$, there's surjection $\bigoplus_\alpha j_{\alpha!}(\Lambda_{U_\alpha}) \to \mathcal{F}$, and we may refine U_α so that $Ext^p(\mathcal{F}, \mu_X^d) \to \bigoplus_\alpha H^p(U_\alpha, \mu_{U_\alpha}^d)$ annihilate some $a \in Ext^p(\mathcal{F}, \mu_X^d)$, using $\varinjlim_{U \ni b} H^p(U, \mu_U^d) = 0$, where b is a geometric point on X.

For the other side, we want to find etale nbhd $V \ni b$ of U_α s.t. $H_c^q(V, \Lambda_V) \to H_c^q(U, \Lambda_U)$ is zero. $\forall q \neq 2d$ For the case of curves, consider the smooth compactification

$$\begin{array}{ccc} V = Y & \to & \bar{Y} \\ \downarrow & & \downarrow \\ U = X & \to & \bar{X} \end{array}$$

and the commutative diagram

$$\begin{array}{ccccccccc} \cdots \to & H^0(F, \mu_n) & \to & H_c^1(Y, \mu_n) & \to & H^1(Y, \mu_n) & \to 0 & & F = \bar{Y} \setminus Y \\ & \downarrow & & \downarrow & & \downarrow & & & \\ \cdots \to & H^0(E, \mu_n) & \to & H_c^1(X, \mu_n) & \to & H^1(X, \mu_n) & \to 0 & & E = \bar{X} \setminus X \end{array}$$

The vertical map on the right is the albanese map,
$Jac(Y) \to Jac(X)$ restricted on n-torsion points
The vertical map on the left is given by

$$H^0(F,\mu_n) = \bigoplus_{p\in F} H^0(p,\mu_n) \to \bigoplus_{\bar p\in E} H^0(\bar p,\mu_n) = H^0(E,\mu_n)$$

$$H^0(p,\mu_n) \begin{cases} \xrightarrow{\cdot e_p} H^0(\bar p,\mu_n) & \text{if } p \mapsto \bar p \in E \\ \to 0 & \text{if } p \text{ maps to } X \end{cases}$$

(e_p ramification degree at p)

We can make these two maps zero (resp.ly) via choosing Y s.t.:

1) $\bar Y \to \bar X$ an etale covering, $Ker(Jac(Y) \to Jac(X))$ contains all n-torsion points
2) $\bar Y \to \bar X$ finite, unramified at b, n divides e_p $\forall p \in E$

Pf. 1) Take Y to be a component of the preimage of X under $Jac(\bar X) \xrightarrow{\cdot n} Jac(\bar X)$
2) Take a rational function ϕ in $K(\bar X)$, regular at b, have simple pole at points in E, consider $K(\bar X)(\sqrt[n]{\phi})/K(\bar X)$

Now consider

$$\begin{tikzcd}
\cdots \arrow[r] & H^0(F',\mu_n) \arrow[r] \arrow[d] & H^1_c(Y',\mu_n) \arrow[r] \arrow[d, "\alpha"] & H^1(Y',\mu_n) \arrow[r] \arrow[d, "0"] & 0 \\
\cdots \arrow[r] & H^0(F,\mu_n) \arrow[r, "\gamma"] \arrow[d, "0"] & H^1_c(Y,\mu_n) \arrow[r, "\delta"] \arrow[d, "\beta"] & H^1(Y,\mu_n) \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & H^0(E,\mu_n) \arrow[r] & H^1_c(X,\mu_n) \arrow[r] & H^1(X,\mu_n) \arrow[r] & 0
\end{tikzcd}$$

$\mathrm{im}\,\alpha \subseteq \ker\delta = \mathrm{im}\,\gamma \subseteq \ker\beta \Rightarrow \alpha\circ\beta = 0$