Last Time:

X: non-sing. proj. curve $/k=\bar{k}$, we have shown:

$H^1(X_{\acute{e}t}, \mu_n) = {}_n\mathrm{Pic}(X) = \{\mathcal{L} \in \mathrm{Pic}(X) : \mathcal{L}^n \cong \mathcal{O}_X\} \subseteq \mathrm{Pic}^0(X)$

For X, one associates an abelian variety J_X: the Jacobian of X. Then $\mathrm{Pic}^0(X) \cong J_X(k)$: closed pt of J_X

Also, $\dim J_X = g(X) = g$. Hence, ${}_n\mathrm{Pic}(X) = \ker(n: J_X \longrightarrow J_X)$
$$P \longmapsto nP$$

$k=\bar{k}$, $\mathrm{char}(k)=p$

Claim: A: abelian variety$/k$ of dim g $n: A \longrightarrow A$, $P \longmapsto nP$ $A_n := \ker(n_A)$

Then for $n \geq 1$, $|A_n| < \infty$ and if $p \nmid n$, then $A_n \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$.

Recall: An abelian variety A/k is a complete alge. gp $/k$.
In fact $\Rightarrow$ A is proj.

Lemma: A: abelian variety $n \in \mathbb{Z}$, then $\forall \mathcal{L}$: invert. sheaf on A
$$n_A^*\mathcal{L} \cong \mathcal{L}^{\otimes \frac{1}{2}n(n+1)} \otimes (-1_A)^*\mathcal{L}^{\otimes \frac{1}{2}n(n-1)}$$

The proof of lemma depends on two theorems:

1. Seasaw theorem: X: complete var. $/k$ T: any variety
$\mathcal{L}$: invert. sheaf. on $X \times T$, then
(i) $T_1 = \{t \in T : \mathcal{L}|_{X \times \{t\}} \cong \mathcal{O}_{X \times \{t\}}\} \subset T$ is closed
(ii) $\exists\, \mathcal{M}$: invert. sheaf on T_1 s.t. $\mathcal{L}|_{X \times T_1} \cong p_2^*\mathcal{M}$.
, where $X \times T_1 \to T_1$ projection.
$\rightsquigarrow$ direct application of coho. and base change.

2. Theorem of Cube: X, Y: complete Z: any var.
$x_0 \in X$, $y_0 \in Y$, $z_0 \in Z$. $\mathcal{L}$: invert. sheaf on $X \times Y \times Z$,
if $\mathcal{L}|_{X \times Y \times \{z_0\}}$, $\mathcal{L}|_{X \times \{y_0\} \times Z}$, $\mathcal{L}|_{\{x_0\} \times Y \times Z}$ is trivial
, then $\mathcal{L}$ is trivial.
$\rightsquigarrow$ By seasaw thm, suffices to prove: $\forall x \in X$, $z \in Z$, $\mathcal{L}|_{\{x\} \times Y \times \{z\}}$ is trivial. Connect x_1 to x_0 by an irred. curve C_1 and consider $C \to C_1$: normalization Reduction to the case X · non-sing. proj. curve.

<u>Cor</u>: X: var. A: abelian var. $f, g, h : X \to A$, then $\forall \mathcal{L}$: invert. sheaf
$(f+g+h)^*\mathcal{L} \cong (f+g)^*\mathcal{L} \otimes (f+h)^*\mathcal{L} \otimes (g+h)^*\mathcal{L} \otimes f^*\mathcal{L}^{-1} \otimes g^*\mathcal{L}^{-1} \otimes h^*\mathcal{L}^{-1}$

<u>pf</u>: $p_i : A \times A \times A \to A$ proj. onto i-th factor.
$m_{ij} = p_i + p_j$, $m = p_1 + p_2 + p_3$. Consider on $A \times A \times A$,
$\mathcal{M} := m^*\mathcal{L} \otimes m_{12}^*\mathcal{L}^{-1} \otimes m_{13}^*\mathcal{L}^{-1} \otimes m_{23}^*\mathcal{L}^{-1} \otimes p_1^*\mathcal{L} \otimes p_2^*\mathcal{L} \otimes p_3^*\mathcal{L}$.
$q : A^2 \to A^3$ $(a, a') \mapsto (0, a', a)$. $(i=2,3)$
$\underset{\substack{\| \\ m_{23}\circ q}}{m \circ q} = \mu$: addition on A, $m_{1i} \circ q = q_i : A \times A \to A$ proj. onto i-th factor.
$p_1 \circ q = 0$, $p_2 \circ q = q_1$, $p_3 \circ q = q_2$
$\rightsquigarrow q^*\mathcal{M} = \mu^*\mathcal{L} \otimes q_1^*\mathcal{L}^{-1} \otimes q_2^*\mathcal{L} \otimes \mu^*\mathcal{L}^{-1} \otimes 0_A^*\mathcal{L} \otimes q_1^*\mathcal{L} \otimes q_2^*\mathcal{L}$ trivial.
$\Rightarrow \mathcal{M}|_{\{0\} \times A \times A} = q^*\mathcal{M}$ is trivial.
Similarly, $\mathcal{M}|_{A \times \{0\} \times A}$, $\mathcal{M}|_{\{0\} \times A \times A}$ trivial.
thm of cube $\Rightarrow$ $\mathcal{M}$: trivial. Pullback $\mathcal{M}$ via
$(f, g, h) : X \longrightarrow A \times A \times A$ □

<u>pf of lemma</u>: Let $f = (n+1)_A$, $g = 1_A$, $h = (-1_A)$ in cor 2
$\Rightarrow (n+1)_A^*\mathcal{L} \cong (n+2)_A^*\mathcal{L} \otimes n_A^*\mathcal{L} \otimes 0_A^*\mathcal{L} \otimes (n+1)_A^*\mathcal{L}^{-1} \otimes \mathcal{L}^{-1} \otimes (-1_A)^*\mathcal{L}^{-1}$
i.e. $(n+2)_A^*\mathcal{L} \otimes (n+1)_A^*\mathcal{L}^{-2} \otimes n_A^*\mathcal{L} \cong \mathcal{L} \otimes (-1)_A^*\mathcal{L}$
$\left((n+2)_A^*\mathcal{L} \overset{\|}{\otimes} (n+1)_A^*\mathcal{L}^{-1}\right) \otimes \left((n+1)_A^*\mathcal{L} \otimes n_A^*\mathcal{L}^{-1}\right)^{-1}$
$\Rightarrow n_A^*\mathcal{L} \cong \mathcal{L}^{\otimes \frac{n(n-1)}{2}} \otimes (-1)_A^*\mathcal{L}^{\otimes \frac{n(n-1)}{2}} \otimes M_1^{\otimes n} \otimes M_2$
$n=0$, $M_2 \cong \mathcal{O}_A$. $n=1 \Rightarrow M_1 \cong \mathcal{L}$ □

<u>pf of claim</u>:

- $|A_n| < \infty$: Pick $\mathcal{L}$: ample inverr. sheaf. For $n \geq 1$,
$n_A^*\mathcal{L} \cong \mathcal{L}^{\otimes \frac{1}{2}\underset{\substack{\vee \\ 0}}{n(n+1)}} \otimes (-1_A)^*\mathcal{L}^{\otimes \frac{1}{2}\underset{\substack{\vee\!| \\ 0}}{n(n-1)}}$ is ample
Thus, if $\dim A_n > 1$, then $((n_A^*\mathcal{L})^{\dim A_n} \cdot A_n)_A > 0$
But $n_A^*\mathcal{L}|A_n$ is trivial $\Rightarrow$ $\divideontimes$. Hence, $\dim A_n = 0$
$\Rightarrow |A_n| < \infty$. $\Rightarrow \dim \operatorname{im}(n_A) = \dim A \Rightarrow n_A : A \twoheadrightarrow A$.
- $A_n \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$ if $p \nmid n$: $n_A : A \twoheadrightarrow A$, $|A_n| = \# f^{-1}(a) = \overset{\substack{\text{sep. deg.} \\ \downarrow}}{\deg_s}(n_A)$
Fix H: ample divisor $D := D + (-1)_A^* H \Rightarrow (-1)_A^* D = D$
$\Rightarrow n_A^* D \sim n^2 D$. Then $(n_A^* D^g)_A = \deg(n_A)(D^g)_A = n^{2g}(D^g)_A$
If $p \nmid n$, $p \nmid \deg(n_A) = n^{2g} \Rightarrow n_A$: sep. $\Rightarrow |A_n| = n^{2g}$
Also, if $m | n \Rightarrow X_m \subset X_n$ with $|X_m| = m^{2g}$ $(m | n \Rightarrow p \nmid m)$
structure thm for finte abelian gp $\Rightarrow A_n \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$.

Cor: X: non-sing. affine curve over $k=\bar{k}$ $\mathrm{char}(k)\nmid n$.

Then $H^i_{ét}(X,\mu_n) = \begin{cases} \mu_n(k), & i=0 \\ (\mathbb{Z}/n\mathbb{Z})^{2g+r-1}, & r=|\bar{X}\setminus X| \text{ for some non-sing. non-sing. compactification } \bar{X} \text{ of } X \\ 0, & i\geq 2. \end{cases}$

pf: Write $X=\bar{X}\setminus\{x_1,\dots,x_r\}$. Apply Kummer seq.

$\Rightarrow 0\to \mu_n(k)\to k^\times \xrightarrow{n} k^\times \to H^1(X_{ét},\mu_n)\to \mathrm{Pic}(X)\xrightarrow{n} \mathrm{Pic}(X)$

$\to H^2(X,\mu_n)\to H^2(X,\mathbb{G}_m)\to\cdots$ And same proof last time

$\Rightarrow H^q(X_{ét},\mathbb{G}_m)=0$ for $q\geq 2$. ∵ $\mathrm{char}(k)\nmid n$ ∴ $k\xrightarrow{n}k\to 0$

$\Rightarrow 0\to H^1(X_{ét},\mu_n)\to \mathrm{Pic}(X)\xrightarrow{n}\mathrm{Pic}(X)\to H^2(X,\mu_n)\to 0$ exact

Also, we have: $0\to R\to \mathrm{Pic}(X)\to \mathrm{Pic}(\bar{X})\to 0$

$R := \langle \mathcal{O}_{\bar{X}}[x_i],\ i=1,\dots,r\rangle$ ∵ $r\geq 1$ ∴ $\mathrm{Pic}(X)\twoheadrightarrow \mathrm{Pic}^0(\bar{X})$

and $n: \mathrm{Pic}^0(\bar{X})\twoheadrightarrow \mathrm{Pic}^0(\bar{X}) \Rightarrow \mathrm{Pic}(X)\xrightarrow{n}\mathrm{Pic}(X)\to 0$

$\Rightarrow H^2(X_{ét},\mu_n)=0$.

$H^1(X_{ét},\mu_n)=\{(\mathcal{L},\alpha):\mathcal{L}\in \mathrm{Pic}(X),\ \alpha:\mathcal{L}^{\otimes n}\xrightarrow{\sim}\mathcal{O}_X\}/\sim$

$=\{(\bar{\mathcal{L}},D,\bar{\alpha})\}/\tilde{R}$

$\bar{\mathcal{L}}\in \mathrm{Pic}^0(\bar{X})$, $D\in \mathrm{Div}(\bar{X})$, $\mathrm{supp}\, D\subset \bar{X}\setminus X$, $\bar{\alpha}:\bar{\mathcal{L}}^{\otimes n}\xrightarrow{\sim}\mathcal{O}_{\bar{X}}(D)$

$\tilde{R}=\{(\mathcal{O}_{\bar{X}}(D'),\ nD',\ 1^{\otimes n})\}$ $\mathrm{supp}\, D'\subset \bar{X}\setminus X$, $\deg D'=0$

⇝

$$0\to H^1(\bar{X},\mu_n)\to H^1(X,\mu_n)\to \bigoplus_{i=1}^r \mathbb{Z}/n\mathbb{Z}\to \mathbb{Z}/n\mathbb{Z}\to 0$$

$$[(\bar{\mathcal{L}},\bar{D},\bar{\alpha})]\mapsto (a_i)_{i=1}^r$$

$$D=\sum_{i=1}^r a_i[x_i]$$

$\Rightarrow H^1(X,\mu_n)=(\mathbb{Z}/n\mathbb{Z})^{2g+r-1}$ □

Def: $\mathcal{F}$: abelian sheaf on $X_{ét}$. $\mathcal{F}$: torsion sheaf if $\forall\, \bar{x}$, $\mathcal{F}_{\bar{x}}$: torsion gp.

$k=k^{sep}$

Thm: X: affine curve (affine var. of dim 1)$/k$, $\mathcal{F}$: torsion sheaf on $X_{ét}$, then $H^q(X_{ét},\mathcal{F})=0$ for $q\geq 2$.

Rmk: (1) We assume that $\mathcal{F}$ has no p-torsion for $p=\mathrm{char}(k)$

(2) Using Dévissage, proper base change, and Spectral seq. ⇒
X: affine scheme of finite type $/ k=k^{sep}$ $\mathcal{F}$: torsion sheaf on X
Then $H^q(X_{ét},\mathcal{F})=0$ $\forall\, q>\dim X$. (Weak Lefschetz thm)

- Constructible Sheaves:

<u>Recall</u>: Λ: abelian gp. We define const. sheaf $\underline{\Lambda}_X$ on $X_{ét}$: the sheafification of the presheaf $X_{ét} \to (Ab)$

$$(U \to X) \mapsto \Lambda$$

Equiv, $\underline{\Lambda}_X : U \longmapsto \Lambda^{\pi_0(U)}$. Then it is obvious that $(X \times \Lambda = \coprod_{\lambda \in \Lambda} X \longrightarrow X) \in X_{ét}$ represents $\underline{\Lambda}_{X_{ét}}$

i.e. $\forall U \to X$, $\underset{Sch/X}{\mathrm{Hom}}(U, X \times \Lambda) \cong \{U \longrightarrow \Lambda : \text{loc. const. fcn}\}$

$$\cong \Lambda^{\pi_0(U_0)} = \underline{\Lambda}_X(U).$$

<u>More example</u>:

(1) $\mathbb{G}_{a,X} : U \longmapsto \Gamma(U, \mathcal{O}_U)$. is represented by $X \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}(\mathbb{Z}[T])$: $\forall U \longrightarrow X$ étale over X

$$\mathrm{Hom}_{Sch/X}(U, X \times_{\mathrm{Spec}\,\mathbb{Z}} \mathbb{Z}[T]) = \mathrm{Hom}_{\mathrm{Spec}\,\mathbb{Z}}(U, \mathbb{Z}[T])$$

$$= \mathrm{Hom}(\mathbb{Z}[T], \Gamma(U, \mathcal{O}_U)) = \Gamma(U, \mathcal{O}_U).$$

Hence, $\mathbb{G}_a(U) = \mathrm{Hom}(U, X \times_{\mathrm{Spec}\,\mathbb{Z}} \mathbb{Z}[T])$

(2) $\mathbb{G}_{m,X} : U \longmapsto \Gamma(U, \mathcal{O}_U)^{\times}$ is rep. by $X \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Z}[T, T^{-1}]$

In terms of usual theory, these space are just "étale space" associated to the sheaf.

Representability lemma (Freitag–Kiehl)

Any sheaf of sets $\mathcal{F}$ is representable by $(U \longrightarrow X) \in \mathrm{Ob}(X_{ét})$ iff (1) $\forall \bar{x}$: geom. pt. $\mathcal{F}_{\bar{x}}$ finite sets

(2) $\forall U \longrightarrow X$ étale $\alpha, \beta \in \mathcal{F}(U)$

$\{u \in U : \exists\, \bar{u} : \underset{\bullet \longmapsto u}{\mathrm{Spec}\,k \longrightarrow U}$ s.t. $\alpha\bar{u} \neq \beta\bar{u}\} \underset{open}{\subset} U$.

↳ Separated condition for U.

<u>Rmk</u>: More generally, M. Artin: Any sheaf $\mathcal{F}$ on $X_{ét}$ can be rep. by an <u>alge. space</u> (cf. Miline Ch.V §1)

↑ outside the category of scheme.

<u>Def'n</u>: $\mathcal{F}$: sheaf of ab. gp on $X_{\acute{e}t}$. $\mathcal{F}$: loc. const. constructible (l.c.c.) if $\exists$ $F \longrightarrow X$ finite, étale s.t. $\mathrm{Hom}_{X_{\acute{e}t}}(U, F) \cong \mathcal{F}(U)$.

<u>Lemma</u>: $\mathcal{F}$: l.c.c. iff $\exists \{U_i \longrightarrow X\}$: étale covering s.t. $\mathcal{F}|_{U_i} \cong \underline{A_i}_{U_i}$, for some finite ab. gps A_i.

(sketch of proof): ($\Rightarrow$) Say $\mathcal{F}$: l.c.c. and rep. by F

$F \longrightarrow X$ finite, étale. Suffices to show: $\exists \{U_i \longrightarrow X\}$ in $X_{\acute{e}t}$

s.t. $F \times_X U_i \longrightarrow U_i$ is trivial i.e. $F \times_X U_i \underset{U_i}{\simeq} \coprod_{f \in F} U_i$.

locally, in affine case, $F = \mathrm{Spec}\, B \to X = \mathrm{Spec}\, A$

i.e. B : étale A-algebra and a finite A-mod. $\mathrm{rk}_A B = n$.

Need to show: $\exists C$: étale A-alge. s.t. $B \otimes_A C \cong C^n$

by induction on n On $B \otimes_A B$, via $B \otimes_A B \longrightarrow B \rightsquigarrow B : B \otimes_A B$-alge.

$$b \otimes b' \longmapsto bb'$$

EGA 4 Pt. 4 18.3.1 $\Rightarrow$ B: proj. $B \otimes_A B$-mod.

(as $B \otimes_A B$-mod.)

$\Rightarrow$ $\delta : B \otimes_A B \twoheadrightarrow B$ has a section $s : B \longrightarrow B \otimes_A B$ s.t. $\delta \circ s = 1_B$, $s(1) = e$

$I = \ker \delta = \langle b \otimes 1 - 1 \otimes b \rangle$, $(b \otimes 1 - 1 \otimes b)e = (b \otimes 1 - 1 \otimes b)s(1) = s(b) - s(b) = 0$

$\Rightarrow$ $Ie = 0$ $\Rightarrow$ $\delta(1 - e) = \delta(1) - \delta(e) = \delta(1) - \delta(s(1)) = 1 - 1 = 0$

$\Rightarrow$ $1 - e \in I$ $\Rightarrow$ $(1 - e)e = 0$ i.e. e: idempotent

$\Rightarrow$ $B \otimes_A B = (B \otimes_A B)e \oplus (B \otimes_A B)(1 - e)$ and $(B \otimes_A B)e \overset{\overset{Ie=0}{\downarrow}}{\cong} B$

Induction hypothesis $\Rightarrow$ $(B \otimes_A B)(1 - e) \otimes_B C \cong \left. C \times \cdots \times C \right\} n-1$

for some étale B-alge. B: étale over A $\Rightarrow$ C : étale / A

$B \otimes_A C = (B \otimes_A B) \otimes_B C \cong (B \oplus (B \otimes_A B)(1 - e)) \otimes_B C$

$\cong \left. C \times \cdots \times C. \right\} n.$

($\Leftarrow$) $\{U_i\}$: étale covering s.t. $\mathcal{F}|_{U_i} \cong \underline{A_i}_{U_i}$. Then as étale sheaf on U_i, $\mathcal{F}|_{U_i}$ is rep. by $U_i \times A_i \xrightarrow[\text{étale}]{\text{finite}} U_i$. Then they forms an effective descent data

Descent theory for schemes and properties of morphisms $\Rightarrow$ Descent to a scheme $F \to X$ which is finite étale

$\square$

Lemma: $\mathcal{F}, \mathcal{G}$: l.c.c. on $X_{ét}$ $\forall \varphi : \mathcal{F} \to \mathcal{G}$, then $\ker(\alpha)$, $\mathrm{coker}(\alpha)$ are l.c.c.

pf: After passing to étale covering $\{U_i \to X\}$, and look at connected compo. ⇝ for U: connected. A, B: finite ab. gp

$$\mathrm{Hom}_{Ab(U_{ét})}(\underline{A}_U, \underline{B}_U) = \mathrm{Hom}_{Ab}(A, B)$$

$\Rightarrow$ $\ker(\varphi : \underline{A}_U \to \underline{B}_U) = \underline{\ker \varphi}_U$ and similar for coker

Def: X: Noeth. scheme. A sheaf $\mathcal{F}$ on $X_{ét}$: constructible if $\exists$ finite $\{X_i \subset X$: loc. closed subscheme$\}$ s.t. $\dot{\bigcup_i} X_i = X$ and $\mathcal{F}|_{X_i}$ is l.c.c.

Prop: [Freitag-Kiehl, prop 4.6]

For $\mathcal{F}$: sheaf of sets on X, TFAE

(1) $\mathcal{F}$: constructible

(2) $\exists$ $\mathcal{G}$: sheaf rep. by $(Y \to X) \in \acute{E}t(X)$ and $\mathcal{G} \twoheadrightarrow \mathcal{F}$

Also, for $\mathcal{F}$: constructible sheaf of abelian gp

Rep. lemma $\Rightarrow$ $\mathcal{F}_{\bar{x}}$ is finite gp $\Rightarrow$ $\mathcal{F}$: torsion sheaf

Fact: (Milne Remark V.1.9)

$\phi : \mathcal{F} \to \mathcal{F}'$ morphism of sheaves on $X_{ét}$.

$\ker(\phi)$ is constructible if $\mathcal{F}$ is

$\mathrm{coker}(\phi)$ " " if $\mathcal{F}'$ is

$\mathrm{im}(\phi)$ " " if $\mathcal{F}$ or $\mathcal{F}'$ is.

⇝ Categ. of constructible sheaves is abelian □

- Extension by Zeros:

Recall: For $f: X \longrightarrow Y$, we have defined $f_*: Ab(X_{ét}) \to Ab(Y_{ét})$

Also, its inverse image $f^{-1}: Ab(Y_{ét}) \longrightarrow Ab(X_{ét})$ s.t.,

$$Hom(f^{-1}\mathcal{G}, \mathcal{F}) = Hom(\mathcal{G}, f_*\mathcal{F}) \quad \mathcal{G} \in Ab(Y_{ét}),\ \mathcal{F} \in Ab(X_{ét})$$

i.e. f_*: right of f^{-1}

Now, for $j: U \longrightarrow X$ étale, just like in classical case $j: U \hookrightarrow X$, we can define $j_!: Ab(U_{ét}) \to Ab(X_{ét})$: extension by zero s.t. $j_!$ is a left adjoint of j^{-1}:

$$Hom(j_!\mathcal{F}, \mathcal{G}) = Hom(\mathcal{F}, j^{-1}\mathcal{G}),\ \mathcal{F} \in Ab(U_{ét}),\ \mathcal{G} \in Ab(X_{ét})$$

$\because$ $j: U \to X$ étale $\therefore$ $j^{-1}\mathcal{G}: (V \to U) \mapsto \mathcal{G}(V \to U \xrightarrow[j]{} X)$

$\rightsquigarrow$ for $\mathcal{F} \in Ab(U_{ét})$,

$$(j_!\mathcal{F})^{psh}: X_{ét} \longrightarrow Ab$$
$$(V \to X) \longmapsto \bigoplus_{\substack{V \to U \\ \searrow \ \swarrow \\ X}} \mathcal{F}(V \to U)$$

and $(j_!\mathcal{F}) = ((j_!\mathcal{F})^{psh})^{\#}$

Lemma: $j: U \longrightarrow X$ étale

(i) j^{-1} and $j_!$ are exact

(ii) $\bar{x}$: geom. pt of X, $(j_!\mathcal{G})_{\bar{x}} = \bigoplus_{(U,\bar{u}) \mapsto (X,\bar{x})} \mathcal{F}_{\bar{u}}$

pf: (ii): $(j_!\mathcal{F})^{psh}_{\bar{x}} = \bigoplus_{\substack{(U,\bar{u}) \\ \mapsto (X,\bar{x})}} \mathcal{F}_{\bar{u}} = (j_!\mathcal{F})_{\bar{x}}$

(i) $j_!$ exact: direct from (ii)

j^{-1} has both left and right adjoint $\Rightarrow$ exact. □

Prop:

$$\begin{tikzcd} U' := U \times_X Y \arrow[r, "j'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ U \arrow[r, "\text{étale}", "j"'] & X \end{tikzcd}$$

(i) $j_!$ commutes with base change: , then $j'_! f'^{-1}\mathcal{F} = f^{-1} j_! \mathcal{F}$, $\forall \mathcal{F} \in Ab(U_{ét})$.

(ii) If $j: U \longrightarrow X$ finite, étale, then $j_! = j_*$

pf: (i) By adjoint property, $\exists$ map $j'_! \circ f'^{-1}\mathcal{F} \longrightarrow f^{-1} j_! \mathcal{F}$

$\forall \bar{y}$: geom. pt. of Y,

$$(j'_! f'^{-1}\mathcal{F})_{\bar{y}} = \bigoplus_{\bar{v}' \mapsto \bar{y}} (f'^{-1}\mathcal{F})_{\bar{v}'} = \bigoplus_{\bar{v} \mapsto f(\bar{y})} \mathcal{F}_{\bar{v}} = (j_!\mathcal{F})_{f(\bar{y})} = (f^{-1} j_! \mathcal{F})_{\bar{y}}$$ □

(ii) $(j_!\mathcal{F})^{psh}(V) = \bigoplus_{V \xrightarrow{\varphi} U} \mathcal{F}(V \xrightarrow{\varphi} U)$ $(j_*\mathcal{F})(V) = \mathcal{F}(V \times_X U)$

$\forall \varphi : V \to U$, $\Gamma_\varphi = (1, \varphi) : V \hookrightarrow V \times_X U$. $\Rightarrow (j_!\mathcal{F})^{psh} \longrightarrow j_*\mathcal{F}$

and sheafification $\Rightarrow$ $j_!\mathcal{F} \longrightarrow j_*\mathcal{F}$. on stalks.

$(j_!\mathcal{F})_{\bar{x}} = \bigoplus_{\bar{u} \mapsto \bar{x}} \mathcal{F}_{\bar{u}}$ $(j_*\mathcal{F})_{\bar{x}} = H^0(\mathrm{Spec}(\mathcal{O}^{sh}_{X,\bar{x}}) \times_X U, \mathcal{F})$

j : finite $\mathcal{O}^{sh}_{X,\bar{x}}$: strictly henselian $\mathrm{Spec}(\mathcal{O}^{sh}_{X,\bar{x}}) \times_X U = \coprod_{\bar{u} \to \bar{x}} \mathrm{Spec}(\mathcal{O}^{sh}_{X,\bar{x}})$

$\Rightarrow (j_*\mathcal{F})_{\bar{x}} = \prod_{\bar{v} \mapsto \bar{x}} \mathcal{F}_{\bar{v}}$. finite product = finite coproduct □

(quasi-cpt)

<u>Cor</u>: X: Noeth. scheme. $j : U \longrightarrow X$ étale, finite-type, then $j_! \underline{\mathbb{Z}/n\mathbb{Z}}_U$ is constructible on X (but not even lcc)

<u>pf</u>: $j : U \longrightarrow X$ étale, then $\exists\ X = \coprod_i X_i$: stratification s.t. $\pi_i : j^{-1}(X_i) \longrightarrow X_i$: finite, étale.

$j_!(\underline{\mathbb{Z}/n\mathbb{Z}}_X)|_{X_i} = (\pi_i)_!(\underline{\mathbb{Z}/n\mathbb{Z}}_{X_i}) = (\pi_i)_*(\underline{\mathbb{Z}/n\mathbb{Z}}_{X_i})$

<u>Note</u>: $\forall Y \to X$ étale, $\pi_* \underline{\mathbb{Z}/n\mathbb{Z}}_X$ is loc. const. since it is rep. by $Y \times \mathbb{Z}/n\mathbb{Z}$. □

$\rightsquigarrow$ With this, we can prove for $\mathcal{F}$: constructible, $j_!\mathcal{F}$ constructible by base change to reduce to the case above.

<u>Cor</u>: X : Noeth. scheme $\mathcal{F}$: torsion sheaf, then $\mathcal{F} = \varinjlim_i \mathcal{F}_i$, $\mathcal{F}_i$: constructible and $\mathcal{F}_i \subset \mathcal{F}_{i+1}$.

<u>pf</u>: Let $j : U \to X$ with U : Noeth. $s \in \mathcal{F}(U)$

Then $\exists n \in \mathbb{N}$ s.t. $ns = 0$ $\Rightarrow$ $\underline{\mathbb{Z}/n\mathbb{Z}}_U \to \mathcal{F}|_U$

$1 \longmapsto s$

$\longleftrightarrow$ $j_! \underline{\mathbb{Z}/n\mathbb{Z}}_U \xrightarrow{\varphi} \mathcal{F}$ with $e \in \Gamma(U, j_! \underline{\mathbb{Z}/n\mathbb{Z}}_U) \mapsto s$

Thus, $\mathrm{im}\,\varphi \subseteq \mathcal{F}$: constructible sheaf and $s \in \mathrm{Im}(\varphi)(U)$

Inductively, we can apply for finitely many sections and $\mathcal{F}$: torsion sheaf iff $\varinjlim_n \ker(\mathcal{F} \xrightarrow{n} \mathcal{F}) = \mathcal{F}$.

□

<u>Recall</u>: $\mathrm{char}(k) = p$

<u>Thm</u>: X: affine curve (affine var. of dim 1)$/k$, $\mathcal{F}$: torsion sheaf on $X_{ét}$, then $H^q(X_{ét}, \mathcal{F}) = 0$ for $q \geq 2$.

<u>pf</u>: Step 1: Reduction to $\mathcal{F}$ is constructible

- Since $\varinjlim$ exchanges with $H^i(X_{ét}, -)$, we may assume $\mathcal{F}$: constructible.

<u>Step 2</u>: Reduction to X: non-sing. $j: U \hookrightarrow X$: open immersion $\mathcal{G}$: l.c.c. on U, $H^q(X_{ét}, j_!\mathcal{G}) = 0$ for $q \geq 2$.

- Choose $U \underset{\text{open}}{\subseteq} X$ s.t. $\mathcal{F}|_U$: l.c.c. and $U \subset X \setminus X_{sing}$.

Consider $0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to Q \to 0$

Hence, $Q_{\bar{x}} = 0$ unless $\bar{x} \in X \setminus U$ and $Q = \bigoplus_{x \in X \setminus U} (i_x)_* Q_x$

<u>Step 3</u> Reduction to X: non-sing. $\mathcal{G}$: l.c.c. on $U \underset{\text{open}}{\subseteq} X$ $\mathcal{G} = j_! \underline{\mathbb{Z}/\ell\mathbb{Z}}_U$ for some prime ℓ, then $H^q(X_{ét}, j_!\mathcal{F}) = 0$ $\forall q \geq 2$.

- Consider

$$\begin{tikzcd} & \tilde{X} \arrow[d, "N"] \\ U \arrow[ur, "j^N"] \arrow[r, hook, "j"'] & X \end{tikzcd}$$

$N: \tilde{X} \to X$ normalization of X

$\because$ N: finite

$\therefore$ $H^*(X_{ét}, j_!\mathcal{G}) = H^*(X_{ét}, j^N_!\mathcal{G})$

($\because$ $R^i N_* = 0$) and hence $N_*(j^N)_!\mathcal{G} = j_!\mathcal{G}$.

<u>Sum up</u>: Reduce to prove: X: non-sing. affine curve $/k$ $j: U \hookrightarrow X$ open immers. $\mathcal{F}$. l.c.c. on $U_{ét}$, then $\forall q \geq 2$, $H^q(X, j_!\mathcal{F}) = 0$.

<u>Step 4</u>: Assume $\mathcal{F}$ is ℓ-torsion for some prime ℓ.

Write $\mathcal{F} = \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_r$ $\mathcal{F}_i$: ℓ_i-primary. ℓ_i: prime

Then consider $0 \to \bigoplus_{i=1}^{r-1} \mathcal{F}_i \to \mathcal{F} \to \mathcal{F}_r \to 0$ exact

apply $j_!$ + long exact seq. in coho. Induction $\Rightarrow$ may assume $\mathcal{F}$ is ℓ-torsion. Say $\mathcal{F} \xrightarrow{\ell^n} \mathcal{F}$ is zero map for some n.

Then consider $0 \to \mathcal{F}[\ell] \to \mathcal{F} \to \mathcal{F}/\mathcal{F}[\ell] \to 0$

Apply $j_!$ + long exact seq. in cohomo. $\Rightarrow$ Assume $\mathcal{F}$ is ℓ-torsion.

Step 5: $\exists$ finite étale $f: V \to U$, $(\deg(f), \ell) = 1$ s.t.
$0 \subset \mathcal{G}_1 \subset \cdots\cdots \subset \mathcal{G}_s = f^{-1}\mathcal{F}$ is a filtration
with $\mathcal{G}_i/\mathcal{G}_{i-1} \cong \underline{\mathbb{Z}/\ell\mathbb{Z}}_V$ $\forall\, i \leq s$.

- $\mathcal{F}$: l.c.c. $\Rightarrow$ $\exists\, h: U' \to U$ finite étale Galois s.t. $h^{-1}\mathcal{F} \cong \underline{A}_{U'}$, $A \cong (\mathbb{Z}/\ell\mathbb{Z})^{\oplus m}$ for some m.

$G = \mathrm{Aut}_U(U')$ and $|G| = \deg(h)$

Take $H \leq G$: ℓ-Sylow and set

$U' \xrightarrow{\pi} V = U'/H \xrightarrow{f} U$ Hence, $\deg(f) = \dfrac{|G|}{|H|}$ is prime to ℓ.

Also, $f^{-1}\mathcal{F}$: l.c.c. on V and

$$\pi^{-1}f^{-1}\mathcal{F} = h^{-1}\mathcal{F} \cong \underline{(\mathbb{Z}/\ell\mathbb{Z})}^{\oplus m}_{U'}$$

and so $H^0(V_{et}, f^{-1}\mathcal{F}) = H^0(U', \underline{(\mathbb{Z}/\ell\mathbb{Z})}^{\oplus m}_{U'})^H = ((\mathbb{Z}/\ell\mathbb{Z})^{\oplus m})^H \neq 0$

$\rightsquigarrow \underline{\mathbb{Z}/\ell\mathbb{Z}}_V \hookrightarrow f^{-1}\mathcal{F}$ Replace $f^{-1}\mathcal{F}$ by quot. by this subsheaf

$\bar{1} \longmapsto s$ and induction $\Rightarrow$ $\exists\, \mathcal{G}_{s-1} \subset f^{-1}\mathcal{F}$

s.t. $f^{-1}\mathcal{F}/\mathcal{G}_{s-1} \cong \underline{\mathbb{Z}/\ell\mathbb{Z}}_V$.

Step 6: "Method of Trace"

$f: Y \to X$ finite, étale $\Rightarrow$ $f_* = f_!$ $\forall\, \mathcal{F} \in \mathrm{Ab}(X_{et})$

$\mathcal{F} \xrightarrow[res]{} f_*f^{-1}\mathcal{F}$ restriction $\quad f_!f^{-1}\mathcal{F} \xrightarrow[tr]{} \mathcal{F}$: trace

The trace map is characterized by:

(1) it commutes with passing to étale nbd of a pt $x \in X$

(2) if $Y = \coprod_{i=1}^{d} X$, then $f_*f^{-1} = \mathcal{F}^{\oplus d} \to \mathcal{F}$.

(Proof: cf. Stack Project §58.65)

Thus, if $\deg f = d$, $\mathcal{F} \xrightarrow{res} f_*f^{-1}\mathcal{F} \to \mathcal{F}$ is just $\mathcal{F} \xrightarrow{d} \mathcal{F}$

$\because$ f: finite $\therefore$ $R^i f_*\mathcal{F} = 0$

$$H^q(X_{et}, \mathcal{F}) \to H^q(Y_{et}, f^{-1}\mathcal{F}) = H^q(X_{et}, f_*f^{-1}\mathcal{F}) \xrightarrow{tr} H^q(X_{et}, \mathcal{F})$$

Observation: if $\mathcal{F} \xrightarrow{d} \mathcal{F}$ induces $H^q(X, \mathcal{F}) \xrightarrow[d\cdot]{\sim} H^q(X, \mathcal{F})$

Then $H^q(X, \mathcal{F}) \hookrightarrow H^q(Y, f^{-1}\mathcal{F})$.

Step 7: Consider normalization Y of X in V

i.e.
$$\begin{tikzcd} V \arrow[r, hook, "j'"] \arrow[d, "f"'] & Y' \arrow[d, "f'\text{: finite}"] \\ U \arrow[r, hook, "j"'] & Y \end{tikzcd}$$

Step 6 $\Rightarrow$ $\underbrace{\mathcal{F} \xrightarrow{res} f_*f^{-1}\mathcal{F} \xrightarrow{tr} \mathcal{F}}_{(\cdot \deg f)}$ is isom. $(\because\ \ell \nmid \deg f)$

∵ f: finite, étale ⇒ $f_* = f_!$ By base change for $f_!$:

$$j_! f_* f^{-1}\mathcal{F} = j_* f_! f^{-1}\mathcal{F} = f'_* j'_! f^{-1}\mathcal{F}$$

Hence, apply $j_!$ to $\mathcal{F} \longrightarrow f_* f^{-1}\mathcal{F} \longrightarrow \mathcal{F}$ gives:

$$j_!\mathcal{F} \longrightarrow f'_* j'_! f^{-1}\mathcal{F} \longrightarrow j_!\mathcal{F}$$

Step 6 ⇒ $H^q(X_{ét}, j_!\mathcal{F}) \hookrightarrow H^q(X_{ét}, f'_* j'_! f^{-1}\mathcal{F})$

∵ f': finite $R^q f'_* = 0$ ∴ $H^q(X_{ét}, f'_* j'_! f^{-1}\mathcal{F})$

$= H^q(Y_{ét}, j'_! f^{-1}\mathcal{F})$.

Combining Step 5-7 ⇒ suffices to show: $H^q(Y_{ét}, j'_! \underline{\mathbb{Z}/\ell\mathbb{Z}}) = 0$

<u>Step 8</u>: X: affine, non-sing. curve. $j: U \hookrightarrow X$ open immersion ℓ: prime Then $\forall q \geq 2$, $H^q(X_{ét}, j_! \underline{\mathbb{Z}/\ell\mathbb{Z}}) = 0$

- Consider $0 \to j_! \underline{\mathbb{Z}/\ell\mathbb{Z}}_U \to \underline{\mathbb{Z}/\ell\mathbb{Z}}_X \to \bigoplus_{x \in X \setminus U} (i_x)_*(\mathbb{Z}/\ell\mathbb{Z}) \to 0$

As last time:

$$H^q(X, (i_x)_* \underline{\mathbb{Z}/\ell\mathbb{Z}}) = H^q(\{x\}, \mathbb{Z}/\ell\mathbb{Z}) = 0 \text{ if } q \geq 1$$

(∵ x: closed pt. ⇒ $R^i(i_x)_* = 0$ and $\{x\} = \operatorname{Spec} k$ no higher coho.)

- If $\ell \neq \operatorname{char}(k)$, then from Cor. in p.3
  ⇒ $H^q(X, \mu_\ell) = H^q(X, \underline{\mathbb{Z}/\ell\mathbb{Z}}_X) = 0$ $\forall q \geq 2$
- If $\ell = \operatorname{char}(k)$, consider Artin-Schreier seq.:

$$0 \to \underline{\mathbb{Z}/\ell\mathbb{Z}}_X \to \mathbb{G}_a = \mathcal{O}_X \longrightarrow \mathcal{O}_X \longrightarrow 0$$
$$x \longmapsto x^p - x - 1$$

Take long exact seq. and $H^q(X_{ét}, \mathcal{O}_{X_{ét}}) = H^q(X_{zar}, \mathcal{O}_{X_{zar}})$

∵ X: affine Cartan's thm B ⇒ $H^q(X_{zar}, \mathcal{O}_{X_{zar}}) = 0$ $\forall q \geq 1$

⇒ $H^q(X, \underline{\mathbb{Z}/\ell\mathbb{Z}}_X) = 0$ for $q \geq 2$.

⇝ $H^q(X_{ét}, j_! \underline{\mathbb{Z}/\ell\mathbb{Z}}_U) = 0$

□

- $\mathbb{Z}_\ell$ / $\mathbb{Q}_\ell$ - sheaves:

Def: X: Noeth. scheme. $\mathcal{F}$: $\mathbb{Z}_\ell$ - sheaf is an inverse system $\{\mathcal{F}_n\}_{n\geq 1}$, where $\mathcal{F}_n$: constructible $\mathbb{Z}/\ell^n\mathbb{Z}$-modules and $\mathcal{F}_{n+1}\to\mathcal{F}_n$ induces $\mathcal{F}_{n+1}\otimes_{\mathbb{Z}/\ell^{n+1}\mathbb{Z}}(\mathbb{Z}/\ell^n\mathbb{Z})\xrightarrow{\sim}\mathcal{F}_n$

$\mathcal{F}$: lisse if $\mathcal{F}_n$: local const.

→ The categ. of $\mathbb{Z}_\ell$ - sheaves on X is abelian.

If $\mathcal{F}=\{\mathcal{F}_n\}_{n\geq 1}$: $\mathbb{Z}_\ell$ - sheaf on X $\bar{x}$: geom. pt of X

$M_n=\{\mathcal{F}_{n,\bar{x}}\}$: inverse system of finite $\mathbb{Z}/\ell^n\mathbb{Z}$-mod.

s.t. $M_{n+1}\to M_n\to 0$ and $M_{n+1}/\ell^n M_{n+1}$

$\Rightarrow$ $M=\varprojlim_n M_n=\varprojlim_n \mathcal{F}_{n,\bar{x}}$: finite $\mathbb{Z}_\ell$ - mod.

Define $M:=\mathcal{F}_{\bar{x}}$: stalk of $\mathcal{F}$ at $\bar{x}$.

Def: $\mathcal{F}$: $\mathbb{Z}_\ell$ - sheaf is torsion if $\ell^n:\mathcal{F}\to\mathcal{F}$ is zero for some n.

($\mathbb{Q}_\ell$ - sheaves) : Ob = $\mathbb{Z}_\ell$ - sheaves and

$$\mathrm{Hom}_{\mathbb{Q}_\ell}(\mathcal{F},\mathcal{G})=\mathrm{Hom}_{\mathbb{Z}_\ell}(\mathcal{F},\mathcal{G})\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$$

Notation: $(\mathbb{Z}_\ell\text{- sheaves})\longrightarrow(\mathbb{Q}_\ell\text{ -sheaves})$

$$\mathcal{F}'\longmapsto\mathcal{F}=\mathcal{F}'\otimes\mathbb{Q}_\ell$$

If $\mathcal{F}'$: $\mathbb{Z}_\ell$-sheaf $\mathcal{F}=\mathcal{F}\otimes\mathbb{Q}_\ell$. $\mathcal{F}_{\bar{x}}:=(\mathcal{F}'_{\bar{x}})\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$

Def:

$$H^i(X,\mathcal{F})\hat{=}\varprojlim H^i(X,\mathcal{F}_n)$$

Note: cohomology does not exchange with inverse limit.