Étale Cohomology : Descents and Cohomology of Curves

- A Sheaf Criterion

<u>Recall</u>: X : Noeth. scheme $X_{\acute{e}t} \subset Sch/X$: fullsubcategory $Ob(X_{\acute{e}t}) = \{U \to X \text{ étale}\}$ and $\mathrm{Hom}_{X_{\acute{e}t}}(U,V) = \mathrm{Hom}_{Sch/X}(U,V) \rightsquigarrow$

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "g\circ f"'] & & V \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

$\because$ $g, g\circ f$: étale $\Rightarrow$ f : étale

$U \hookrightarrow X$. open immersion $\Rightarrow \{U \hookrightarrow X\} \in Ob(X_{\acute{e}t})$ and

$U \to X$ étale $\Rightarrow$ $U \to X$: open (only need flat + loc. of finite type)

A presheaf $\mathcal{F}$ on $X_{\acute{e}t}$: $X_{\acute{e}t}^{op} \to (Ab)$ contravariant functor

$U \in Ob(X_{\acute{e}t})$, $\{U_i \xrightarrow[f_i]{} U\}$: (étale) covering if $\{U_i \to U\}$: finite family of étale morphism and $\bigcup_i f_i(U_i) = U$.

A presheaf $\mathcal{F}$ is a sheaf on $X_{\acute{e}t}$ if $\forall \{U_i \to U\}$: covering

$$(*) \quad \mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i,j} \mathcal{F}(U_i \times_U U_j) \text{ is exact}$$

<u>Prop 1</u>: A presheaf $\mathcal{F}$ on $X_{\acute{e}t}$ is a sheaf iff

(1) $(*)$-condition holds for $\{U_i \hookrightarrow U\}$: Zariski open covering i.e. $\mathcal{F}$ is a sheaf on X_{zar}.

(2) $(*)$ condition holds for $\{V \to U\}$ with V, U : affine.

<u>pf</u>: ($\Rightarrow$) is obvious. For ($\Leftarrow$), if $U = \coprod_i U_i$ $U_i \subset U$: open subscheme , then $U_i \times_U U_j = \phi \Rightarrow \mathcal{F}(U) = \prod_i \mathcal{F}(U_i)$.

Thus, given $\{U_i \to U\}$, consider $\coprod_i U_i \to U$, then

$$(*) \quad \mathcal{F}(U) \longrightarrow \mathcal{F}(\coprod_i U_i) = \prod_i \mathcal{F}(U_i) \longrightarrow \mathcal{F}((\coprod_i U_i) \times_U (\coprod_j U_j)) = \mathcal{F}(\coprod_{i,j} U_i \times_U U_j) = \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$$

Thus, (2) $\Rightarrow$ $(*)$ is exact if U_i, U : affine since $\coprod_i U_i$ is affine

Now, let $f : U' := \coprod_i U_i \to U$ Write $U = \bigcup_k V_k$ V_k : affine, open

$f^{-1}(V_k) = \bigcup_j U_{kj}$ and $f(U_{kj}) \subset_{open} V_k$ $\because$ V_k : affine $\Rightarrow$ quasi-cpt.

$\exists$ $J(k)$: finite set s.t. $\{U_{kj} \to V_k\}_{j \in J(k)}$: covering

$\rightsquigarrow$ May assume: $U = \bigcup_k V_k$, $U' = \bigcup_{j,k} U_{kj}$ V_k, U_{kj} : affine.

and $\forall$ k, $\{U_{kj} \to V_k\}$: finite covering.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{F}(U') \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathcal{F}(U' \times_U U') \arrow[d] \\ \prod_k \mathcal{F}(V_k) \arrow[r] \arrow[d] & \prod_j \prod_k \mathcal{F}(U_{jk}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod_k \prod_{j,j'} \mathcal{F}(U_{jk} \times_U U_{j'k}) \\ \prod_{k,k'} \mathcal{F}(V_k \cap V_{k'}) \arrow[r] & \prod_{\substack{j,k \\ j',k'}} \mathcal{F}(U_{jk} \cap U_{j'k'}) & \end{tikzcd}$$

(1) $\Rightarrow$ columns exact

(2) $\Rightarrow$ middle row is exact

diagram chasing

$\Rightarrow$ top row exact $\square$

• Example of Sheaves on $X_{ét}$:

(1) Constant Sheaf: Λ: abelian group.

$\underline{\Lambda}_X : X_{ét} \to (Ab)$ given by: $U \mapsto \Lambda^{\pi_0(U)}$ $\pi_0(U)$: connected component of U.

(2) Étale sheaf defined by a quasi-coh. $\mathcal{O}_X$-sheaf.

$\mathcal{M} \in Qcoh(X)$. $\forall \varphi: U \to X$ étale, $\varphi^*\mathcal{M} \in Qcoh(U)$.

Define $\mathcal{M}^{ét}: X_{ét} \to (Ab)$ by: $U \mapsto \Gamma(U, \varphi^*\mathcal{M})$

$$\begin{tikzcd} V\times_U V \arrow[r, "p_1"] \arrow[d, "p_2"'] & V \arrow[d, "\psi"] \\ V \arrow[r, "\psi"'] & U \end{tikzcd}$$

In view of prop 1 $\Rightarrow$ $\mathcal{O}_{X_{ét}}$ satisfies (1).

For (2), Consider $V = \operatorname{Spec} B \xrightarrow{\psi} U = \operatorname{Spec} A$. $\because$ $V \to U$ étale

$\Rightarrow$ $A \to B$ étale (and hence flat) and $V \to U$ surj. $\Rightarrow$ $A \to B$ faithfully flat

$$\Gamma(U, \varphi^*\mathcal{M}) = M \to \Gamma(V, \psi^*\varphi^*\mathcal{M}) = M\otimes_A B \rightrightarrows \Gamma(V\times_U V, p_i^*\psi^*\varphi^*\mathcal{M}) = M\otimes_A B \otimes_A B$$

equiv to show:

<u>Descent Lemma</u>: $A \xrightarrow{f} B$ faithfully flat, M: A-mod.

$$0 \to M \to M\otimes_A B \to M\otimes_A B\otimes_A B \text{ is exact}$$
$$m\otimes b \longmapsto m\otimes 1\otimes b - m\otimes b\otimes 1$$

pf: If $\exists\, g: B \to A$ s.t. $gf = 1_A$, then $M\otimes_A B\otimes B \longrightarrow M\otimes_A B$
$m\otimes b\otimes b' \longmapsto m\otimes (b\cdot fg(b'))$

Then $k(m\otimes 1\otimes b - m\otimes b\otimes 1) = m\otimes(fg(b) - b)$. Hence, if $m\otimes 1\otimes b - m\otimes b\otimes 1 = 0$, then $k(m\otimes 1\otimes b - m\otimes b\otimes 1) = m\otimes fg(b) - m\otimes b = 0$

$\Rightarrow m\otimes b = f(g(b))m\otimes 1$. (If $\exists g$, no need faithful flatness)

$\because$ $A \to B$ faithfully flat $\therefore$ suffices to prove the exactness of:

$$\underset{N}{(M\otimes_A B)} \to \underset{N\otimes_A B}{(M\otimes_A B)\otimes_A B} \to \underset{N\otimes_A B\otimes_A B}{(M\otimes_A B\otimes_A B)\otimes_A B}$$

But $B \to B\otimes_A B$ has a section given by $B\otimes_A B \to B$
$b \mapsto b\otimes 1$ $\qquad$ $b\otimes b' \mapsto bb'$ $\square$

Thus, if $\mathcal{O}_{X_{ét}}: U \mapsto \Gamma(U, \mathcal{O}_U)$. Descent lemma + prop 1

$\Rightarrow$ $\mathcal{O}_{X_{ét}}$: étale sheaf

Similarly, $\mathcal{O}_{X_{ét}}^\times = \mathbb{G}_{m,X}: U \mapsto \Gamma(U, \mathcal{O}_U)^\times$ defines a sheaf on $X_{ét}$.

<u>Descent Theory</u>: $\{U_i \xrightarrow{f_i} U\}_{i\in I}$: family of fpqc morphism.

A descent datum for quasi-coh. sheaves w.r.t. $\{U_i \to U\}$ consists of $(\mathcal{F}_i, \varphi_{ij})_{i,j\in I}$, where

(1) $\forall i$, $\mathcal{F}_i \in Qco(U_i)$, and

(2) $\forall i,j$, $\varphi_{ij} : p_1^*\mathcal{F}_i \xrightarrow{\sim} p_2^*\mathcal{F}_j$ on $U_i \times_U U_j$ s.t.

$$\begin{tikzcd} p_1^*\mathcal{F}_i \arrow[rr, "p_{12}^*\varphi_{ij}"] \arrow[dr, "p_{13}^*\varphi_{ik}"'] & \arrow[d, phantom, "\circlearrowright"] & p_2^*\mathcal{F}_j \arrow[dl, "p_{23}^*\varphi_{jk}"] \\ & p_3^*\mathcal{F}_k & \end{tikzcd}$$ on $U_i \times_U U_j \times_U U_k$.

Then $\exists\, \mathcal{F} \in Qco(U)$ and $\varphi_i : f_i^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}_i$ s.t. $\varphi_{ij} = p_2^*\varphi_j \circ p_1^*\varphi_i^{-1}$

<u>Rmk</u>: The proof of descent theory is similar to previous case.

Step 1: Reduction to the case $\{\operatorname{Spec} B \to \operatorname{Spec} A\}$, where B : fp $/A$

Step 2: Show that $B \to B\otimes_A B \to B\otimes_A B\otimes_A B \to \cdots$ is exact

$b \mapsto 1\otimes b - b\otimes 1$

$b\otimes b' \longmapsto 1\otimes b\otimes b' - b\otimes 1\otimes b + b\otimes b'\otimes 1$

(Similar to our descent lemma)

<u>Step 3</u>: $A \to B$ $N \in B\text{-Mod}$ $\varphi : N\otimes_A B \xrightarrow{\sim} B\otimes_A N$ as $B\otimes_A B$-mod.

s.t. $$\begin{tikzcd} N\otimes_A B\otimes_A B \arrow[rr, "\varphi_{12}"] \arrow[dr, "\varphi_{13}"'] & \arrow[d, phantom, "\circlearrowright"] & B\otimes_A N\otimes_A B \arrow[dl, "\varphi_{23}"] \\ & B\otimes_A B\otimes_A N & \end{tikzcd}$$

, then $\exists\, M \in A\text{-mod}$ s.t. $N \cong (B\otimes_A M)$ and

$\varphi_{can} : (B\otimes_A M)\otimes B \longrightarrow B\otimes_A(B\otimes_A M)$

$b\otimes m\otimes b' \longmapsto b\otimes b'\otimes m$

s.t. two descent data iso.

<u>Application</u>: $H^1(X, \mathbb{G}_m) = Pic(X)$.

$\mathcal{L}$: invertible sheaf on X, $U \xrightarrow{\varphi} X$ étale $\varphi^*\mathcal{L}$. inverTible sheaf on U.

Also, define $U \mapsto \mathcal{L}^*(U) = Isom_U(\mathcal{O}_U, \varphi^*\mathcal{L})$

For an étale sheaf $\mathcal{G}$ on $X_{ét}$, if $\forall\, U \to X$, $\mathcal{G}(U)$. $\Gamma(U, \mathcal{O}_U)$-mod. and compatible with restriction $\to \mathcal{G}$: $\mathcal{O}_{X_{ét}}$-mod

$\mathcal{G}$: étale invertible sheaf if $\exists\, \{U_i \to X\}$ s.t. $\mathcal{G}|_{U_i} \cong \mathcal{O}_{U_i, ét}$

$Pic_{ét}(X)$ = Isom. classes of étale line bundle w/ gp str from tensor prod

Then $\check{H}^1(X, \mathbb{G}_m) = H^1(X, \mathbb{G}_m) = Pic_{ét}(X)$.

$\mathcal{L} \mapsto \mathcal{L}^*$ defines a gp homo $Pic(X) \to Pic_{ét}(X)$

Descend theory $\Rightarrow$ $Pic(X) \cong Pic_{ét}(X)$

Recall: Stalks

$\bar{x}$: geom. point of X if $\mathrm{Spec}(K) \xrightarrow{\bar{x}} X$ and $x := \mathrm{im}(\mathrm{Spec} K)$, K: sep. closed

alge. geom. pt if $K/k(x)$: alge.

Étale nbd of $\bar{x}$:

$$\begin{tikzcd} & U \arrow[d, "\text{étale}"] \\ \bar{x} \arrow[ur] \arrow[r] & X \end{tikzcd}$$

strict localization: $\mathcal{O}^{sh}_{\bar{x},X} := \varinjlim \Gamma(U, \mathcal{O}_U)$ over all étale nbd of $\bar{x}$

$\rightsquigarrow$ $\mathcal{O}_{\bar{x},X}$: strict henselian local ring and $k(\bar{x}) := \mathcal{O}^{sh}_{\bar{x},X}/m_{\bar{x},X}$: sep. closure of $k(\bar{x})$ in K

For $\mathcal{F}$: sheaf on $X_{\acute{e}t}$, define stalk $\mathcal{F}_{\bar{x}} := \varinjlim \mathcal{F}(U)$ over all étale nbd U of $\bar{x}$ and we have the following fact:

Fact: $\mathcal{F}, \mathcal{G}, \mathcal{H}$: étale sheaf on $X_{\acute{e}t}$ TFAE:

(1) $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is exact

(2) $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0$ is exact, $\forall\, U \xrightarrow{\varphi} X$ étale

(3) $0 \to \mathcal{F}_{\bar{x}} \to \mathcal{G}_{\bar{x}} \to \mathcal{H}_{\bar{x}} \to 0$, $\forall\, \bar{x}$: geom. pt of X.

Recall: Higher Direct Image.

$f: X \longrightarrow Y$ morphism $\mathcal{F}$: sheaf on $X_{\acute{e}t}$.

$f_*\mathcal{F}: (V \xrightarrow[\text{étale}]{} Y) \longmapsto \mathcal{F}(X \times_Y V \longrightarrow X)$: a presheaf on $Y_{\acute{e}t}$.

However, for any étale covering $\{U_i \to U\}$ of $Y_{\acute{e}t}$,

$\{X \times_Y U_i \longrightarrow X \times_Y U\}$: étale covering in $X_{\acute{e}t}$. $\rightsquigarrow$ sheaf condition on $\mathcal{F} \Rightarrow$ sheaf condition on $f_*\mathcal{F}$ i.e. $f_*\mathcal{F}$: sheaf on $Y_{\acute{e}t}$.

f_* is left exact $\rightsquigarrow$ $R^i f_*$: right der. functor of f_*.

$\rightsquigarrow$ $R^q f_* \mathcal{F}$ is the sheafification of the presheaf

$$(V \longrightarrow Y) \longmapsto H^q(V \times_Y X, \mathcal{F}|_{V \times_Y X})$$

$\Rightarrow$ For $\bar{y} \in Y$: geom. pt, $(R^q f_* \mathcal{F})_{\bar{y}} = \varinjlim H^q(V \times_Y X, \mathcal{F}|_{V \times_Y X})$

, where direct limit

Now, let $\mathcal{O}^{sh}_{\bar{y},Y}$: strict loc. of Y at geom. pt $\bar{y}$.

$X \times_Y \mathrm{Spec}(\mathcal{O}^{sh}_{\bar{y},Y}) \longrightarrow X$, then $(R^q f_* \mathcal{F})_{\bar{y}} = H^q(X \times_Y \mathrm{Spec}(\mathcal{O}^{sh}_{\bar{y},Y}), \mathcal{F}|_{X \times_Y S})$

, where $S = \mathrm{Spec}(\mathcal{O}^{sh}_{\bar{y},Y})$.

Prop: $f: X \to Y$ finite morphism, then $\forall q \geq 1$, $\forall \mathcal{F}$: abelian sheaf on $X_{\acute{e}t}$,
$R^q f_* \mathcal{F} = 0$

pf: $X^{sh}_{\bar{y}} = X \times_Y \mathrm{Spec}(\mathcal{O}^{sh}_{Y,\bar{y}})$. Since f: finite $\Rightarrow X^{sh}_{\bar{y}} \to \mathrm{Spec}(\mathcal{O}^{sh}_{Y,\bar{y}})$ finite
$\Rightarrow X^{sh}_{\bar{y}} = \mathrm{Spec}\, A$ for some finite $\mathcal{O}^{sh}_{Y,\bar{y}}$-mod.
$\because \mathcal{O}^{sh}_{Y,\bar{y}}$: finite $\therefore A = \prod_{i=1}^{r} A_i$, A_i: henselian local ring.
$\kappa(\mathcal{O}^{sh}_{Y,\bar{y}})$: sep. closed $\quad k(\mathcal{O}^{sh}_{Y,\bar{y}}) \subseteq k(A_i)$: finite $\forall i$
$\Rightarrow k(A_i)$: sep. closed $\Rightarrow A_i$: strictly henselian
and $X^{sh}_{\bar{y}} = \coprod_{i=1}^{r} \mathrm{Spec}(A_i)$. The statements follows from below lemma. $\square$

Lemma: R: strictly henselian ring $S = \mathrm{Spec}\, R$, then $\Gamma(S, -): S_{\acute{e}t} \to (Ab)$ is exact. In particular, $H^p_{\acute{e}t}(S, \mathcal{F}) = 0$ $\forall p \geq 1$, $\mathcal{F}$: sheaf on $S_{\acute{e}t}$.

pf: $\mathcal{U} = \{f_i: U_i \to S\}_{i \in I}$: étale covering. $s \in S$: closed pt of S
$\exists i$ s.t. $s \times_S U_i \neq \phi$,

$$\begin{array}{ccc} s \times_S U_i & \longrightarrow & U_i \\ \downarrow{\scriptstyle f_i'} & & \downarrow{\scriptstyle f_i} \\ s = \mathrm{Spec}(k(R)) & \xrightarrow[s]{} & S \end{array}$$

$\because s \times_S U_i \xrightarrow{f_i'} \mathrm{Spec}(k(R))$ étale $\therefore s \times_S U_i = \mathrm{Spec}\left(\prod_{i=1}^{r} L_i\right)$
L_i : sep. ext. of $k(R)$

But R: strictly henselian $k(R)$: sep. closed $\Rightarrow L_i \cong k(R)$, $\forall i$
Thus, $s \times_S U_i = \mathrm{Spec}\, k(R)^r \Rightarrow \exists$ section $g: s \to s \times_S U_i$
Then pick affine nbd $\mathrm{Spec}\, A$ of $u_i = g(s)$

$$\begin{array}{ccc} R & \longrightarrow & A \\ \downarrow & & \downarrow \\ k(R) & \longrightarrow & k(u_i) \end{array}$$

Again, $k(R)$: sep. closed and $k(R) \subset k(u_i)$: finite sep.
$\Rightarrow k(R) \simeq k(u_i)$.

$\Rightarrow A \cong R \times A' \Rightarrow A \to R$ projection

$$\rightsquigarrow \begin{tikzcd} \mathrm{Spec}\, A \arrow[r, hook] \arrow[dr, bend left] & U_i \arrow[d] \\ & S \arrow[ul, bend left, "\text{section}"] \end{tikzcd}$$

Thus, if $0 \to \mathcal{F} \to \mathcal{G} \xrightarrow{\alpha} \mathcal{H} \to 0$ SES in $X'_{\acute{e}t}$
Given $s \in \Gamma(S_{\acute{e}t}, \mathcal{H}) = \mathcal{H}(S)$, $\exists$ covering $\mathcal{U}$ and $s_i \in \mathcal{G}(U_i)$ s.t. $\alpha(U_i)(s_i) = s|_{U_i}$
But $S \to \mathrm{Spec}\, A$ lifts s locally to U_i
$\underset{\text{glueing}}{\rightsquigarrow}$ $0 \to \Gamma(S_{\acute{e}t}, \mathcal{F}) \to \Gamma(S_{\acute{e}t}, \mathcal{G}) \to \Gamma(S_{\acute{e}t}, \mathcal{H}) \to 0$

done.

- **Kummer Sequence**: If $\operatorname{char} k(x) \nmid n$, $\forall n \in \mathbb{N}$.

Define $n_X : \mathcal{O}^*_{X_{\acute{e}t}} \longrightarrow \mathcal{O}^{\times}_{X_{\acute{e}t}}$ by: $n_X(U) : \Gamma(U, \mathcal{O}_U)^{\times} \longrightarrow \Gamma(U, \mathcal{O}_U)^{\times}$
$$t \longmapsto t^n$$

$\mu_{n,X} = \ker(n_X) \rightsquigarrow 0 \to \mu_n \to \mathbb{G}_m \xrightarrow{n_X} \mathbb{G}_m \to 0$ Kummer seq.

remains to show: n_X is surj. By checking on stalks over gem. pt. $\bar{x}$

$\rightsquigarrow \quad 0 \to \mu_n(A) \to A^{\times} \xrightarrow{n} A^{\times} \to 0$, where $A = \mathcal{O}_{\bar{x}, X}$: strict local ring

But $\frac{d}{dT}(T^n - a) = nT^{n-1} \neq 0$ in $k(A)[T]$

$\because$ A: hensel $\therefore$ $T^n - a$ splits in $A[T]$. $\square$

- **Étale Cohomology for Spec(K)**:

Let K: field $\bar{K} = K^{sep}$: separable closure of K

$G = \operatorname{Gal}(\bar{K}/K)$ $X = \operatorname{Spec} K$ Any $Y = \operatorname{Spec} A \longrightarrow X = \operatorname{Spec} K$ finite étale

$\rightsquigarrow$ A: finite étale K-alge. $\Rightarrow A = \prod_i L_i$ L_i/K : sep. ext.

$\mathcal{F}$: étale sheaf of abelian groups over X

$\mathcal{F}(\bar{K}) = \varinjlim_{\substack{K \subset L \subset \bar{K} \\ [L:K] < \infty}} \mathcal{F}(L)$ and since L/K : sep. $\exists\, K \subset L \subset \tilde{L}$, $\tilde{L}/K$: Galois ext.

Hence, $\mathcal{F}(\bar{K})$: conti. G-module and hence we can define the group cohomology $H^{\cdot}(G, \mathcal{F}(\bar{K}))$.

<u>Fact</u>: $\{$étale sheaves over $X = \operatorname{Spec} k\} \xrightarrow[\text{of categ.}]{\text{equiv}} \{$conti. G-mod$\}$
$$\mathcal{F} \longmapsto \mathcal{F}(\bar{K}) = \mathcal{F}_{\operatorname{Spec}(\bar{K})}$$

For the reverse arrow:

Given G-mod. M, $L_i = \bar{K}^{G_i}$

$\mathcal{F}(A) = \mathcal{F}(\prod_i \bar{K}^{G_i}) = \bigoplus M^{G_i}$ for some $G_i \leq G$ Then

$\rightsquigarrow H^q(X_{\acute{e}t}, \mathcal{F}) = H^q(G, \mathcal{F}(\bar{K}))$

<u>Example</u>:

(1) $\mathbb{G}_{m,X} \longleftrightarrow \bar{K}^{\times}$ with natural group action G.

(2) $\mu_{n,X} \longleftrightarrow \mu_n(\bar{K})$: n-th roots of unity in $\bar{K}$

<u>Hilbert theorem 90"</u>: L/K: Galois ext. $G = \operatorname{Gal}(L/K)$, then
$$H^1(G, L^{\times}) = 0$$

(Note: $\bar{K} = K^{sep}/K$ is Galois)

Hence, we have: $H^1(X_{\acute{e}t}, \mathbb{G}_m) = H^1(G, L^{\times}) = 1$.

· Brauer Group and Tsen's thm:

<u>Recall</u>. K: field A: f.d. associative unital alge.

A: Central simple alge./K (CSA) if TFAE

(a) A has no 2-sided ideal and $C(A)=K$.

(b) $\exists\, L/K$: finite Galois s.t. $A_L = A\otimes_K L \cong M_\cdot(L)$

(c) $A \cong_K M_\cdot(\Delta)$, Δ: division alge. with $C(\Delta)=K$.

From (c), $A\sim A'$, where $A\cong_K M_\cdot(\Delta)$, $A'\cong M_\cdot(\Delta')$, if $\Delta\cong_K \Delta'$.

$Br(K)$: set of equiv. classes. of CSA alge./K with group op. $\otimes_K$.

$1=[M_\cdot(K)] \rightsquigarrow Br(K)$: Brauer group.

<u>Fact</u>: $Br(K)\cong H^2(G,\bar{K}^\times)$

<u>Fact</u>: K: field $\bar{K}=K^{sep.}$ $G=Gal(\bar{K}/K)$ If $Br(K')=0$, for any K'/K: finite ext., then $H^q(G,\bar{K}^\times)=0$.

(cf. Serre Galois Cohomology, §3 Prop 5)

<u>Def</u>: K: field. K is C_1 if $\forall\, f(x_1,\dots,x_n)\in k[x_1,\dots,x_n]$. non-const. homogen. poly. of degree $d<n$, then f has a non-trivial zero.

<u>Lemma</u>: K: C_1-field, then $Br(K)=0$.

<u>pf</u>: Suffices to show: any division alge. Δ/K with $C(\Delta)=K$, $\Delta\cong K$.

Observe: $\Delta\otimes_K\bar{K}\cong M_r(\bar{K})$ non-canonically (up to $Aut(M_r(\bar{K}))= GL_r(\bar{K}) \curvearrowright M_r(\bar{K})$ by conjugation)

Consider $\det: M_r(\bar{K})\to\bar{K}$ $\because$ Similar matrices have the same det

$\therefore$ descends to $N:\Delta\to K$ $\dim_K\Delta=r^2$.

Pick K-basis of Δ $\{e_1,\dots,e_{r^2}\}$, $x=\sum x_ie_i$. $N(x)=N(x_1,\dots,x_{r^2})$: homogen. poly of deg r^2

For $d\in\Delta^\times$, $N(d)\cdot N(d^{-1})=1 \Rightarrow N^{-1}(0)=0\in\Delta$.

However, if $r>1$ and K: C_1-field, $N(x_1,\dots,x_{r^2})$ has a non-trivial sol'n ($\because r^2>r>1$) $\Rightarrow$ $\cancel{\times}$

Hence, $r=1$ and $\Delta=K$.

$\square$

Thm (Tsen) $k=\bar{k}$: alge. closed field. K/k and $\mathrm{tr.deg}_k K = 1$, then K is C_1-field.

pf: Case 1: $K = k(X)$ i.e. $K = K(\mathbb{P}^1_K)$

$$f(t_1,\dots,t_n) = \sum a_{i_1\dots i_n} t_1^{i_1}\cdots t_n^{i_n} \quad d := \sum_{j=1}^{n} i_j < n,\ a_I = a_{i_1\dots i_n} \in k(X)$$

By cleaning the denominator, we may assume $a_I \in k[X]$.

and $r := \sup_I \deg(a_I)$.

Now, replace t_i by $c_{i0} + c_{i1}X + \cdots\cdots + c_{iN}X^N$, $i=1,\dots,n$

Let $\phi(c_{ij}) = f\left(\sum_{j=0}^{N} c_{1j}X^j, \dots, \sum_{j=0}^{N} c_{nj}X^j\right)$ $\deg_X \phi = r + Nd$

$$= \phi_0(c_{ij}) + X\phi_1(c_{ij}) + \cdots + X^{r+Nd}\phi_{r+Nd}(c_{ij})$$

$f(t_1,\dots,t_n) = 0 \Leftrightarrow \phi_0 = 0 = \cdots = \phi_{r+Nd}$

So, there are $r+Nd+1$-many eqns in $n\times(N+1)$-variables.

$\because k=\bar{k}$ $\therefore$ This has non-trivial sol'n $c_{ij}\in k$ if $n(N+1) > r+Nd+1$

$\because n>d$ $\therefore$ For $N \gg 0$, $n(N+1) > r+Nd+1$.

Hence, $\exists\, c_{ij}\in k$, not all zero s.t. $t_i(X) = \sum_{j=0}^{N} c_{ij}X^j$

$(t_1(X),\dots,t_n(X)) \in k[X]^n \subset k(X)^n$ is a non-trivial sol'n of f.

Case 2: If $K/k(X)$: alge.

Let $f(t_1,\dots,t_n)$: homogen. poly of deg $d<n$. with coeff. in K

Replace K by subfield $k(X)$ adjoing all coeff. of f, may assume

$K/k(X)$: finite. Let $s := [K : k(X)]$ $\{e_1,\dots,e_s\}$: $k(X)$-basis of K.

Introduce variables u_{ij} by : $t_i = \sum_{j=1}^{s} u_{ij}e_j$.

Consider $g = N_{K/k(X)}(f(t_1,\dots,t_n)) \rightsquigarrow$ plug t_i by $\sum u_{ij}e_j$

$g(t_1,\dots,t_n) = 0 \Leftrightarrow \phi(u_{ij}) = N_{K/k(X)}(f(\sum u_{1j}e_j, \dots, f(\sum u_{sj}e_j))$

$\phi(u_{ij})$ has degree sd in ns variables $\because d<n$

$\therefore \phi$ has non-trivial sol'n in $k(X)$ $\Rightarrow N_{K/k(X)}(f(\sum u_{ij}e_j)) = 0$

$\Rightarrow f$ has a non-trivial sol'n $t_i = \sum u_{ij}e_j$ in K $\square$

In fact, we prove any finite ext. of C_1-field is C_1 in case 2.

Cor: $k=\bar{k}$. K/k: transcendental deg. 1. Then $H^q(\mathrm{Spec}K, \mathbb{G}_m) = 0$ $\forall q>0$.

($q=1$: Satz 90 $q=2$. Tsen thm $q>2$. fact + Tsen)

- Étale Cohomology for non-sing. proj. curve.

X: non-sing. proj. curve / $k=\bar{k}$

Thm 1:

$$H^i(X_{ét}, \mathbb{G}_m) = \begin{cases} k^\times, & i=0 \\ \mathrm{Pic}(X), & i=1 \\ 0, & i\geq 2. \end{cases}$$

Let η: generic pt. of X, $j: \eta = \mathrm{Spec}(K(X)) \hookrightarrow X$

$\mathbb{G}_{m,\eta} = \mathbb{G}_m$ on $\eta_{ét} = K(X)^\times$. For $x \in X$: closed pt, i.e. $k(x)=k=\bar{k}$

$\mathbb{Z}_x := \underline{\mathbb{Z}}_x$ and $i_x : x \longrightarrow X$

By def'n. $j_*\mathbb{G}_{m,\eta} : U \mapsto K(U)^\times$

and $(i_x)_*\mathbb{Z}_x : U \mapsto k(x)$ i.e. skyscraper sheaf in étale.

$\overset{\shortparallel}{\mathrm{Div}_X}: \ U \longmapsto \mathrm{Div}(U)$

$\rightsquigarrow \ \forall U \to X, \quad 0 \to \Gamma(U, \mathcal{O}_U^\times) \to K(U)^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(U) \to 0$

$\Rightarrow \quad 0 \to \mathbb{G}_m \longrightarrow j_*\mathbb{G}_{m,\eta} \longrightarrow \bigoplus_{\substack{x\in X \\ \text{closed pt.}}} (i_x)_*\mathbb{Z}_x \to 0$ exact (divisor seq.)

Lemma 1: $R^q j_* \mathbb{G}_{m,\eta} = 0$, $\forall q>0$

pf: $\bar{x} : \mathrm{Spec}(K) \longrightarrow X$ $x = \bar{x}(\mathrm{Spec}\, K)$. Let $U = \mathrm{Spec}\, A$: affine nbd of x

$K = \mathrm{Frac}(A)$ s.t. $\mathrm{Spec}(\mathcal{O}^{sh}_{\bar{x},X}) \times_X \eta = \mathrm{Spec}(\mathcal{O}^{sh}_{\bar{x},X} \otimes_A K)$

A: Noeth. local ring. A^h or A^{sh} is also noeth.

$\Rightarrow \mathcal{O}^{sh}_{\bar{x},X}$: DVR and $\mathcal{O}^{sh}_{\bar{x},X} \otimes_A K$: localization of it.

$\Rightarrow \mathcal{O}^{sh}_{\bar{x},X} \otimes_A K = \mathrm{Frac}(\mathcal{O}^{sh}_{\bar{x},X})$ and $\mathrm{Frac}(\mathcal{O}^{sh}_{\bar{x},X})$: transcendental deg. 1

Cor of Tsen's thm $\Rightarrow (R^q j_* \mathbb{G}_{m,\eta})_{\bar{x}} = H^q(\mathrm{Spec}(\mathrm{Frac}(\mathcal{O}^{sh}_{\bar{x},X})), \mathbb{G}_m) = 0$ □

deg. case of Léray spectral seq.

$\therefore \ R^q j_* \mathbb{G}_{m,\eta} = 0 \ \forall q>0, \ H^q(X, j_*\mathbb{G}_{m,\eta}) = H^q(\eta, \mathbb{G}_{m,\eta}) \ \forall q \geq 0$

Tsen's thm again $\Rightarrow H^q(\eta, \mathbb{G}_{m,\eta}) = 0 \ \forall q>0$.

Lemma 2: $H^q(X, \bigoplus_{x\in X} (i_x)_*\mathbb{Z}_x) = 0 \ \forall q>0$.

pf: x: closed point $i_x : x \hookrightarrow X$ closed immersion $\Rightarrow i_x$: finite.

$R^q i_{x*}\mathbb{Z} = 0$. Again, $H^q(X, (i_x)_*\mathbb{Z}) = H^q_{ét}(\{x\}, \mathbb{Z})$

$\because x$: closed pt. $x = \mathrm{Spec}\, k$, $k=\bar{k}$ $\Rightarrow H^q_{ét}(\{x\}, \mathcal{F}) = 0 \ \forall \mathcal{F}$: sheaf on étale. □

Divisor seq. $\Rightarrow$

$$1 \to \underset{\underset{k^\times}{\parallel}}{H^0(X, \mathbb{G}_m)} \to \underset{\underset{K(X)^\times}{\parallel}}{H^0(X, j_*\mathbb{G}_{m,\eta})} \to \underset{\underset{\mathrm{Div}(X)}{\parallel}}{H^0(X, \bigoplus_{x\in X}(i_x)_*\mathbb{Z}_x)} \to \underset{\underset{\mathrm{Pic}(X)}{\parallel}}{H^1(X, \mathbb{G}_m)} \to 1$$

and $H^q(X, \mathbb{G}_m) = 0$, for $q > 1$.

Thm 2: If $g(X) = g$, $\mathrm{char}(k) = p$. $p \nmid n$.

$$H^q(X_{\acute{e}t}, \mu_n) = \begin{cases} \mu_n, & q=0 \\ (\mathbb{Z}/n\mathbb{Z})^{2g} & q=1 \\ \mathbb{Z}/n\mathbb{Z}, & q=2 \\ 0, & q>2. \end{cases}$$

Under $\mu_n \cong \mathbb{Z}/n$ (up to choice of n-th root of unity in k)

$$H^q(X_{\acute{e}t}, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} \mathbb{Z}/n\mathbb{Z}, & q=0,2 \\ (\mathbb{Z}/n\mathbb{Z})^{2g}, & q=1 \\ 0, & q>2. \end{cases}$$

$\updownarrow$

X: sm. proj. curve/$\mathbb{C}$

$H^q(X, \mathbb{Z}) = \mathbb{Z}, \mathbb{Z}^{2g}, \mathbb{Z}$, for $q = 0,1,2$.

pf: From Kümmer seq. $0 \to \mu_n \to \mathbb{G}_m \xrightarrow{n_X} \mathbb{G}_m \to 0$

$\rightsquigarrow 0 \to H^0(X_{\acute{e}t}, \mu_n) \to k^\times \xrightarrow{n} k^\times \to H^1(X_{\acute{e}t}, \mu_n) \to \mathrm{Pic}(X) \xrightarrow{n} \mathrm{Pic}(X)$

$\to H^2(X_{\acute{e}t}, \mu_n) \to 0$

$\because k = \bar{k}$ and $\gcd(p,n) = 1$ $\therefore$ $k^\times \xrightarrow{n} k^\times$ is surj. with

$\ker(n) = H^0(X_{\acute{e}t}, \mu_n) = \mu_n$ and $H^q(X_{\acute{e}t}, \mu_n) = 0$, for $q > 2$.

For $0 \to H^1(X_{\acute{e}t}, \mu_n) \to \mathrm{Pic}(X) \xrightarrow{n} \mathrm{Pic}(X) \to H^2(X_{\acute{e}t}, \mu_n) \to 0$

Also, from $0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z} \to 0$

$J(X)$: Jacobian variety of X. Then $\mathrm{Spec}\, k \to J(X)$

$\leftrightarrow$ $\mathrm{Pic}^0(X/k) = \mathrm{Pic}^0(X)$ i.e. $\mathrm{Pic}^0(X)$: k-rat'l pt of $J(X)$.

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Pic}^0(X) & \to & \mathrm{Pic}(X) & \to & \mathbb{Z} & \to & 0 \\ & & \downarrow & & \downarrow n & & \downarrow n & & \\ 0 & \to & \mathrm{Pic}^0(X) & \to & \mathrm{Pic}(X) & \to & \mathbb{Z} & \to & 0 \end{array}$$

$\rightsquigarrow H^2(X_{\acute{e}t}, \mu_n) = \mathrm{coker}(\mathbb{Z} \xrightarrow{n} \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$. and $H^1(X_{\acute{e}t}, \mu_n) = \underset{\overset{\{\mathcal{L} \in \mathrm{Pic}^0(X) : \mathcal{L}^n = \mathcal{O}_X\}}{\parallel}}{\mathrm{Pic}_n(X)}$

Now, consider $A = J(X)$: abelian var. of dim g $\quad \begin{array}{c} A \xrightarrow{n} A \\ P \longmapsto nP \end{array}$

Claim: (Will be proved next Tuesday)

$X_n = \ker(n_X) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$ if $p \nmid n$.

With this, we get $H^1(X_{\acute{e}t}, \mu_n) = \mathrm{Pic}_n(X) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$

$\square$