Algebraic Geometry II Homework
Chapter V Surfaces

A course by prof. Chin-Lung Wang

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Exercise 0 (by Kuan-Wen)**.**

This is an example of proof.

Remark. This is an example for how to write in this format.

1 Geometry on a Surface

Exercise 1 (by Shi-Xin)**.**

By Riemann-Roch theorem on surface, we have $\chi(\mathcal{L}^{-1}) = \frac{1}{2}C.(C+K) + \chi(\mathcal{O}_X), \chi(\mathcal{M}^{-1}) = \frac{1}{2}D.(D+K) + \chi(\mathcal{O}_X)$ and $\chi((\mathcal{L}\otimes\mathcal{M})^{-1}) = \frac{1}{2}(C+D).(C+D+K) + \chi(\mathcal{O}_X)$. Thus,

$$C.D = \chi(\mathcal{O}_X) - (\chi(\mathcal{L}^{-1}) + \chi(\mathcal{M}^{-1})) - \chi((\mathcal{L}\otimes\mathcal{M})^{-1})$$

Exercise 2 (by Chi-Kang)**.**

We have

$$\frac{1}{2}az^2 + bz + c = P(z) = \chi(\mathcal{O}(zH)) = (\frac{1}{2}H^2)z^2 - (\frac{1}{2}H.K)z + (1+p_a).$$

So we have $a = H^2, b = -\frac{1}{2}H.K, c = 1 + p_a$, and by adjunction formula $H.K + H^2 = 2\pi - 2$, thus $b = -\frac{1}{2}H.K = \frac{1}{2}H^2 + 1 - \pi$.

Let C be a curve in X, H be a hyperplane, and $D := H|_C$, then we have $P_C(z) = \chi(zD) = (H.C)z + 1 - g(C)$, hence deg$C$=$H.C$.

Exercise 3 (by Tzu-Yang Tsai)**.**

(a) Consider the exact sequence

$$0 \longrightarrow \mathcal{L}(-D) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_D \longrightarrow 0$$

Then take Euler symbol, by Riemann-Roch theorem we get

$$\chi(\mathcal{O}_D) = \chi(\mathcal{O}_X) - \chi(\mathcal{L}(-D)) = 1 + p_a(X) - (\frac{1}{2}D.(D+K) + 1 + p_a(X) = -\frac{1}{2}D.(D+K)$$

Thus $2p_a(D) - 2 = -2\chi(\mathcal{O}_D) = D.(D+K)$

(b) Since RHS of (a) is the sum of intersections, which only depends on the linear equivalence class of D, $p_a(D)$ also only depends on the linear equivalence class of D.

(c) $p_a(D) + p_a(-D) = \frac{1}{2}D.(D+K) + \frac{1}{2}D.(D-K) + 2 = D^2 + 2$
$\Rightarrow p_a(-D) = D^2 + 2 - p_a(D)$
$p_a(C+D) = \frac{1}{2}(C+D).(C+D+K) + 1 = \frac{1}{2}C.(C+K) + \frac{1}{2}D.(D+K) + 2 + C.D - 1$
$= p_a(C) + p_a(D) + C.D - 1$

Exercise 4 (by Yi-Tsung Wang)**.**

(a) Note that $\omega_X \cong \mathcal{O}_X(d-4)$, we may write $K \sim (d-4)$ for some hyperplane H, then $C.H = \deg C = 1$. By adjunction formula, $C^2 = 2g(C) - 2 - C.K = 2 - d$.

(b) For $d = 1$, consider $X : x + y = 0$. Clearly X is a nonsingular surface of degree 1 containing $x = y = 0$. For $d \geq 2$, consider $X : x^{d-1}z + xz^{d-1} + y^{d-1}w + yw^{d-1} = 0$, which is a surface of degree d containing $x = y = 0$. Note that the system

$$\begin{cases} (d-1)\,x^{d-2}z + z^{d-1} = 0 \\ (d-1)\,y^{d-2}w + w^{d-1} = 0 \\ x^{d-1} + (d-1)\,xz^{d-2} = 0 \\ y^{d-1} + (d-1)\,yw^{d-2} = 0 \end{cases}$$

has only a solution $(0,0,0,0) \notin \mathbb{P}^3$. Therefore X is nonsingular.

Exercise 5 (by Ping-Hsun Chuang)**.**

(a) Since X is a surface of degree d in $\mathbb{P}^3$, we have $K_X = (d-4)\,H$. Then, $K^2 = (d-4)^2\,H^2$. Also, using the exercise V.1.2, we have $H^2 = d$. Thus, $K_X^2 = d\,(d-4)^2$.

(b) By the exercise II.8.3, we have $K_X = p_1^*K_C + p_2^*K_{C'}$, where $\begin{array}{ccc} & X = C \times C' & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ C & & C' \end{array}$ are projection.

Since $\deg K_C = 2g - 2$, we have $p_1^*K_C = (2g-2)\,(\{\mathrm{pt}\} \times C') = (2g-2)\,C'$. Similarly, we have $p_2^*K_{C'} = (2g'-2)\,C$. Also, we know that $C^2 = 0$, $C'^2 = 0$, and $C.C' = 1$. Thus, we have

$$(K_X)^2 = (p_1^*K_C + p_2^*K_{C'})^2 = (2g-2)^2\,C'^2 + 2\,(2g-2)\,(2g'-2)\,C.C' + (2g'-2)^2\,C^2 = 8\,(g-1)\,(g'-1)\,.$$

Exercise 6 (by Tzu-Yang Chou)**.**

(a) Let p, q be projections from $C \times C$ to its two factors respectively. Note that $(C \times \{pt\}) \cdot \Delta = 1 = (\{pt\} \times C) \cdot \Delta$. We know $\Delta \simeq C$, so $\deg K_\Delta = \deg K_C = 2g - 2$
On the other hand, adjunction formula says that $\deg K_\Delta = (K_X + \Delta) \cdot \Delta = (p^* K_C + q^* K_C) \cdot \Delta + \Delta^2 = \Delta^2 + 4g - 4$ and hence $\Delta^2 = 2 - 2g$.

(b) If $al + bm + c\Delta = 0$, we intersect it with l, m and Δ respectively and we obtain $a = b = c = 0$.

Exercise 9 (by Yu-Ting Huang).

(a) Let $D' = H^2 D - (H.D)H$. $D'.H = 0$ If $D' \equiv 0$, $D'.D = 0$, then the equality holds. Otherwise, if $D' \not\equiv 0$, by Hodge Index Theorem, we have $0 > (D')^2 = H^4 D^2 - 2H^2(H.D)^2 + (H.D)^2 H^2$. i.e. $(D^2)(H^2) < (D.H)^2$. Now we can conclude that $(D^2)(H^2) \leq (D.H)^2$.

(b) Let $H = l + m$. H is ample, since every curve on X intersects H, and $H^2 = l^2 + 2l.m + m^2 = 2 > 0$. Let $E = l - m$ and $D' = (H^2)(E^2)D - (E^2)(D.H)H - (H^2)(D.E)E$. Then $E^2 = -2$ and $H.E = 0$. By (a), $2(16D^2 - 32ab) = (D')^2(H^2) \leq (D.H)^2 = 0$. Therefore, $D^2 \leq 2ab$. And also by (a), the equality hold if and only if $0 \equiv D' = -4D + 2(a+b)(l+m) - 2(a-b)(l-m)$. i.e. $D \equiv bl + am$.

Exercise 10 (by Yu–Chi Hou).

Let C be curve of genus g defined over $k = \overline{\mathbb{F}_q}$, let $C(\mathbb{F}_q) := C \times_{\mathrm{Spec}(k} \mathrm{Spec}(\mathbb{F}_q)$ be the $\mathbb{F}_q$−rational points of C, and let $N := \#C(\mathbb{F}_q)$ be the number of $\mathbb{F}_q$−rational points. Consider the k−linear Frobenius morphism $f : C_q \to C$, which $C_q \cong C$ as schemes. Recall that $f : C_q \to C$ corresponds to the extension $K(C) \subset K(C)^{1/q}$ and thus f is a finite morphism of degree q. From now on, we identify C_q as C.

On $X := C \times C$, let $\Gamma \subset X$ be the graph of f and $\Delta \subset X$ be the diagonal. It is obvious that each closed point of $\Gamma \cap \Delta$ is one–to–one corresponding to fixed point of f. Also, $f^\# : \mathcal{O}_{C,P} \to \mathcal{O}_{C,P}$ induces field homomorphism $k(P) \to k(P)$ given by $\bar{x} \to \bar{x}^q$. If P is a fixed point of f, then $\bar{x}^q = \bar{x}$ and thus $\bar{x} \in \mathbb{F}_q$. Therefore, $k(P) = \mathbb{F}_q$ and P is a $\mathbb{F}_q$–rational point. Conversely, if P is a $\mathbb{F}_q$–rational point, say $\phi : \mathrm{Spec}(\mathbb{F}_q) \to X$ with $\phi(\{0\}) = P$, then $k(P) \subseteq \mathbb{F}_q$ and thus P is a fixed point of f.

Thus, $\Gamma.\Delta$ is the number of fixed point of f, counted multiplicities. However, C is non–singular and f is purely inseparable morphism, we claim that:

Claim. *The scheme intersection $\Gamma \cap \Delta \subset X$ is reduced. That is, Γ and Δ intersect transversally.*

Proof. Let $(P, P) \in \Gamma \cap \Delta$ be a closed point, Δ has local equation $(x - y)$ in $\mathcal{O}_{X,(P,P)}$, where x is the local parameter of $\mathcal{O}_{C,P}$ of the first coordinate, y is the local paramter of $\mathcal{O}_{C,P}$ of the second coordinate. On the other hand, Γ has local equation $x^q - y$ in $\mathcal{O}_{X,(P,P)}$. Thus, the local multiplicities $(\Gamma.\Delta)_{(P,P)} = \mathrm{length}(\mathcal{O}_{C,P}/(x^q - x)$. Let $\bar{x}$ be the image of x in $k(P)$. Since P is a fixed point and hence $\bar{x}^q - \bar{x} = 0$ has no multiple roots. We find that each $(\Gamma.\Delta)_{(P,P)} = 1$ and hence $\#(\Gamma \cap \Delta) = \Gamma.\Delta$ □

Therefore, $\Gamma.\Delta = N$. Let $l = \{pt\} \times C$ and $m := C \times \{pt\}$. From Ex.V.1.6, we know that $l.\Delta = m.\Delta = 1$ and $\Delta^2 = 2 - 2g$. Similarly, $\Gamma.l = 1$ and $\Gamma.m = \deg(f) = q$. Also, $\Gamma \cong C$ and hence adjunction formula gives

$$2g - 2 = \Gamma(\Gamma + K_X) = \Gamma^2 + \Gamma.K_X,$$

and Ex.V.1.5 shows that $K_X = (2g-2)(l+m)$. Hence, $\Gamma^2 = (2g-2)q$. Now, for any $r, s \in \mathbb{Z}$, we consider $D := r\Gamma + s\Delta$ and

$$\begin{aligned} a &:= D.l = r\Gamma.l + s\Delta.l = r+s. \\ b &:= D.m = r\Gamma.m + s\Delta.m = rq+s. \end{aligned}$$

Hence, Castelnuovo–Severi inequality (Ex.V.1.9 (b)) shows that $D^2 \leq 2ab$. Thus, one has

$$D^2 = r^2\Gamma^2 + 2rs\Gamma.\Delta + s^2\Delta^2 = r^2q + 2rsN + s^2(2-2g) \leq 2(rq+s)(r+s).$$

Therefore, we have $qgr^2 - (N-(q+1))rs + gs^2 \geq 0$. Since the inequality works for any r, s, we must have $(N-(q+1))^2 - 4qg^2 \geq 0$ and thus $|N-(q+1)| \leq 2g\sqrt{q}$.

Exercise 11 (by Shuang-Yen Lee)**.**

(a) Let $E = \operatorname{Num} X \otimes_{\mathbb{Z}} \mathbb{R}$, then the bilinear pairing $\operatorname{Num} \times \operatorname{Num} \to \mathbb{Z}$ induces a bilinear form on E. By Hodge Index Theorem, we can choose a basis $\{h, e^1, \ldots, e^r\}$ such that the bilinear form is represented by $\operatorname{diag}(1, -1, \ldots, -1)$, where $h = H/\sqrt{H^2}$. If $C \equiv c_0 h + c_i e^i$ is an irreducible curve, then

$$c_0 = \langle h, C\rangle = \frac{1}{\sqrt{H^2}}\langle H, C\rangle \geq \frac{1}{\sqrt{H^2}}.$$

Let $K \equiv k_0 h + k_i e^i$, by adjunction formula, $C.(C+K) = 2p_a(C) - 2$, so

$$c_0(c_0+k_0) - \sum_i c_i(c_i+k_i) = 2p_a(C) - 2 \geq -2$$

which implies

$$c_0^2 + \sum_i c_i^2 \leq c_0^2 + c_0k_0 + 2 + \frac{1}{2}k_0^2 \leq Ac_0^2,$$

where $A = \sqrt{H^2}|k_0| + H^2(2 + \sum_i k_i^2/2)$ is a constant. Consider the set

$$S = \{\sum_i a_i C_i \mid C_i \text{ irred}, \ a_i \geq 0\} \subseteq E,$$

it's a cone in E. For any $\vec{v} \in S$, we have $\sum_i v_i^2 \leq Av_0^2$. So

$$S \subseteq \{\vec{v} \in E \mid \sum_i v_i^2 \leq Av_0^2, \ v_0 \geq 0\}.$$

Let $D_d = \{D \mid D \geq 0 \text{ and } H.D = d\}/\sim$, then

$$D_d \subseteq S \cap (\{d\sqrt{H^2}h\} \times \langle e^1, \ldots, e^r\rangle_{\mathbb{R}}) \subseteq \{\vec{v} \in E \mid \sum_i v_i^2 \leq A(d\sqrt{H^2})^2, \ v_0 = d\sqrt{H^2}\}$$

So $\#D_d < \infty$.

(b) If $\Gamma \equiv \Delta$, then $\Gamma.\Delta = \Delta^2 = 2 - 2g < 0$, this means $\Gamma = \Delta$. Let $H = C \times \{\text{pt}\} + \{\text{pt}\} \times C = \ell + m$, then H is ample, and $\Gamma.H = 2$ for each Γ since $\sigma \in \text{Aut}(C)$. This means the number of Γ is finite by (a).

Exercise 12 (by Pei-Hsuan Chang)**.**

By Nakai Moishezon criterion, ampleness is only depend on the numerical equivalence class.

To give a example to explain that very ampleness is not a numerical equivalence invariant. We first consider on a nonsingular genus g curve C, with $g > 2$. Take a divisor on C with degree $2g$,then $\ell(D) = g+1$ and $\forall P, Q \in C$, $\ell(D - P - Q) = g - 1 + \ell(K - D + P + Q)$ by Riemann-Roch theorem on curve. Thus, D is very ample $\Leftrightarrow$ D is not of the form $K + P + Q$ for some $P, Q \in C$. So we can take $D = K + P + Q$, then D is not very ample. Notice that the set $\{K + P + Q \mid P, Q \in C\}$ has dimension at most $2 < g + 1$. Thus, $\exists D' \in |D|$ which is very ample. Also, $D' \sim D \Rightarrow D' \equiv D$. Finally, let $X = C \times \mathbb{P}^1$, $\tilde{D} = D \times \mathbb{P}^1$, and $\tilde{D'} = D' \times \mathbb{P}^1$. Then $\tilde{D} \equiv \tilde{D'}$ but $\tilde{D'}$ is very ample, and $\tilde{D}$ is not.

2 Ruled Surfaces

Exercise 1 (by Shuang-Yen Lee)**.**

It suffices to show that if $C \times \mathbb{P}^1$ is birationally equivalent to $C' \times \mathbb{P}^1$, then $C \cong C'$.

Let $f : C \times \mathbb{P}^1 \dashrightarrow C' \times \mathbb{P}^1$ and g be its inverse. If both C and C' are $\mathbb{P}^1$ then we are done, so we may assume that C' is not $\mathbb{P}^1$. For any $x \in C$, consider the image of $f(\{x\} \times \mathbb{P}^1) \subseteq C' \times \mathbb{P}^1 \to C'$. Note that there are only finitely many x such that the image is empty. So for almost all x, there's a morphism $\mathbb{P}^1 \to C'$ by extension theorem. Since $C' \neq \mathbb{P}^1$, the map is constant, let it be $\widetilde{f}(x) \in C'$, then we have a map $C \to C'$ by $x \mapsto \widetilde{f}(x)$ by extension theorem. It is a morphism since it coincides with the map $C \times \{\text{pt}\} \subseteq C \times \mathbb{P}^1 \dashrightarrow C' \times \mathbb{P}^1 \to C'$.

If $C \neq \mathbb{P}^1$, similarly we have a map $\widetilde{g} : C' \to C$. Since f and g are inverse to each other, so is $\widetilde{f}$ and $\widetilde{g}$, hence $C \cong C'$.

If $C = \mathbb{P}^1$, then $\widetilde{f}$ is a constant, which means f is a constant, a contradiction.

Exercise 2 (by Yu–Chi Hou)**.**

($\Rightarrow$): If $\mathscr{E}$ is decomposable, then say $\mathscr{E} = \mathscr{L}_1 \bigoplus \mathscr{L}_2$. Then the surjections $\mathscr{E} \to \mathscr{L}_i \to 0$ corresponds to $\sigma_i : C \to X$ with $C_i = \sigma_i(C)$, for $i = 1, 2$. Therefore,

$$0 \to \mathscr{L}_1 \to \mathscr{E} \to \mathscr{L}_2 \to 0 \text{ and } \mathscr{L}_1 = \ker(\mathscr{E} \to \mathscr{L}_2).$$

Now, $\pi^*\sigma_1^*\mathcal{O}_X(1) = \pi^*\mathscr{L}_1 = \mathcal{O}_x(1) \otimes \mathcal{O}_X(-C_2)$ and hence

$$\mathcal{O}_X(1)|_{C_1} = \pi^*\sigma_1^*\mathcal{O}_X(1)|_{C_1} = \mathcal{O}_X(1)|_{C_1} \otimes \mathcal{O}_X(-C_2)|_{C_1} \Rightarrow \mathcal{O}_X(-C_2)|_{C_1} \cong \mathcal{O}_{C_1}.$$

Therefore, $\deg_{C_1}(\mathcal{O}_X(C_2)|_{C_1}) = C_1.C_2 = 0$. Since C_1, C_2 are irreducible non–singular curve in X,

$$C_1.C_2 = 0 \Rightarrow C_1 \cap C_2 = 0.$$

($\Leftarrow$): If there exists $\sigma_i : C \to X$ with $C_i = \sigma_i(C) \subset X$, for $i = 1, 2$, and $C_1 \cap C_2 = \emptyset$. This corresponds to surjections $\mathscr{E} \to \mathscr{L}_i \to 0$ and $\mathscr{N}_i := \ker(\mathscr{E} \to \mathscr{L}_i)$, for $i = 1, 2$. From Proposition V.2.6, we know that $\pi^*\mathscr{N}_i \cong \mathcal{O}_X(1) \otimes \mathcal{O}_X(-C_i)$, for $i = 1, 2$. Now, consider $\mathscr{N}_1 \bigoplus \mathscr{N}_2 \to \mathscr{E} \to \mathscr{F} \to 0$, where $\mathscr{F} := \operatorname{coker}(\mathscr{N}_1 \bigoplus \mathscr{N}_2 \to \mathscr{E})$. Thus, after applying π^* to the exact sequence,

$$\bigoplus_{i=1}^{2} \mathcal{O}_X(1) \otimes \mathcal{O}_X(-C_i) \xrightarrow{\psi} \mathcal{O}_X(1) \to \pi^*\mathscr{F} \to 0.$$

Notice that it suffices to prove that $\bigoplus_{i=1}^{2} \mathcal{O}_X(-C_i) \xrightarrow{\phi} \mathcal{O}_X$ is surjective, then after twisting $\mathcal{O}_X(1)$ shows that ψ is surjectve and hence $\pi^*\mathscr{F} = 0$. Since π is surjective onto a non–singular curve, π is a flat morphism. Moreover, $\pi^* : \mathfrak{Coh}_C \to \mathfrak{Coh}_X$ is an faithful functor, for one can check locally on affine chart and the fact that a local homomorphism is flat if and only if it is faithfully flat (cf. Matsumura, Commutative Ring theory, p.48). Therefore, $\pi^*\mathscr{F} = 0$ implies $\mathscr{F} = 0$. Now, both sides are locally free sheaves of rank 2 and $\mathscr{N}_1 \oplus \mathscr{N}_2 \to \mathscr{E}$ is surjective, $\mathscr{E} \cong \mathscr{N}_1 \oplus \mathscr{N}_2$.

Now, at each point $x \in X$, if $x \notin C_1 \cup C_2$, then $(\mathcal{O}_X(-C_i))_x = \mathcal{O}_{X,x}$, for both $i = 1, 2$. Therefore, ϕ_x is obviously surjective in this case. For $x \in C_1$, let f_1 be local equation of C_1 at x, then $(\mathcal{O}_X(-C_1))_x = f_1\mathcal{O}_{X,x} \subset \mathcal{O}_{X,x}$. Since $C_1 \cap C_2 = 0$, $(\mathcal{O}_X(-C_2))_x = \mathcal{O}_{X,x}$. Thus, for $a \in \mathcal{O}_{X,x}$, we can take $a \in (\mathcal{O}_X(-C_2))_x = \mathcal{O}_{X,x}$, $0 \in (f_1)$, then $\psi_x(0, a) = 0 + a = a$. Similar for $x \in C_2$, we shows that ϕ_x is surjective, for any $x \in X$ and thus ϕ is surjective.

Exercise 3 (by Yu-Chi Hou)**.**

(a) Given a locally free sheaf of rank r on a curve C. We take $n \gg 0$ such that $\mathcal{E}^\vee(n)$ is generated by global section. Then exercise II.8.2 shows that there exists $\mathcal{O}_C \to \mathcal{E}^\vee(n)$ such that

$$0 \to \mathcal{O}_C \to \mathcal{E}^\vee(n) \to \mathcal{E}' \to 0$$

with $\mathcal{E}'$ is locally free. Equivalently,

$$0 \to (\mathcal{E}')^\vee(n) \to \mathcal{E} \to \mathcal{O}_C(n) \to 0.$$

Then let $\mathcal{E}_{r-1} = (\mathcal{E}')^\vee(n) \subset \mathcal{E}$ be the locally free subsheaf and $\mathcal{E}/\mathcal{E}_{r-1} \cong \mathcal{O}_C(n)$. We then proceed by induction on rank r.

(b) Suppose $\Omega^1_{\mathbb{P}^2}$ is extension of invertible sheaves. That is,

$$0 \to \mathcal{O}_{\mathbb{P}^2}(n) \to \Omega^1_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(m) \to 0.$$

Then the long exact sequence of cohomology of above short exact seqeunce gives

$$\cdots \to H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n)) \to H^1(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}) \to H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m) \to \cdots\cdots$$

Since $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n)) = H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m)) = 0$ and hence $H^1(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}) = 0$. On the other hand, consider Euler sequence

$$0 \to \Omega^1_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^2} \to 0,$$

and from the long exact of cohomology of it, we have

$$H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}) = H^2(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}) = 0; \quad H^1(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}) \cong k.$$

This gives the contradction.

Exercise 4 (by Pei-Hsuan Chang)**.**

To shows is even: let $D \equiv C_0 + bf$ be a section, then $D^2 = C_0^2 + 2bC_0.f + b^2f^2 = -e + 2b$. Example V. 2.11.1 says that $e = 0$, so $D^2 = 2b$ is even.

(a) By computation above, just choose $D = C_0$, then $D^2 = 0$.

For $r \geq g + 1$, by exercise IV. 6.8, there excist a divisor E of degree r such that $|E|$ is base point free. Thus, we have a morphism $C \to \mathbb{P}^1$, and the graph of it is a section in X correspond to $D = C_0 + Ef$.

(b) if there exist $D = C_0 + Ef$ such that $D^2 = 2 \cdot 1 = 2$, then it induce a degree 1 morphism $C \to \mathbb{P}^1$, but $g(C) = 3 \to\leftarrow$. So $r = 1$ is impossible.

If C is hyperelliptic, then $\exists! g^1_2$ which induce a morphism $C \to \mathbb{P}^1$ of degree $2 \Rightarrow r = 2$ is possible. Now, suppose there is a morphism $C \to \mathbb{P}^1$ of degree 3 , then there exists a effective divisor D of degree 3. By Riemann-Roch theorem, $\ell(D) = 3 + 1 - 3 + \ell(K - D) \Rightarrow D$ is special. By Clifford's theorem, $\dim |D| \leq \frac{1}{2} \deg D = \frac{3}{2} \Rightarrow 2 \leq \ell(D) \leq \frac{5}{2} \Rightarrow \ell(D) = 2$. Notice that $D + (K - D) = K = g^1_2 + g^1_2$, so we can assume $D = P_1 + P_2 + P_3$, $K - D = P_4$, then D must contain a g^1_2. But $\dim |D - P_3| = 1 = \ell(D) - 1 = \dim |D| \Rightarrow D$ is not base point free, which is a contradiction. Hence, $r = 3$ is impossible in hyperelliptic case.

If C is nonhyperelliptic, then there is no degree 2 morphism $C \to \mathbb{P}^1$, so $r = 2$ is impossible in this case. On the other hand, consider $K_C - P$. It is base point free since K_C is very ample, and

$$\dim |K_C - P| = \dim |K_C| - 1 = (g - 1) - 1 = 3 - 1 - 1 = 1.$$

(The first equality follows form $|K_C - P|$ is base point free.) So $r = 3$ is possible in this case.

Exercise 6 (by Yu-Ting Huang)**.**

We shall prove by induction. The case rank 1 is trivial. Assume that the statement holds for rank $r - 1$. Let $\mathcal{E}$ be a locally free sheaf of rank r. When $n >> 0$, $H^0(\mathbb{P}^1, \mathcal{E}(-n)) = 0$. Let k be the largest number such that $\mathcal{E}(-k)$ does not vanish. Then we have a morphism induced by section: $0 \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{E}(-k)$. Twisting by 1, $0 \longrightarrow \mathcal{O}(k) \longrightarrow \mathcal{E} \longrightarrow \operatorname{coker}(\mathcal{O}(k) \longrightarrow \mathcal{E}) \longrightarrow 0$. By exercise II.8.2, $\mathcal{F} = \operatorname{coker}(\mathcal{O}(k) \longrightarrow \mathcal{E})$ is a locally free sheaf of rank $r - 1$. By the induction hypothesis, we may assume that $\mathcal{F} = \bigoplus_{i=1}^{r-1} \mathcal{O}(f_i)$. Next, twist the previous exact sequence by $(-k - 1)$, then obtain

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{E}(-k - 1) \longrightarrow \bigoplus_{i=1}^{r-1} \mathcal{O}(f_i - k - 1) \longrightarrow 0.$$

Take their global section, since $H^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$, for every i, $H^0(\mathbb{P}^1, \mathcal{E}(-k-1)) \longrightarrow H^0(\mathbb{P}^1, \mathcal{O}(f_i - k - 1))$ is surjective. However, $H^0(\mathbb{P}^1, \mathcal{E}(-k-1)) = 0$, so $H^0(\mathbb{P}^1, \mathcal{O}(f_i - k - 1)) = 0$. i.e. $f_i \leq k$.
Now,

$$Ext^1(\mathcal{F}, \mathcal{O}(k)) = \bigoplus_{i=1}^{r-1} Ext^1(\mathcal{O}(f_i), \mathcal{O}(k)) = \bigoplus_{i=1}^{r-1} H^1(\mathbb{P}^1, \mathcal{O}(k - f_i)) = \bigoplus_{i=1}^{r-1} H^0(\mathbb{P}^1, \mathcal{O}(-k + f_i - 2)) = 0.$$

Thus, the sequence $0 \longrightarrow \mathcal{O}(k) \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$ splits, then $\mathcal{E} = \mathcal{F} \oplus \mathcal{O}(k)$.

Exercise 7 (by Po-Sheng Wu)**.**

Let the elliptic ruled surface be given by the unique indecomposable extension $0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O}(P) \to 0, P \in C$. If $D^2 = 1$ for some section D, then by Prop 2.9., $D \sim C_0 + (\mathfrak{d} - P)f$ for some divisor $\mathfrak{d}$ on C, so $D^2 = C_0^2 + 2\deg(\mathfrak{d} - P) \Rightarrow deg(\mathfrak{d}) = 1$, so $\mathfrak{d}$ is equivalent to some $Q \in C$. Conversely, given $Q \in C$, we want to construct a section D_Q such that $D \sim C_0 + (Q - P)f$. By Prop 2.9., this is equivalent to finding a surjection $\mathcal{E} \to \mathcal{O}(Q) \to 0$. But in the proof of Thm 2.15., we have already constructed $0 \to \mathcal{O} \to \mathcal{E}' \cong \mathcal{E} \otimes \mathcal{O}(R - P) \to \mathcal{O}(Q) \to 0$, where $2R \sim P + Q$, then by tensoring $\mathcal{O}(R - P)$, and replacing Q, R with $2Q - P, Q$, then we obtain $0 \to \mathcal{O}(Q - P) \to \mathcal{E} \to \mathcal{O}(Q) \to 0$, and this gives us such section. The section is unique due to the uniqueness of the extension E', and D_Q are not equivalent to each others.

Exercise 8 (by Tzu-Yang Chou)**.**

(a) Decompose $\mathscr{E}$ as $\mathscr{F} \oplus \mathscr{G}$ and consider the short exact sequences: $0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{E} \longrightarrow \mathscr{G} \longrightarrow 0$ and $0 \longrightarrow \mathscr{G} \longrightarrow \mathscr{E} \longrightarrow \mathscr{F} \longrightarrow 0$. Looking at the slopes we see that $\mathscr{E}$ can not be stable.

(b) "$\Rightarrow$" Consider $0 \longrightarrow \mathcal{O}_C \longrightarrow \mathscr{E} \longrightarrow \mathscr{L} \longrightarrow 0$ and hence $\mu(\mathscr{E}) > \mu(\mathcal{O}_C) = 0$. So $\deg \mathscr{E} > 0$. The semistable case is totally the same.
"$\Leftarrow$" Assume that $\mathscr{E}$ is normalized and $\deg \mathscr{E} > 0$. If $\mathscr{E}$ is unstable, say $\exists \mathscr{F} \subseteq \mathscr{E}$: an invertible subsheaf such that $\mu(\mathscr{F}) \geq \mu\mathscr{E}$, that is $\deg \mathscr{F} \geq \frac{\deg \mathscr{E}}{2} > 0$. The map $0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{E}$ gives $0 \longrightarrow \mathcal{O} \longrightarrow \mathscr{E} \otimes \mathscr{F}^{-1}$ but $H^0(\mathscr{E} \otimes \mathscr{F}^{-1}) = 0$ by the normalized assumption.

(c) This is just the proof of Theorem 2.12, combined with (b).

Exercise 10 (by Chi-Kang)**.**

Use the very-ample divisor $D = C_0 + nf$, we want to find a curve Y on X s,t, Y is a canonical curve under the embedding i,e, $D|_Y = K_Y$. By adjunction formula we have $K_X + Y|_Y = K_Y$, so we want $D\ K_X + Y$. So consider the linear system $|D - K_X|$, if there is an smooth irreducible curve $Y \in |D - K_X|$, then Y is what we want. Note that $K_X = -2C_0 + (-e - 2)f$, so $D - K_X\ 3C_0 + (n + e + 2)f$, by corollary 2.18, this linear system has an smooth irreducible curve iff $e \geq 0$ and $n + e + 2 \geq 3e$, since we assume $n \geq 2e - 2$ and $e \geq 0$, this linear system must contains an smooth irreducible curve. and we have $\deg(K_Y) = D.Y = (C_0 + nf).(3C_0 + (n + e + 2)f) = 4n + e + 2 - 3e = 4n - 2e + 2 = 2d + 2$, hence $g(Y) = d + 2$. In particular, this method constructs canonical curves for any $g(Y) \geq 4$. Finally, we have $f.Y = 3$ and $\dim |f| = 1$, so $f|_Y$ is a g^1_3.

Exercise 14 (by Yi-Heng).

(a) Let $\widetilde{Y}$ be the normalization of Y, and $\varphi : \widetilde{Y} \to Y \to C$. For $2 \leq a \leq p-1$, we have $Y.(Y+K) \geq a(2g-2)$ by Hurwitz's formula, Adjunction formula and Ex IV.1.8. Thus, $b \geq \frac{1}{2}ea$ by direct computation. Next, for $p \leq a$, we only have $p_a(Y) \geq g$. Therefore, $b \geq \frac{1}{2}ea + 1 - g$. Last, for $a = 1$, since Y is a section corresponding to $\mathscr{E} \to \mathscr{L} \to 0$, $deg(\mathscr{L}) \geq deg(\mathscr{E}) \Rightarrow b \geq 0$.

(b) If D is ample, $a = D.f > 0$ and $D^2 > 0 \Rightarrow b > \frac{1}{2}ae$. Conversely, for all Y numerical equivalent to $aC_0 + bf$, we have $D.Y > 0$ by (a). Hence, D is ample by Nakai-Moishezon criterion.

3 Monoidal Transformations

Exercise 1 (by Shuang-Yen Lee).

It suffices to show that $H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \cong H^i(X, \mathcal{O}_X)$ for all i. By Ex III 8.1, we need to show that $\pi_*\mathcal{O}_{\widetilde{X}} \cong \mathcal{O}_X$ and $R^i\pi_*\mathcal{O}_{\widetilde{X}} = 0$ for $i > 0$. Since $\pi : \widetilde{X} \to X$ is a birational projective morphism and X is normal, $\pi_*\mathcal{O}_{\widetilde{X}} \cong \mathcal{O}_X$ follows from the proof of III 11.4.

Since $\widetilde{X} - \widetilde{Y} \xrightarrow{\pi} X - Y$ is an isomorphism, $\mathscr{F}^i := R^i\pi_*\mathcal{O}_{\widetilde{X}}$ is 0 outside Y. Let $y \in Y$, by formal function theorem, $\widehat{\mathscr{F}^i}_y \cong \varprojlim H^i(E_n, \mathcal{O}_{E_n})$, where E_n is the closed subscheme of $\widetilde{X}$ defined by $\mathscr{I}$ and $\mathscr{I}$ is the ideal sheaf of E, the inverse image of y of π. We have the exact sequence

$$0 \longrightarrow \mathscr{I}^n/\mathscr{I}^{n+1} \longrightarrow \mathcal{O}_{E_{n+1}} \longrightarrow \mathcal{O}_{E_n} \longrightarrow 0,$$

so we get the long exact sequence

$$\cdots \longrightarrow H^i(E, \mathscr{I}^n/\mathscr{I}^{n+1}) \longrightarrow H^i(E, \mathcal{O}_{E_{n+1}}) \longrightarrow H^i(E, \mathcal{O}_{E_n}) \longrightarrow \cdots.$$

Since $\mathscr{I}^n/\mathscr{I}^{n+1} \cong \mathrm{Sym}^n(\mathscr{I}/\mathscr{I}^2) \cong \mathcal{O}_E(n)$ and $E \cong \mathbb{P}^{r-1}$, where r is the codimension of Y in X, we have $H^i(E, \mathscr{I}^n/\mathscr{I}^{n+1}) \cong H^i(\mathbb{P}^{r-1}, \mathcal{O}_{\mathbb{P}^{t-1}}(n)) = 0$ for all $i > 0$, $n \geq 0$, this implies $H^i(E_n, \mathcal{O}_{E_n}) = 0$ by induction on n. So $\widehat{\mathscr{F}^i}_y = 0$, and hence $\mathscr{F}^i{}_y = 0$. Since y is arbitrary, $\mathscr{F}^i = 0$ for all $i > 0$. Thus, $p_a(X) = p_a(\widetilde{X})$.

Exercise 2 (by Tzu-Yang Tsai).

By property, $\tilde{C} = \pi^*C - \mu_P(C)E, \tilde{D} = \pi^*D - \mu_P(D)E$
$\Rightarrow \tilde{C}.\tilde{D} = (\pi^*C - \mu_P(C)E).(\pi^*D - \mu_P(D)E) = \pi^*C.\pi^*D - (\mu_P(C)E.\pi^*D + \mu_P(D)E.\pi^*C) + \mu_P(C)\mu_P(D)E^2$
$= C.D - \mu_P(C)\mu_P(D)$
For the second problem, first we assume that C, D are irreducible. Then by theorem, we know that we can blow up several points $\{P_i\}_{i=1}^r$ such that the strict transform of C, D, called it C', D', become nonsingular. By formula, $C'.D' = C.D - \sum_{i=1}^r \mu_{P_i}(C)\mu_{P_i}(D)$, and since C', D' are nonsingular, they intersect transversally, and $\#(C' \cap D') = C.D$, thus $C.D = \sum_{i=1}^r \mu_{P_i}(C)\mu_{P_i}(D) + \#(C' \cap D') = \sum_{i=1}^r \mu_{P_i}(C)\mu_{P_i}(D) + \sum_{Q \in C \cap D} 1 \times 1 = \sum \mu_P(C)\mu_P(D)$, where P sum over all intersection of C, D, including infinitely near ones.
Now consider the general case. By definition, a curve is an effective Cartier divisor, so it must be sum of irreducible curves. Observe that the LHS of equation obeys the distribution law, so does the RHS owing to the definition of multiplicity, thereby complete the proof.

Exercise 3 (by Yu-Ting Huang).

We will apply Nakai-Moishezon Criterion. Since D is very ample, $D^2 > 0$ and for every curve C on X, $D.C = 0$. Then, $(2\pi^*D - E)^2 = 4(\pi^*D)^2 - 4(\pi^*D.E) + E^2 = 4D^2 - 1 > 0$. Except exceptional curves, for every curves $\tilde{C}$ on $\tilde{X}$, we can write $\tilde{C} = \pi^*C + \mu_p(C)E$. Then, $(2\pi^*D - E).\tilde{C} = 2\pi^*D.\pi^*C + 2\mu_p(C)\pi^*D.E - E.\pi^*C - \mu_p(C)E^2 = 2D.C + 1 > 0$. Consider the case of exceptional curve: $(2\pi^*D - E).E = 1 > 0$. By Nakai-Moishezon Criterion, we conclude that $2\pi^*D - E$ is ample on $\tilde{X}$.

Exercise 4 (by Yu–Chi Hou).

(a) Given a Noetherian local ring (A, m), consider the associated graded ring $S := \mathrm{gr}_m(A) := \bigoplus_{d\geq 0} m^d/m^{d+1}$ of (A, m). Notice that $S_0 = A/m = k(m)$, the residue field of A. Since A is Noetherian, let $m = A\langle x_1, \dots, x_r\rangle$, for some $x_1, \dots, x_r \in m$. We denote $\bar{x}_i$ by the image of x_i in m/m^2. Then $S := k(m)[\bar{x}_1, \dots, \bar{x}_r]$ is a finitely generated $k(m)$-algebra. Also, from the exact sequence

$$0 \to m^d/m^{d+1} \to A/m^{d+1} \to A/m^d \to 0,$$

we have $\mathrm{length}(A/m^d) = \mathrm{length}(A/m) + \cdots + \mathrm{length}(m^{d-1}/d^d)$ and each m^i/m^{i+1} are finite $k(m)$-vector spaces. Thus, $\mathrm{length}_A(m^i/m^{i+1}) = \dim_{k(m)}(m^i/m^{i+1})$ is finite and thus $\psi(l) := \mathrm{length}(A/m^l)$ is well–defined. Now, apply Hilbert–Serre theorem (in the form of Atiyah–Macdonald Theorem 11.1 or Theorem 1.78 in Professor's Note of last semester) to $\psi(l)$, we can find a numerical polynomial $P_A(Z) \in \mathbb{Q}[z]$ such that $P_A(l) = \psi(l)$, for $l \gg 0$.

(b) This is just dimension theory for Noetherian local ring (cf.Atiyah-Macdonald, Theorem 11.4 or Theorem 1.82 of Professor's lecture note of last semester), which asserts that for a Noetherian local ring (A, m),

$$\deg P_A(n) = \dim A = \text{ number of minimal generators of certain } m\text{-primary ideal } q.$$

(c) By (b), we know that $n := \dim A = \deg P_A$. For a local ring (A, m) with Hilbert-Samuel function $P_A(z)$ associated to ψ_A as above, then we define the multiplicity $\mu(A)$ by $n!$ times the leading coefficients of P_A. For a Noetherian scheme X, we define the multiplicity $\mu_P(X)$ of X at P by $\mu(\mathcal{O}_{X,P})$.

(d) Let X be a surface, C be an irreducible curve on X, and $P \in C \subset X$ be a closed point. Let f be local equation of C in $\mathcal{O}_{X,P}$ and thus $f \in m_{X,P}$. Let r be the minimal $r \in \mathbb{N}$ such that $f \in m^r_{X,P}$ but $f \notin m^{r-1}_{X,P}$. Since X is non–singular, $\mathcal{O}_{X,P}$ is regular local ring. Let x, y be regular sequence of $\mathcal{O}_{X,P}$. Namely, $m_{X,P} = \mathcal{O}_{X,P}\langle x, y\rangle$ and y is not a zero divisor in $\mathcal{O}_{X,P}/\langle x\rangle$.

Now, let $A := \mathcal{O}_{C,P} = \mathcal{O}_{X,P}/\langle f\rangle$, $\dim A = 1$ with maximal ideal $m := m_{C,P} = m_{X,P}/\langle f\rangle = A\langle \bar{x}, \bar{y}\rangle$, where $\bar{x}, \bar{y}$ are the image of x, y in A respectively. Since $A/m = k$, $\mathrm{length}_A(A/m^l) = \dim_k(A/m^l)$. Write the Hilbert-Samuel polynomial $P_A(z) = \mu_P(C)z + c$, where $c = P_A(0)$. Hence, for $n \gg 0$,

$$P_A(n+1) - P_A(n) = \mu_P(C) = \mathrm{length}_A(A/m^{n+1}) - \mathrm{length}(A/m^n)$$

Now, consider the following sequence of $k-$vector spaces

$$\mathcal{O}_{X,P}/m^{n-r}_{X,P} \xrightarrow{\phi} \mathcal{O}_{X,P}/m^n_{X,P} \xrightarrow{\psi} \mathcal{O}_{X,P}/(m^n_{X,P} + f) = \mathcal{O}_{C,P}/m^n_{C,P},$$

where $\psi(\bar{g}) = \bar{f}g$ and ϕ is just quotient. Since $f \in m_{X,P}^r$, $fg \in m_{X,P}^n$ if $g \in m_{X,P}^{n-r}$. Thus, the sequence is exact. As a result, we have

$$\begin{aligned}\dim_k(A/m^n) &= \dim_k(\mathcal{O}_{X,P}/m_{X,P}^n) - \dim_k(\mathcal{O}_{X,P}/m_{X,P}^{n-r}) \\ &= \dim_k(k[x,y]/(x,y)^n) - \dim_k(k[x,y]/(x,y)^{n-r}) \\ &= (n-r+1) + \cdots + n = nr - \frac{r(r-1)}{2}.\end{aligned}$$

Thus, $\mu_P(C) = (n+1)r - nr = r$. Recall that we previously define r as the multiplicity of C at P and hence two definitions coincide.

(e) Let $Y = \mathrm{Proj}(k[x_0, \ldots, x_n]/I)$ be a projective variety in $\mathbb{P}^n$ of degree d with homogenous ideal I. Recall that the degree $\deg Y$ of Y is defined to be the $(\dim Y)!$ times the leading coefficient of Hilbert polynomial P_Y for the graded ring $S(Y) = k[x_0, \ldots, x_n]/I = k[\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_n]$, where $\bar{x}_i$ is the image of x_i in S and has degree 1. Now, consider the projective cone X over Y. Its local ring at vertex P_0 is the same as the affine cone $X_0 := \mathrm{Spec}(k[\bar{x}_0, \ldots, \bar{x}_n])$ at vertex $P_0 = (\bar{x}_0, \ldots, \bar{x}_n)$. Hence, $A := \mathcal{O}_{X_0,P_0} = k[\bar{x}_0, \ldots, \bar{x}_n]_{(\bar{x}_0,\ldots,\bar{x}_n)}$, where $m = A\langle \bar{x}_0, \ldots, \bar{x}_n\rangle$. Thus, $\mathrm{gr}_m(A)_0 = A/m = k$ and $\mathrm{gr}_m(A) = k[\bar{x}_0, \ldots, \bar{x}_n]$. As a result, the Hilbert-Samuel function of (A, m) is the same as the Hilbert function for the graded ring $S(Y)$. Thus, $P_A(z) = P_H^Y(z)$. Also, $\deg P_A = \dim A = \dim Y$, hence $\mu_{P_0}(X) = \mu_{P_0}(X_0) = \deg(Y) = d$.

Exercise 5 (by Shuang-Yen Lee)**.**

Let $f(x) = \prod_i (x - \alpha_i)$, then the singularities of $y^2 = f(x)$ on $\mathbb{A}^2$ are the commom solutions of

$$f(x) - y^2 = f'(x) = 2y = 0,$$

which is impossible (when $\mathrm{char}(k) = 2$) since $f(x)$ and $f'(x)$ has no common solution. Take the affine chart $y \neq 0$, we have $z^{r-2} = \prod_i (x - a_i z)$, then the multiplicity at P is $r-2$.

Let $z = wx$, then the curve is defined by $w^{r-2} = \prod_i (1 - a_i w)x^2$ after blowing-up, the singularity only happens when $(w, x) = (0, 0)$.

Let $x = wz$, then the curve is defined by $1 = \prod_i (w - a_i)z^2$, so $(w, z) = (0, 0)$ is not on the curve. Therefore we don't need to consider this case.

Now the multiplicity is 2.

Let $x = wt$, then we get $w^{r-4} = \prod_i (1 - a_i w)t^2$, again, the only singular point is $(t, w) = (0, 0)$. If we blow-up on the other chart, $w = tx$, then we still have $(t, x) = (0, 0)$ is not on the curve. Replace t by x, we get $w^{r-4} = \prod_i (1 - a_i w)x^2$, keep blowing-up by letting $x = wt$, then we will get $w^e = \prod_i (1 - a_i w)x^2$, $e = r\%2$. There are no singularities in both case so we get $\widetilde{Y}$ now. Then

$$\delta_P = \frac{1}{2}(r-2)(r-3) + \frac{1}{2} \cdot 2 \cdot 1 \cdot \left\lfloor \frac{r-2}{2} \right\rfloor = \frac{1}{2}(r-2)(r-3) + \left\lfloor \frac{r-2}{2} \right\rfloor$$

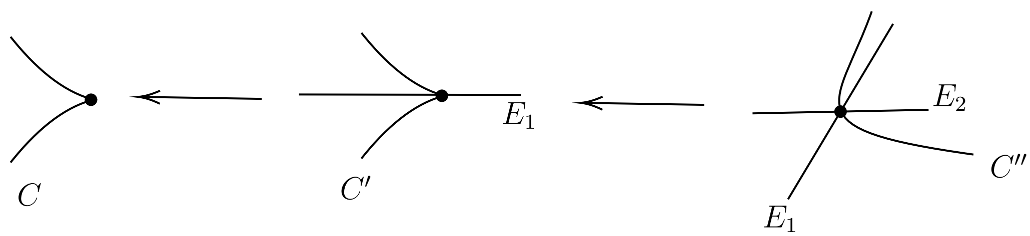
and

$$p_a(\widetilde{Y}) = p_a(Y) - \delta_P = \frac{1}{2}(r-1)(r-2) - \frac{1}{2}(r-2)(r-3) - \left\lfloor \frac{r-2}{2} \right\rfloor = \left\lfloor \frac{r-1}{2} \right\rfloor$$

Remark. In fact, the blow-up above tells us that we can change the coordinate near P by $(1/x, y/x^{g+1})$.

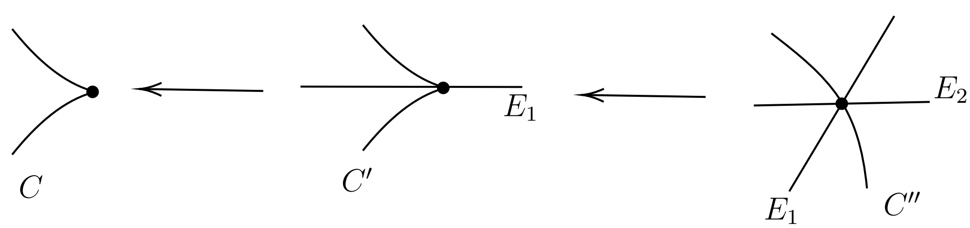
Exercise 7 (by Ping-Hsun Chuang)**.**

(a) Let C be the curve $x^3 + y^5 = 0$. The singularity $(0, 0)$ has multiplicity 3. Blowing up at $(0, 0)$, the resulting curve has only one singularity. Taking the chart $(y, s) = (y, x/y)$, the curve become $s^3 + y^2 = 0$ with $E_1 = \{y = 0\}$. This curve has only singularity at $(0, 0)$ with multiplicity 2. Then, Blowing up at $(0, 0)$ and taking the chart $(v, s) = (y/s, s)$, we compute that the curve become $s + v^2 = 0$ with $E_1 = \{v = 0\}$ and $E_2 = \{s = 0\}$. This curve is non-singular. The resolution of singularity can be expressed in the following graph:



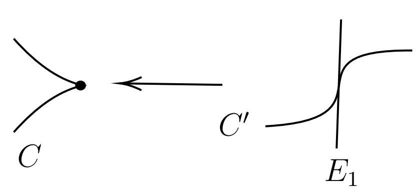
Also, $\delta_p = \dfrac{1}{2}3 \cdot 2 + \dfrac{1}{2}2 \cdot 1 = 4$.

(b) Let C be the curve $x^3 + x^4 + y^5 = 0$. The singularity $(0, 0)$ has multiplicity 3. Blowing up at $(0, 0)$, the resulting curve has only one singularity. Taking the chart $(y, s) = (y, x/y)$, the curve become $s^3 + s^4y + y^2 = 0$ with $E_1 = \{y = 0\}$. This curve has only singularity at $(0, 0)$ with multiplicity 2. Then, Blowing up at $(0, 0)$ and taking the chart $(v, s) = (y/s, s)$, we compute that the curve become $s + vs^3 + v^2 = 0$ with $E_1 = \{v = 0\}$ and $E_2 = \{s = 0\}$. This curve is non-singular. The resolution of singularity can be expressed in the following graph:



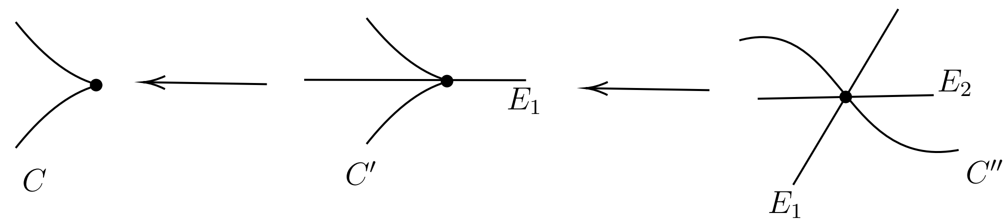
Also, $\delta_p = \dfrac{1}{2}3 \cdot 2 + \dfrac{1}{2}2 \cdot 1 = 4$.

(c) Let C be the curve $x^3 + y^4 + y^5 = 0$. The singularity $(0, 0)$ has multiplicity 3. Blowing up at $(0, 0)$, the resulting curve is non-singular. Taking the chart $(y, s) = (y, x/y)$, the curve become $s^3 + y + y^2 = 0$ with $E_1 = \{y = 0\}$. The resolution of singularity can be expressed in the following graph:



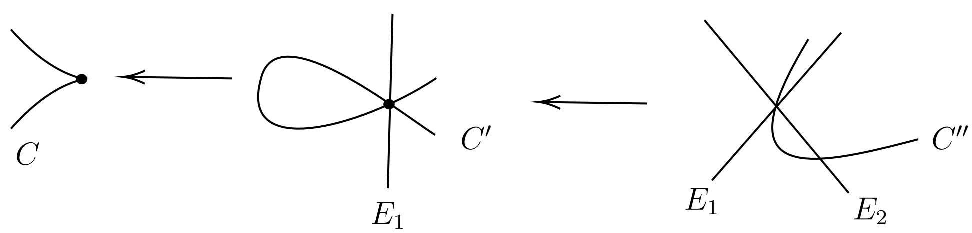
Also, $\delta_p = \frac{1}{2} 3 \cdot 2 = 3$.

(d) Let C be the curve $x^3 + y^5 + y^6 = 0$. The singularity $(0,0)$ has multiplicity 3. Blowing up at $(0,0)$, the resulting curve has only one singularity. Taking the chart $(y,s) = (y, x/y)$, the curve become $s^3 + y^2 + y^3 = 0$ with $E_1 = \{y = 0\}$. This curve has only singularity at $(0,0)$ with multiplicity 2. Then, Blowing up at $(0,0)$ and taking the chart $(v,s) = (y/s, s)$, we compute that the curve become $s + v^2 + sv^3 = 0$ with $E_1 = \{v = 0\}$ and $E_2 = \{s = 0\}$. This curve is non-singular. The resolution of singularity can be expressed in the following graph:



Also, $\delta_p = \frac{1}{2} 3 \cdot 2 + \frac{1}{2} 2 \cdot 1 = 4$.

(e) Let C be the curve $x^3 + xy^3 + y^5 = 0$. The singularity $(0,0)$ has multiplicity 3. Blowing up at $(0,0)$, the resulting curve has only one singularity. Taking the chart $(y,s) = (y, x/y)$, the curve become $s^3 + sy + y^2 = 0$ with $E_1 = \{y = 0\}$. This curve has only singularity at $(0,0)$ with multiplicity 2. Then, Blowing up at $(0,0)$ and taking the chart $(v,s) = (y/s, s)$, we compute that the curve become $s + v + v^2 = 0$ with $E_1 = \{v = 0\}$ and $E_2 = \{s = 0\}$. This curve is non-singular. The resolution of singularity can be expressed in the following graph:



Also, $\delta_p = \frac{1}{2} 3 \cdot 2 + \frac{1}{2} 2 \cdot 1 = 4$.

By the resolution graph, we conclude that the sigularity of $(0,0)$ of $(a)\,(b)\,(d)$ are equivalent and $(c)\,(e)$ are different from the others.

Exercise 8 (by Yi-Heng)**.**

(a) Blowing up $x^4 - xy^4$ at $(0,0)$ by $xy_1 = x_1 y$, the singularities occur at $(0,0)$ of $x_1^4 - x_1 y$ in the chart $y_1 = 1$, which is a node. Thus, $\delta = 7$.

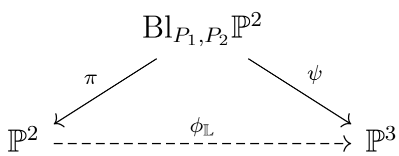
(b) Blowing up $x^4 - x^2y^3 - x^2y^5 + y^8$ at $(0,0)$ by $xy_1 = x_1y$, the singularities occur at $(0,0)$ of $x_1^4 - x_1^2y^3 - x_1^2y^3 + y^4$ in the chart $y_1 = 1$, which is of multiplicity 3. Then, blowing up $x_1^4 - x_1^2y^3 - x_1^2y^3 + y^4$ at $(0,0)$ by $x_1y_2 = x_2y$, then the strict transformation is non-singular. Thus, $\delta = 9$.

I guess the equation in (b) should be $x^4 - x^2y^2 - x^2y^5 + y^8$, then after the first blowing up, it has a cusp. As a result, (a) and (b) is not equivalent while their δ's are both 7.

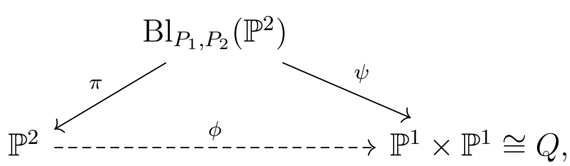
4 The cubic Surface in $\mathbb{P}^3$

Exercise 1 (by Yu–Chi Hou)**.**

Consider the linear system $\mathbb{L} := |2L - P_1 - P_2|$ and this gives a rational map $\phi := \phi_{\mathbb{L}} : \mathbb{P}^{\nvDash} \dashrightarrow \mathbb{P}^3$. By a projective linear transformation, we may assume $P_1 = [0:1:0]$, $P_2 = [0:0:1]$, then we can write down the rational map ϕ as following. Observe that $V = \{ax_0^2 + bx_0x_1 + cx_0x_2 + dx_1x_2 : a,b,c,d \in k\}$ is the subspace of $\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ such that $\mathbb{P}(V) = \mathbb{L}$. Then $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ is given by $[x_0 : x_1 : x_2] \to [x_0^2 : x_0x_1 : x_0x_2 : x_1x_2]$. Thus, ϕ is undefined only at P_1 and P_2 and its image is contained in $Q := V(Z_0Z_3 - Z_1Z_2) \subset \mathbb{P}^3$, which is a non–singular quadric surface. We now resolve the indeterminacy of this rational map by blowing up P_1 and P_2 on $\mathbb{P}^2$, we have:



Then $\mathrm{im}(\psi) \subset Q$ and $\mathrm{Bl}_{P_1,P_2}(\mathbb{P}^2)$ is irreducible of dimension 2, and hence $\psi : \mathrm{Bl}_{P_1,P_2}(\mathbb{P}^2) \to Q \subset \mathbb{P}^3$. Since$|2L - P_1 - P_2|$ has no unasigned based point and $\dim|2L - P_1 - P_2| = \dim|L - P_1| + \dim|L - P_2|$, $|L - P_1| \times |L - P_2| \to |2L - P_1 - P_2|$ is bijective. Therefore, one also sees that the strict transform L' of L_{P_1,P_2}, the line through P_1 and P_2, on $\mathrm{Bl}_{P_1,P_2}(\mathbb{P}^2)$ has self–intersection number $L'^2 = \pi^*L_{P_1,P_2}^2 - E_1^2 - E_2^2 = -1$, where E_1 and E_2 are exceptional divisors of P_1 and P_2 respectively. Moreover, ϕ also has the following diagram.



where $\mathbb{P}^1 \times \mathbb{P}^1 \cong Q$ via Segre embedding.

Since there is no (-1)-curves on $\mathbb{P}^1 \times \mathbb{P}^1$, ψ must contract only L'. Thus, ψ is just blowing up of a point on Q.

Addendum: Let $l = \mathbb{P}^1 \times \{pt\}, m = \{pt\} \times \mathbb{P}^1$. Then for any curve Y in $\mathbb{P}^1 \times \mathbb{P}^1$, $Y \equiv al + bm$ in $\mathrm{Num}(\mathbb{P}^1 \times \mathbb{P}^1)$, for some $a, b \in \mathbb{Z}$. Since $H := l + m$ is ample, $H.Y = b + a > 0$. If $Y^2 = ab < 0$, then this leads to a contradiction. Thus, there is no $(-1)-$curve on $\mathbb{P}^1 \times \mathbb{P}^1$.

Exercise 2 (by Yu-Ting Huang)**.**

We may assume $P_1 = (1,0,0)$, $P_2 = (0,1,0)$, and $P_3 = (0,0,1)$. Let C be defined by $f(x_0, x_1, x_2)$ and $\deg f = d$. φ is defined by $(y_0, y_1, y_2) = (x_1x_2, x_0x_2, x_0x_1)$, where y_0, y_1, y_2 are the new coordinates. Since $\mu_{P_1}(C) = r_1$, r_1 is the largest number such that $f \in \mathfrak{m}_{P_1}^{r_1} = (x_1, x_2)^{r_1}$. We can write

$$f(x_0, x_1, x_2) = f_{r_1}(x_1, x_2)x_0^{d-r_1} + \cdots + f_d(x_1, x_2)$$

, where $\deg f_j = j$. Now,

$$\begin{aligned} f(x_1x_2, x_0x_2, x_0x_1) &= f_{r_1}(x_0x_2, x_0x_1)(x_1x_2)^{d-r_1} + \cdots + f_d(x_0x_2, x_0x_1) \\ &= x_0^{r_1}(f_{r_1}(x_2, x_1)(x_1x_2)^{d-r_1} + \cdots + x_0^{d-r_1}f_d(x_2, x_1)) \end{aligned}$$

Thus, $x_0^{r_1} | f(x_1x_2, x_0x_2, x_0x_1)$. Similarly, we have $x_1^{r_2} | f(x_1x_2, x_0x_2, x_0x_1)$ and $x_2^{r_3} | f(x_1x_2, x_0x_2, x_0x_1)$. Let the curve C' be defined by g.

$$g = \frac{f(x_1x_2, x_0x_2, x_0x_1)}{x_0^{r_1}x_1^{r_2}x_2^{r_3}} = f_{r_1}(x_2, x_1)(x_1x_2)^{d-r_1}x_1^{-r_2}x_2^{-r_3} + \cdots + x_0^{d-r_1}f_d(x_2, x_1)x_1^{-r_2}x_2^{-r_3}.$$

$d - r_2 - r_3$ is the largest number such that $g \in \mathfrak{m}_{Q_1}$. i.e. $\mu_{Q_1}(C') = d - r_2 - r_3$. Similarly, we have $\mu_{Q_2}(C') = d - r_1 - r_3$ and $\mu_{Q_3}(C') = d - r_2 - r_1$.

Exercise 3 (by Shuang-Yen Lee)**.**

Let P be a singularity, then we can write C as $f(x, y) = g_r + g_{r+1} + \cdots = 0$ locally, where g_j is a homogeneous polynomial of degree j. Write $g_r(x, y) = \prod_i(\lambda_i x - \mu_i y)^{e_i}$, we need to make these e_i's into 1. Choose two lines L_1, L_2 through P such that the slope of the lines are different with the tangent lines at P and meets C transversally outside P. Take L_3 such that $\mathrm{Sing}(C) \cap L_3 = \varnothing$, L_3 meets C transversally and $Q_1 = L_1 \cap L_3$, $Q_2 = L_2 \cap L_3$ are not on C. Then we do the quadratic transformation centered at P, Q_1, Q_2. Blowing-up at Q_1, Q_2 doesn't changes the singularity, the singularities from the blown-down are ordinary. So we can concentrate at blowing-up P_1, we only need to see one of the $[\mu_i : \lambda_i]$ on E_1. May assume $\lambda = 1$ and let $x = (t + \mu)y$, then we get the equation

$$t^e y^r h_r(t) + \sum_{k=1}^{\infty} t^{e'_k} y^{r+k} h_{r+k}(t) = 0,$$

where $h_i(t) = g_i(t + \mu, 1)/t^{e'_{k-r}}$, $e'_k = \mathrm{ord}_t(g_i(t + \mu, 1))$. So we have

$$t^e h_r(t) + \sum_{k=1}^{\infty} t^{e'_k} y^k h_{r+k}(t) = 0.$$

If none of the $e'_k = 0$, then we can divide t on both side to let e be smaller, so assume that there's a k_0 such that $e'_k = 0$, let k_0 be the smallest, note that the term is of degree k.

The multiplicity at $(0,0)$ is $\min\{e, e'_k + k\}$. If there's a k such that $e'_k + k < e$, then the multiplicity is less than e, then we are done. If not, then $e \le e'_k + k$ for all k, then the equation at degree e is

$$c_r t^e + \sum_{e'_k + k = e} c_{r+k} t^{e'_k} y^k. \qquad (\spadesuit)$$

Now we are done if there's a k such that $e'_k + k = e$ since we can factor (♠) into linear equations such that every direction has multiplicity less than e. So now $e < e'_k + k$ for all k.

Blowing up again, let $t = sy$, then we get

$$s^e h_r(sy) + \sum_{k=1}^{\infty} s^{e'_k} y^{e'_k + k - e} h_{r+k}(sy) = 0.$$

Now the minimal term such that $e'_k = 0$ is $k = k_0$ and the term is of degree $k_0 - e < k_0$. So induction on this then we are done.

Exercise 4 (by Tzu-Yang Chou)**.**

(a) Consider the cubics $C' := PP' + QQ' + RR'$ and $C" := L + L' + P"Q"$. Intersect them with C we find that $C' \cap C = C" \cap C$ and the nine points $P, Q, R, P', Q', R', P", Q", R"$ all belong to them. In particular, $R" \in C"$ and hence $R" \in P"Q"$.

(b) Let P, Q, U be three arbitrary points on the cubic C and define the following points: $PQ \cap C = P, Q, R; P_0R \cap C = P_0, R, T; TU \cap C = T, U, V; P_0V \cap C = P_0, V, W; QU \cap C = Q, U, X; PV \cap C = P, V, Y$.
Now let $L := PQR, L' := TUV$, then part (a) says that P_0, X, Y are colinear. Combining all these together, we have $(P + Q) + U = T + U = W = P + Y = P + (Q + U)$.

Exercise 5 (by Yu-Ting Huang)**.**

Let C be the conic. We have three cubics: $X_1 = \overline{AB'} + \overline{CA'} + \overline{BC'}$, $X_2 := \overline{A'B} + \overline{C'A} + \overline{B'C}$, and $X_3 = C + \overline{PQ}$. By Corollary 4.5, $X_1.X_3 = X_1.X_2 = \{A, B, C, A', B', C'P, Q, R\}$. Thus $R \in \overline{PQ}$. i.e. P, Q, R are colinear.

Exercise 7 (by Yi-Tsung Wang)**.**

Write $D = D_1 - D_2$ with D_i effective. By proposition 5.4.8(b) and $\deg D = \deg D_1 + \deg D_2$, we see that if $D \sim a\ell - \sum b_i e_i$, then $\deg D = 3a - \sum b_i$. Then

$$2p_a(D) = D^2 + D.K = a^2 - \sum b_i^2 - \deg D = \frac{1}{9}\left(\left(d + \sum b_i\right)^2 - 9\sum b_i^2 - 9d\right)$$

To show that $p_a(D) \le \frac{1}{6}(d-1)(d-2) + \frac{2}{3}$, it suffices to show that $(d + \sum b_i)^2 - 9\sum b_i^2 - 9d \le 3d^2 - 9d$, and it is equivalent to show that $2d\sum b_i + (\sum b_i)^2 - 9\sum b_i^2 \le 2d^2$. Since $6\sum b_i^2 \ge (\sum b_i)^2$, we have

$$2d^2 \ge 2d\sum b_i - \frac{1}{2}\left(\sum b_i\right)^2 \ge 2d\sum b_i + \left(\sum b_i\right)^2 - 9\sum b_i^2$$

Therefore $p_a(D) \le \frac{1}{6}(d-1)(d-2) + \frac{2}{3}$. If $d \equiv 1, 2 \pmod 3$, since $p_a(D) \in \mathbb{Z}$ and $\frac{1}{6}(d-1)(d-1) \in \mathbb{Z}$, we have $p_a(D) \le \frac{1}{6}(d-1)(d-2)$. Now for $d \equiv 0 \pmod 3$, consider $D \sim d\ell - \sum \frac{d}{3} e_i$, then $\deg D = d$ and $p_a(D) = \frac{1}{6}(d-1)(d-2) + \frac{2}{3}$. For $d \equiv 1 \pmod 3$ with $d > 4$, consider

$$D \sim d\ell - \frac{d-1}{3}e_1 - \frac{d-1}{3}e_2 - \frac{d-1}{3}e_3 - \frac{d-1}{3}e_4 - \frac{d+2}{3}e_5 - \frac{d+2}{3}e_6$$

Then $\deg D = d$ and $p_a(D) = \frac{1}{6}(d-1)(d-2)$. For $d \equiv 2 \pmod 3$ with $d > 5$, consider

$$D \sim d\ell - \frac{d-2}{3}e_1 - \frac{d-2}{3}e_2 - \frac{d-2}{3}e_3 - \frac{d-2}{3}e_4 - \frac{d+1}{3}e_5 - \frac{d+1}{3}e_6$$

Then $\deg D = d$ and $p_a(D) = \frac{1}{6}(d-1)(d-2)$. For $d = 1$ and 2, a line and a conic achieve the maximum, respectively. For $d = 4$, I don't know QAQ. For $d = 5$, consider

$$D \sim 4\ell - 2e_1 - e_2 - e_3 - e_4 - e_5 - e_6$$

Then $\deg D = d$ and $p_a(D) = 2 = \frac{1}{6}(5-1)(5-2)$. Hence for every $d > 0$ (possibly except $d = 4$), this maximum is achieved by some irreducible nonsingular curve on cubic surface.

Exercise 10 (by Shuang-Yen Lee)**.**

Observe that if $x + y = z + w$ and $|x - y| > |z - w|$, then $x^2 + y^2 > z^2 + w^2$.

WLOG let $b_1 \geq b_2 \geq \cdots \geq b_6$, then we have $a > b_1 + b_2$ and $a > \frac{1}{2}(b_1 + \cdots + b_5)$. So we may assume that $a = \max\{b_1 + b_2, \frac{1}{2}(b_1 + \cdots + b_5)\}$ and prove that $a^2 \geq \sum_i b_i^2$.

If $b_1 + b_2 \geq b_3 + b_4 + b_5$, then $a = b_1 + b_2$. Make b_3, b_4, b_5 larger we may assume that $b_1 + b_2 = b_3 + b_4 + b_5$. Replace b_1, b_2 and b_6 by $b_1 + b_2 - b_3$, b_3 and b_5, then by the observation above we may assume that $b_2 = b_3$, $b_5 = b_6$. So we get $b_1 = b_4 + b_5 \geq b_2$ and

$$a^2 - \sum_i b_i^2 = 2b_2b_4 + 2b_2b_5 - b_2^2 - b_4^2 - 2b_5^2.$$

It's a quadratic polynomial in b_2, so we only need to check the cases $b_2 = b_4$ and $b_2 = b_4 + b_5$, which are $2b_4b_5 - 2b_5^2 \geq 0$ and $2b_4b_5 - b_5^2 \geq 0$.

If $b_1 + b_2 \leq b_3 + b_4 + b_5$, then $a = \frac{1}{2}(b_1 + \cdots + b_5)$ and

$$a^2 - \sum_i b_i^2 = -\frac{3}{4}b_1^2 + \frac{1}{2}(b_2 + \cdots + b_5)b_1 + *,$$

so the minimum of this w.r.t. b_1 is when $b_1 = b_3 + b_4 + b_5 - b_2$ since

$$\frac{-\frac{1}{2}(b_2 + b_3 + b_4 + b_5)}{-\frac{3}{4} \cdot 2} - b_2 - (b_3 + b_4 + b_5 - b_2) = \frac{1}{3}(2b_2 - (b_3 + b_4 + b_5)) \leq 0,$$

so we reduce to the case when $b_1 + b_2 = b_3 + b_4 + b_5$.

Exercise 11 (by Pei-Hsuan Chang)**.**

(a) Let $\begin{array}{rccc} \varphi: & \mathcal{A}_n & \to & \Sigma_n \\ & x_i & \mapsto & (i, i+1) \end{array}$. It is easy to check $\varphi(x_i)^2 = (i, i+1)^2 = 1 = \varphi(1) = \varphi(x_i^2)$, $\varphi(x_i, x_j)^2 = 1$ if $j \notin \{i+1.i-1\}$, and $\varphi(x_i x_{i+1})^3 = 1$. Thus, φ is a homomorphism. Since $\Sigma =< (12), (23), \ldots, (n-1, n) >$, φ is surjective.

Now, consider $\mathcal{A}_{n-1}$ as a subgroup of $\mathcal{A}_n$. Since $\{\mathcal{A}_{n-1}, x_{n-1}\mathcal{A}_{n-1}, x_{n-1}x_{n-2}\mathcal{A}_{n-1}, \ldots, x_1 \cdots x_{n-1}\mathcal{A}_{n-1}\}$ form a partition of $\mathcal{A}_n$. So, $[\mathcal{A}_n : \mathcal{A}_{n-1}] = n$. Notice that $\mathcal{A}_2 =< x_1 \mid x_1^2 >$, so $\mathcal{A}_2 = \{1, x_1\}$. Thus, $|\mathcal{A}_n| = n!$. Hence, φ is an isomorphism.

(b) Let $\varphi : E_6 \to G$ as in the question. It is clearly that $\varphi(x_i)^2 = 1, \forall i$. Quadratic transformation is of order $2 \Rightarrow \varphi(y)^2 = 1$. Also, quadratic transformation doesn't depend on the order of P_1, P_2, P_3, so $\varphi(y)\varphi(x_i) = \varphi(x_i)\varphi(y), \forall i = 1, 2, 4, 5$. As for $\varphi(y)\varphi(x_3)$, we can calculate that $\varphi(y)(E_i) = F_{jk}$, for $\{i, j, k\} = \{1, 2, 3\}$, $\varphi(y)(E_i) = E_i$, for $i = 4, 5, 6$, $\varphi(y)(F_{ij}) = F_{ij}$, if $i \in \{1, 2, 3\}, j \in \{4, 5, 6\}$, $\varphi(y)(F_{ij}) = G_k$ for $\{i, j, k\} = \{4, 5, 6\}$, and $\varphi(y)(G_i) = (G_i)$ for $i = 1, 2, 3$. Hence, $(\varphi(x_3)\varphi(y))^3 = 1$. So, φ is a homomorphism. Finally, the proof of proposition V. 4.10 in the textbook says permutation of $\{E_1, \ldots, E_6\}$ and quadratic transformation generate G, so φ is surjective.

(c)

Exercise 12 (by Shi-Xin Wang)**.**

Since D is ample, by Theorem 4.11, D is very ample. Then we can consider it as a irreducible subvaritey of X. Then from the short exact sequence $0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$, we obtain

$$0 \to H^0(X, \mathcal{O}_X(-D)) \to H^0(X, \mathcal{O}_X) \to H^0(D, \mathcal{O}_D) \to H^1(X, \mathcal{O}_X(-D)) \to H^1(X, \mathcal{O}_X) \to \cdots$$

Clearly, $H^0(X, \mathcal{O}_X) \cong H^0(D, \mathcal{O}_D) \cong k$. Since X is a hypersurface in $\mathbb{P}^3$, $H^1(X, \mathcal{O}_X) = 0$ by the cohomology of projective space. Moreover, the ampleness of D implies $H^0(X, \mathcal{O}_X(-D)) = 0$. Thus, we conclude $H^1(X, \mathcal{O}_X(-D)) = 0$.

Exercise 14 (by Ping-Hsun Chuang)**.**

By the theorem V.4.13, if $D \sim a\ell - \sum b_i e_i$, then D is very ample if and only if $b_i > 0$ for all i, $a > b_i + b_j$ for all i, j, and $2a > \sum_{j \neq i} b_i$ for all j. To show the existence of non-singular curve in $\mathbb{P}^3$, we only need to find such a and b_i with the corresponding equality. Note that $\deg D = 3a - \sum b_i$ and $g_a(D) = \frac{1}{2}(a-1)(a-2) - \frac{1}{2}\sum b_i(b_i - 1)$. Thus, from the following table, we may find the non-singular curve on cubic surface with desired degree and genus:

$d = \deg D$	$p_a(D)$	a	$(b_1, \cdots, b_6)$
8	6	8	$(4, 3, 3, 2, 2, 2)$
8	7	8	$(3, 3, 3, 3, 2, 2)$
9	7	9	$(4, 4, 4, 2, 2, 2)$
9	8	9	$(4, 4, 3, 3, 2, 2)$
9	9	9	$(4, 3, 3, 3, 3, 2)$
10	8	10	$(6, 3, 3, 3, 3, 2)$
10	9	12	$(5, 5, 5, 5, 3, 3)$
10	10	11	$(5, 5, 4, 3, 3, 3)$
10	11	10	$(5, 3, 3, 3, 3, 3)$

Combine the results in section IV.6, we conclude all possible genus $g \leq 12$ for curves of degree $d \leq 10$ in $\mathbb{P}^3$.

Exercise 16 (by Po-Sheng Wu)**.**

Assume $\mathrm{char}(k) \neq 2$. Note that $(x_0+x_1)(x_0+\omega x_1)(x_0+\omega^2 x_1) = (x_2+x_3)(x_2+\omega x_3)(x_2+\omega^2 x_3)$, where $\omega^3 = 1$, so $\{x_0+\omega^i x_1 = x_2+\omega^j x_3 = 0\}$ gives us 9 lines on X. By permuting the coordinates we obtain all 27 lines. Also notice that $(1, \omega^i, 0, 0)$ and its permutations are passed through by 3 lines, and $(1, -\omega^i, \omega^j, -\omega^k)$ and its permutations are passed through by two lines. They are the only intersections of these lines because they produces $18 \times 3 + 27 \times 3 = 27 \times 10 \div 2$ angles in total. To determine the automorphism group of X (induced from $\mathrm{Aut}(\mathbb{P}_3)$), notice that $\{x_i = 0\}$ are the only planes passing through 9 of the 18 three-line intersections, so the automorphisms of X permute $\{x_i = 0\}$, hence permute e_i. Therefore, an automorphism of X is a composition of $\mathrm{diag}(1, \omega^i, \omega^j, \omega^k)$ with a permutation on coordinates.

5 Birational Transformations

Exercise 1 (by Po-Sheng Wu).

Let $(f)_0$ and $(f)_\infty$ be the (effective) divisor given by the zeros and poles of f, so that $(f) = (f)_0 - (f)_\infty$. Suppose P is an intersection of $(f)_0$ and $(f)_\infty$, then we consider a blowup $\pi : \widetilde{X} \to X$ at P. WLOG assume that $\mu_P((f)_0) \geq \mu_P((f)_\infty)$, then $(\pi^*(f)) = \widetilde{(f)_0} - \widetilde{(f)_\infty} + (\mu_P((f)_0) - \mu_P((f)_\infty))E$, thus $(\pi^*(f))_0 \cdot (\pi^*(f))_\infty = \widetilde{(f)_0} \cdot \widetilde{(f)_\infty} + (\mu_P((f)_0) - \mu_P((f)_\infty))(\mu_P((f)_\infty)) = (f)_0 \cdot (f)_\infty - (\mu_P((f)_\infty))^2$ by Ex.3.2.. Thus the intersections between poles and zeros decreases. Repeat this process and eventually we obtain a birational morphism $g : X' \to X$ such that $g^*(f)$ has disjoint poles and zeros. $(g^*(f))_0$ then provides a base-point-free linear system which induces a morphism onto $\mathbb{P}^1$.

Exercise 2 (by Yi-Tsung Wang).

Let $Y^2 = -a < 0$. Choose a very ample divisor H on X such that $H^1(X, \mathscr{L}(H)) = 0$ and $H.Y = ak$ with $k \geq 2$. Let $\mathscr{M} = \mathscr{L}(H + kY)$. Consider an exact sequence

$$0 \to \mathscr{L}(H + (i-1)Y) \to \mathscr{L}(H + iY) \to \mathscr{L}(H + iY)|_Y \to 0$$

Since $(H + iY).Y = a(k - i)$, $\mathscr{L}(H + iY)|_Y = \mathcal{O}_{\mathbb{P}^1}(a(k-i)) \Rightarrow H^1(Y, \mathscr{L}(H + iY)|_Y) = 0$ for $i \leq k$. Therefore

$$H^1(X, \mathscr{L}(H + (i-1)Y)) \twoheadrightarrow H^1(X, \mathscr{L}(H + iY))$$

is surjective. By induction and $H^1(X, \mathscr{L}(H)) = 0$, we see that $H^1(X, \mathscr{L}(H + iY)) = 0$ for all $i = 0, \ldots, k$. Since H is very ample, $|H + kY|$ has no base points away from $Y \Rightarrow \mathscr{M}$ is generated by global section off Y. Consider an exact sequence

$$0 \to \mathscr{M} \otimes \mathscr{I}_Y \to \mathscr{M} \to \mathscr{M}|_Y \to 0$$

and note that $\mathscr{M} \otimes \mathscr{I}_Y \cong \mathscr{L}(H + (k-1)Y) \Rightarrow H^1(X, \mathscr{M} \otimes \mathscr{I}_Y) = 0 \Rightarrow H^0(X, \mathscr{M}) \twoheadrightarrow H^0(Y, \mathscr{M}|_Y)$ is surjective. Since $(H + kY).Y = 0$, we have $\mathscr{M}|_Y = \mathcal{O}_{\mathbb{P}^1}$, which is generated by the global section 1. By Nakayama lemma, $\mathscr{M}$ is also generated by global section at every point of Y, hence it gives a morphism $f : X \to \mathbb{P}^N$ with $\mathscr{M} \cong f^*\mathcal{O}(1)$. Since $\mathscr{M}|_Y$ is of degree 0, Y is mapped to one point P. Finally, since H is very ample, $|H + kY|$ separates points and tangent vectors away from Y and separates points in Y from points not in $Y \Rightarrow f$ gives an isomorphism between $X \backslash Y$ and $f(X) \setminus \{p\}$. Therefore Y is contractible to a point on a projective variety.

Exercise 3 (by Tzu-Yang Chou)**.**

First, we have $H^i(\tilde{X}, \pi^*\Omega_X) \simeq H^i(X, \pi_*\pi^*\Omega_X) \simeq H^i(X, \Omega_X)$ by Ex(III.8.1) and projection formula: $\pi_*(\pi^*\Omega_X \otimes \mathcal{O}_{\tilde{X}}) \simeq \Omega_X \otimes \pi_*\mathcal{O}_{\tilde{X}} \simeq \Omega_X$. Next, we claim that there's an exact sequence: $0 \longrightarrow \pi^*\Omega_X \longrightarrow \Omega_{\tilde{X}} \longrightarrow \Omega_E \longrightarrow 0$, that is, $\Omega_{\tilde{X}/X} \simeq \Omega_E$. They are both supported on $E \simeq \mathbb{P}^1$, so let $[t : u]$ be coordinates of E and (x, y) be local coordinate at the center of blow-up. But at $P \in E$, the sequence becomes $0 \longrightarrow < dx, dy > \longrightarrow < dx, dy, dy > /(dx - tdy - ydt) \longrightarrow < dt > \longrightarrow 0$.
Now since $H^0(E, \Omega_E) = 0, H^1(E, \Omega_E) = k$, the cohomology sequence proves the assertion since $H^0(X, \Omega_X) \simeq H^0(\tilde{X}, \Omega_{\tilde{X}})$ and by Serre duality we have the H^2 part is also an isomorphism.

Exercise 4 (by Pei-Hsuan Chang)**.**

(a) By theorem V. 5.3, f factor through $X_1 = \mathrm{Bl}_P X'$, and we can repeat it until $X_n \cong X$, then Y is one of the exceptional curves, since Y is irreducible. Thus, $Y \cong \mathbb{P}^1$.

Take an ample divisor H on X'. Then $(f^*H)^2 = H^2 > 0$. So by exercise V. 1.9(a), we have

$$(Y^2)(f^*H)^2 \leq (Y.(f^*H))^2 = 0.$$

Hence, $Y^2 \leq 0$.

(b) (Method I)

By remark V. 1.9.1, intersection pairing induce a nondegenerate billinear pairing $\mathrm{Num}\, X \times \mathrm{Num}\, X \to \mathbb{Z}$. We can consider vector space $\mathrm{Num}\, X \otimes_{\mathbb{Z}} \mathbb{R}$ over $\mathbb{R}$, and the induced sysmmetric billinear form. Sylvester theorem says that this matrix can be diagonalized with ± 1's on diagonal. Hodge index theorem says there is exactly one $+1$, corresponding to a real multiple of ample divisor f^*H. So, the others correspond to $(Y_i.Y_j)_{ij}$. Thus, it is negative definite.

(Method II)

[Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity*]

Let H_1' and H_2' be two hyperplane section of X', H_1' passing through P and H_2' not. Let $(f) = H_1' - H_2'$. Let H_1 be the strict transform of H_1' and H_2 be the total transform of H_2'. Then $H_2 \sim H_1 + \sum m_i Y_i$ where $m_i = \mathrm{ord}_{Y_i} f > 0$. Let $S = (Y_i.Y_j)_{ij}$, $S' = (m_i Y_i.m_y Y_j)_{ij}$, and $M = \mathrm{diag}(m_1, \ldots m_r)$, then $S' = MSM$, so S is negative definite $\Leftrightarrow$ S' is negative definite. Notice that $S_{ij} \geq 0, \forall i \neq j$, and

$$\sum_j S'_{ij} = \sum_j m_i Y_i.m_j Y_j = \sum_j (H_2 - H_1).m_j Y j = -H_1.m_j Y_j \leq 0, \forall j.$$

Now, for $v = (\alpha_1, \ldots \alpha_r)$,

$$v^T S' v = \sum_{i,j} \alpha_i \alpha_j S'_{ij} = \sum_i \alpha_i^2 S'_{ii} + 2 \sum_{i<j} \alpha_i \alpha_j S'_{ij} = \sum_j (\sum_i S'_{ij}) \alpha_j^2 - \sum_{i<j} S'_{ij} (\alpha_i - \alpha_j)^2 \leq 0.$$

So S'_{ij} is negative semi-definite. Since H_1 pass through some Y_j, $\sum_j S'_{ij} < 0$, for some j, and $\cup Y_i$ is connected. So we can not spilt $(1, \ldots, r) = (i_1, \ldots i_k) \cup (j_1, \ldots j_{r-k})$ such that $S'_{i_a j_b} = 0, \forall a, b$. Thus, if

$$v^T S' v = \sum_j (\sum_i S'_{ij}) \alpha_j^2 - \sum_{i<j} S'_{ij} (\alpha_i - \alpha_j)^2 = 0,$$

then $\alpha_j = 0$ and $\alpha_i = \alpha_j, \forall i \Rightarrow v = 0$. This complete the proof.

Remark. In exercise V. 1.9, we assume H is ample, but actually we just need to assume $H^2 > 0$. Same as what we do in the method I of part (b), Sylvester theorem tell us we can use any divisor with $H^2 > 0$ to start our diagonalization process, so we may assume $H = (1, 0, \dots, 0)$, then it is easy to see $(D^2)(H^2) \leq (D.H)^2$.

Exercise 7 (by Po-Sheng Wu)**.**

Let H_0, H_1 be the hyperplane sections on X_0 so that H_0 contains P while H_1 does not. Then $\overline{f}^*(H_0) = \overline{f^*(H_0 - \{P\})} + mY$ for some $m > 0$. Thus $0 = f^*(H_1) \cdot Y = f^*(H_0) \cdot Y = f^*(H_0 - \{P\}) \cdot Y + mY^2 \geq mY^2 \Rightarrow Y^2 = 0$.

Exercise 8 (by Yi-Heng)**.**

(a) Since z is a prime in A, it suffices to show A_z is an UFD by Nagata's criterion. Indeed, $A_Z = (k[x,y,z]/(x^2+y^3+z^5))_z = k[x,y,z,z^{-1}]/(x^2+y^3+z)$ is an UFD.

(b) (i) Blowing up $x^2+y^3+z^5$ at $(0,0,0)$ by $xy_1 = x_1y, yz_1 = y_1z, zx_1 = z_1x$, the singularity occurs at $(0,0,0)$ of $x_1^2 + zy_1^3 + z^3$ in the chart $z_1 = 1$.

(ii) Blowing up $x_1^2 + zy_1^3 + z^3$ at $(0,0,0)$ by $x_1y_2 = x_2y_1, y_1z_2 = y_2z, zx_2 = z_2x_1$, the singularity occurs at $(0,0,0)$ of $x_2^2 + z_2y_1^2 + y_1z^3$ in the chart $y_2 = 1$.

(iii) Blowing up $x_2^2 + z_2y_1^2 + y_1z^3$ at $(0,0,0)$ by $x_2y_3 = x_3y_1, y_1z_3 = y_3z_2, z_2x_3 = z_3x_2$, the singularities occur at $(0,0,0)$ of $x_3^2 + z_3y_1 + y_1^2z_3^3$ in the chart $y_3 = 1$ and $(0,0,0)$ of $x_3^2 + z_2y_3^2 + y_3z_2^2$ in the chart $z_3 = 1$.

(iv) Blowing up $x_3^2 + z_3y_1 + y_1^2z_3^3$ at $(0,0,0)$ by $x_3y_4 = x_4y_1, y_1z_4 = y_4z_3, z_3x_4 = z_4x_3$, then the strict transformation is non-singular.

(v) Blowing up $x_3^2 + z_2y_3^2 + y_3z_2^2$ at $(0,0,0)$ by $x_3y_4 = x_4y_3, y_3z_4 = y_4z_2, z_2x_4 = z_4x_3$, the singularities occur at $(0,0,0)$ of $x_4^2 + z_4y_3 + y_3z_4^2$ in the chart $y_4 = 1$, $(0,0,0)$ of $x_4^2 + z_2y_4^2 + y_4z_2$ in the chart $z_4 = 1$ and $(0,0,-1)$ of $x_4^2 + z_4y_3 + y_3z_4^2$ in the chart $y_4 = 1$.

(vi) Blowing up $x_4^2 + z_4y_3 + y_3z_4^2$ at $(0,0,0)$ by $x_4y_5 = x_5y_3, y_3z_5 = y_5z_4, z_4x_5 = z_5x_4$, then the strict transformation is non-singular.

(vii) This is similar to (vi).

(viii) Blowing up $x_4^2+z_4y_3+y_3z_4^2$ at $(0,0,-1)$, that is, blowing up $x_4^2-z_4y_3+y_3z_4^2$ by $x_4y_5 = x_5y_3, y_3z_5 = y_5z_4, z_4x_5 = z_5x_4$, then the strict transformation is non-singular.

In conclusion, the inverse image of P is a union of eight projective lines corresponding to $\mathbb{E}_8$.

6 Classification of Surfaces