Algebraic Geometry II Homework
Chapter IV Curves

A course by prof. Chin-Lung Wang

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Exercise 0 (by Kuan-Wen)**.**

This is an example of proof.

Remark. This is an example for how to write in this format.

1 Riemann-Roch Theorem

Exercise 1 (by Chi-Kang)**.**

This is equivalent to show that there exists $f \in K(X)$ s,t, $f \in H^0(X, nP)$ for some $n > 0$ and f is non-constant. By Riemann-Roch we have for any natural number $n \geq 2g-1$, $h^1(nP) = 0$, thus

$$\chi(nP) = h^0(nP) = n + 1 - g$$

take $n \geq g+1$ we have $h^0(nP) \geq 2$, hence there is a non-constant function $f \in H^0(X, nP)$.

Exercise 2 (by Chi-Kang)**.**

Use induction on r, $r = 1$ is just exercise 1.1. If the consequence holds for $r-1$, then for r, there exists some f s,t, f has pole at $P_1, ..., P_{r-1}$ and regular elsewhere. And as 1.1 there is g s,t, g has pole at P_r and regular elsewhere. hence $f + g$ is a function has pole at $P_1, ..., P_r$ and regular elsewhere.

Exercise 3 (by Yi-Tsung Wang)**.**

Proof. By Nagata theorem (remark 2.7.17.2), X can be embedding as an open subset of a complete curve $\overline{X}$, then in this case $\overline{X} \backslash X$ is just a finite set, say $\overline{X} \backslash X = \{p_1, \ldots, p_r\}$. By Exercise 4.1.2, take $f : \overline{X} \to \mathbb{P}^1$ such that f has poles at each of the p_i and regular elsewhere. Since f is not constant, f must be surjective, then $f^{-1}(\mathbb{A}^1) = X$. Moreover, f is a finite morphism, hence an affine morphism, and then $X = f^{-1}(\mathbb{A}^1)$ is affine. □

Exercise 4 (by Yi-Tsung Wang)**.**

Proof. Let X be a separated one-dimensional scheme of finite type over k. By Exercise 3.3.1, we may assume X is reduced. By Exercise 3.3.2, we may furthermore assume X is irreducible, hence X is integral and is not proper over k. Let Y be the normalization of X, and the natural map $\pi : Y \to X$. π is finite since X is of finite type over k by Exercise 2.3.8 and is surjective since X is integral (locally, it is going-up). If Y is proper over k, by Exercise 2.4.4, $X = \pi(Y)$ is also proper over k, contradiction. Now note that Y is also integral, separated, one-dimensional scheme of finite type over k, and is regular since Y is furthermore normal, by Exercise 4.1.3, Y is affine. By Chevalley's theorem (Exercise 3.4.2), X is also affine. □

Exercise 5 (by Shuang-Yen Lee)**.**

By Riemann-Roch Theorem, we have

$$\dim|D| = \ell(D) - 1 = \ell(K-D) + \deg(D + 1 - g = \deg(D) + (\ell(K-D) - g) \leq \deg(D)$$

since $K - D \leq K \implies \ell(K-D) \leq \ell(K) = g$. If $g = 0$, then

$$\deg(K-D) \leq \deg(K) = -2 \implies \ell(K-D) - g = 0$$

so the equality holds. If $g \neq 0$, then $D = 0 \implies \ell(K-D) - g = \ell(K) - g = 0$. Suppose $D \neq 0$, say $D = \sum n_i P_i$, then $K - D \leq K - P_1 \leq K$, so

$$0 = \ell(K-D) - g \leq \ell(K-P_1) - g \leq \ell(K) - g = 0 \implies \ell(K-P_1) = g.$$

By Riemann-Roch Theorem, $\ell(P_1) = \ell(K-P_1) + 2 - g$. So $\ell(P_1) = 2$, which is impossible since $g > 0$.

Exercise 6 (by Shi-Xin)**.**

Let P be a point on X, and let g denote $g(X)$. Consider the divisor $D = (g+1)P$. By Riemann-Roch Theorem, we have

$$\ell(D) \geq \deg D + 1 - g > 1.$$

Therefore, there is a $f \in K(X)$ such that $(g+1)P + div(f) \geq 0$, i.e. f has pole at P with order $\leq g+1$ and is regular everywhere else. Thus it induces a finite morphism $\tilde{f} : X \to \mathbb{P}^1$ by $x \mapsto f(x)$ which is of degree $\leq g+1$ since $\deg \tilde{f} \cdot \deg \infty = \deg D$.

Exercise 8 (by Shi-Xin)**.**

(a) From $0 \to \mathcal{O}_X \to f_*\mathcal{O}_{\tilde{X}} \to \sum_{p\in X} \tilde{\mathcal{O}}_p/\mathcal{O}_p \to 0$ where $f : \tilde{X} \to X$ is the normalization of X, we obtain

$$\begin{aligned} 0 &\to H^0(X, \mathcal{O}_X) \to H^0(X, f_*\mathcal{O}_{\tilde{X}}) \cong H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to H^0(X, \sum_{p\in X} \tilde{\mathcal{O}}_p/\mathcal{O}_p) \\ &\to H^1(X, \mathcal{O}_X) \to H^1(X, f_*\mathcal{O}_{\tilde{X}}) \cong H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to H^1(X, \sum_{p\in X} \tilde{\mathcal{O}}_p/\mathcal{O}_p) = 0 \end{aligned}$$

Since $H^0(X, \mathcal{O}_X) \cong H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong k$, we have

$$0 \to H^0(X, \sum_{p\in X} \tilde{\mathcal{O}}_p/\mathcal{O}_p) \to H^1(X, \mathcal{O}_X) \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to 0$$

Thus by *ex.iii*.5.3, $p_a(X) = p_a(\tilde{X}) + \sum_{p\in X} length(\tilde{\mathcal{O}}_p/\mathcal{O}_p) = p_a(\tilde{X}) + \sum_{p\in X} \delta_p$.

(b) If $p_a(X) = 0$, then it forces $p_a(\tilde{X}) = 0$ and $\delta_p = 0$ for any $p \in X$. So f is an isomorphism, i.e. $X \cong \tilde{X} \cong \mathbb{P}^1$ which is given by Riemann-Roch Theorem.

(c)

Exercise 9 (by Ping-Hsun Chuang)**.**

Proof. (a) Let $f : X_{\text{reg}} \to X$ be the inclusion. We have the short exact sequence

$$0 \longrightarrow \mathcal{O}(D) \longrightarrow f_*\mathcal{O}_{\text{reg}}(D) \longrightarrow \sum_{P\in X} \left(f_*\mathcal{O}_{\text{reg}}(D)\right)_P / \mathcal{O}(D)_P \longrightarrow 0$$

Note that $\mathcal{O}(D)_P = \mathcal{O}_{X,P}$ since $\mathcal{O}(D)$ is an invertible sheaf. Also, $\left(f_*\mathcal{O}_{\text{reg}}(D)\right)_P = \mathcal{O}_{\widetilde{X},P}$ since X_{reg} is normal. Moreover,

$$\delta_P = \text{length}\left(\mathcal{O}_{\widetilde{X},P}/\mathcal{O}_{X,P}\right) = h^0\left(X, \left(f_*\mathcal{O}_{\text{reg}}(D)\right)_P / \mathcal{O}(D)_P\right).$$

Finally, since χ is a additive function, we have

$$\begin{aligned}\chi(\mathcal{O}(D)) &= \chi\left(f_*\mathcal{O}_{\text{reg}}(D)\right) - \sum_{P\in X} \chi\left(f_*\mathcal{O}_{\text{reg}}(D)_P / \mathcal{O}(D)_P\right) \\ &= \deg D + 1 - p_a(X_{\text{reg}}) - \sum_{P\in X} \delta_P = \deg D + 1 - p_a(X).\end{aligned}$$

(b) Since X is projective, take a very ample divisor R on X. Then, there exists $n > 0$ such that $\mathcal{O}(nR + D)$ is generated by global section. Then, since R is very ample, $\mathcal{O}(nR + D + R)$ is also very ample. Now, $D = M - (n+1)R$, where $M = D + (n+1)R$ which is very ample.

(c) Using the result in (b), it suffice to show the result in case that $\mathcal{L}$ is very ample. Write $\mathcal{L} = f^*\mathcal{O}(1)$ for some embedding $f : X \to \mathbb{P}^N$. Since X has finitely may irregular points, there exists a hyperplane H in $\mathbb{P}^N$ such that $H \cap X \subseteq X_{\text{reg}}$. Now, take $D = H \cap X$.

(d) Since X is locally complete intersection, we may apply the Sere duality. We then get

$$\begin{aligned}H^1(X, \mathcal{O}(D)) &\cong \text{Ext}^0_X\left(\mathcal{O}(D), \omega^\circ_X\right)^\vee \\ &\cong \text{Ext}^0_X\left(\mathcal{O}_X, \mathcal{O}(-D) \otimes \omega^\circ_X\right)^\vee \\ &\cong H^0\left(\mathcal{O}_X, \mathcal{O}(-D) \otimes \omega^\circ_X\right)^\vee\end{aligned}$$

Then, $\chi(\mathcal{O}(D)) = h^0(X, \mathcal{O}(D)) - h^1(X, \mathcal{O}(D)) = \ell(D) - \ell(K - D)$. Finally, using the result in (a), we get the required formula.

□

Exercise 10 (by Ping-Hsun Chuang)**.**

Proof. Apply exercise 4.1.9 to $D = K$ and get

$$\ell(K) - \ell(0) = \deg K + 1 - g_a = \deg K.$$

Also, we have

$$\ell(K) = h^0(X, \omega_X^\circ) = h^1(X, \mathcal{O}_X) = p_a.$$

Note that the second equality above holds by the Serre duality since we assume X is locally complete intersection. Thus, we get $\deg K = p_a - 1 = 0$.

Now, for any $D \in \mathrm{Pic}^0 X$, apply exercise 4.1.9 to $D = D + P_0$ and get

$$\ell(D + P_0) - \ell(K - D - P_0) = \deg(D + P_0) + 1 - p_a = 1.$$

Also, we have $\deg(K - D - P_0) = \deg K - 1 = -1$ and thus $\ell(K - D - P_0) = 0$. Hence, $\ell(D + P_0) = 1$, that is, there exists a unique $R > 0$ such that $R \sim D + P_0$. Therefore, for any $D \in \mathrm{Pic}^0 X$, we find a unique R such that $D \sim R - P_0$ and thus $X_{\mathrm{reg}} \to \mathrm{Pic}^0 X$ is bijection. □

2 Hurwitz's Theorem

Exercise 1 (by Pei-Hsuan Chang)**.**

Induction on n. For $n = 1$, it is Example IV.2.5.3 in Hartshorne. So let's deal with the case $n > 1$. Let $f : X \to \mathbb{P}^n$ be an étale covering. We may assume that X is connected. For each hyperplane $H \cong \mathbb{P}^{n-1}$ in $\mathbb{P}^n$, $f : f^*H \to H$ is an étale covering of H. By induction hypothesis, f^*H is disjoint union of copies of H.

Now, we are going to showing that f^*H is connected, and conclude f^*H is isomorphic to H via f. To show this, we want to show that X is normal and f^*H is ample with codimension 1, then by Corollary III. 7.9, f^*H will be connected. Notice that H is ample, so f^*H is ample since f is finite. Also, an étale covering is smooth, so X is smooth over k and thus, is normal. Hence, $f^*H \cong H$, and $f|_{f^*H}$ is an isomorphism. Now, $\deg f = \deg f|_{f^*H} = 1$. An étale covering with degree 1 is an isomorphism, so $X = \mathbb{P}^n$. This complete the prove.

Exercise 2 (by Yu-Chi Hou)**.**

(a) From Exercise 1.7, we know that any curve X of genus 2 is hyperelliptic whose the degree 2 morphism $f := \phi_{|K_X|} : X \longrightarrow \mathbb{P}^1$ coming from the canonical system. Using Riemann-Hurwitz formula, one computes directly that $\deg(R) = 6$. If P is branched point of f, then $e_P = 2$, for any $P \in f^{-1}(Q)$. Since $\mathrm{char}(k) \neq 2$, any ramification point $P \in X$ is tamely ramified,

$$R = \sum_{P \in X} (e_P - 1), \text{ and } \deg(R) = 6.$$

Hence, f is ramified exactly at 6 points.

(b) Let $h(x) := (x-\alpha_1)\cdots(x-\alpha_6) \in k[x]$ and $K := (k(x))[z])/(z^2-h)$. Since $K/k(x)$ is an algebraic extension of degree 2, K/k has transcendental degree 1. This together with $[K:k(x)] = 2$ determines a non-singular projective curve X and a morphism $f : X \to \mathbb{P}^1$ of degree 2. On the affine open chart $U_0 = \operatorname{Spec} k[x] \subset \mathbb{P}^1$, there exists a morphism from the affine open set $V \subseteq f^{-1}(U_0) \to U_0$ correpsonding to the inclusion $k[x] \hookrightarrow k[x, \bar{z}] := k[x,z]/(z^2-h)$. Hence, the function field $K(V) = K(X) = K$.

Since $k = \bar{k}$, any closed point $P \in U_0 \subset \mathbb{P}^1$ correponds to the maximal ideal $(x-\alpha) \subset k[x]$. If $\alpha \notin \{\alpha_1, \ldots, \alpha_6\}$, then $h(\alpha) \neq 0$ and thus $(x-\alpha, \bar{z} \pm \sqrt{h(\alpha)}$ is a maximal ideal in $k[x,z]/(z^2-h)$. In other words, $\#f^{-1}(P) = 2$ if $P \in U_0$ not corresponding to $\alpha_1, \ldots, \alpha_6$. Thus, we have shown that $f^{-1}(U_0) \to U_0$ is only branched at $\alpha_1, \ldots, \alpha_6 \in k \cong U_0 \subset \mathbb{P}^1$.

Next, we need to check that f does not brached at $\infty \in \mathbb{P}^1$. To see this, we first localizing

$$k[x]_{(x)} = k[x, x^{-1}] \hookrightarrow (k[x,z]/(z^2-h))_{(x)} = k[x, x^{-1}, z]/(z^2-h),$$

which correponds to $f^{-1}(U_0 \cap U_1) \to U_0 \cap U_1$. On $k[x, x^{-1}, z]$, we first assume that $\alpha_1, \ldots, \alpha_6 \in k^*$, then

$$z^2 - h(x) = z^2 - (x-\alpha_1)\cdots(x-\alpha_6) = z^2 - x^6(1-\frac{\alpha_1}{x})\cdots(1-\frac{\alpha_6}{x})$$
$$= z^2 - \alpha_1\cdots\alpha_6 x^6(\frac{1}{\alpha_1}-\frac{1}{x})\cdots(\frac{1}{\alpha_6}-\frac{1}{x}) = x^6(x^{-6}z^2 - \alpha_1\cdots\alpha_6\tilde{h}(1/x)),$$

where $\tilde{h}(1/x) := \prod_{i=1}^6 \left(\frac{1}{\alpha_i} - \frac{1}{x}\right) \in k[x^{-1}]$. Since x is a unit in $k[x, x^{-1}, z]$, $(z^2 - h(x)) = (\tilde{z}^2 - \alpha_1\cdots\alpha_6\tilde{h}(1/x))$, where $\tilde{z} = x^{-3}z$. Hence, we have

$$k[x, x^{-1}, z]/(z^2-h) = k[x, \tilde{x}, \tilde{z}]/(\tilde{z}^2 - \alpha_1\cdots\alpha_6\tilde{h}(1/x)).$$

Let $y = 1/x$, $k[x, \tilde{x}, \tilde{z}]/(\tilde{z}^2 - \alpha_1\cdots\alpha_6\tilde{h}(1/x)) = k[y, y^{-1}, \tilde{z}]/(\tilde{z}^2 = \alpha_1\cdots\alpha_6\tilde{h}(y))$. Thus, on $U = \operatorname{Spec} k[y]$, $f^{-1}(U_1) \to U_1$ is defined by the correponding morphism from $k[y] \hookrightarrow k[y, \tilde{z}]/(\tilde{z}^2 = \alpha_1\cdots\alpha_6\tilde{h})$. Same argument as above shows that f is only branched at $y - \alpha_i \in \operatorname{Spec} k[y]$ for $i = 1, \ldots, 6$. Now,, if $\alpha_6 = 0$, $\alpha_1, \ldots \alpha_5 \neq 0$ (since $\alpha_1, \cdots, \alpha_6$ are distinct),

$$h(x) = x(x-\alpha_1)\cdots(x-\alpha_5) = \alpha_1\cdots\alpha_5 x^6(\frac{1}{\alpha_1}-\frac{1}{x})\cdots(\frac{1}{\alpha_5}-1/x).$$

Repeating above argument shows that f is not branched at ∞. Thus, f is only ramified over 6 points with each ramification index 2 (since f is of degree 2). Using Riemann-Hurwitz formula, $2g(X) - 2 = 2(0-2) + 6 = 2 \Rightarrow g(x) = 2$. Moreover, let $P \in X$ such that $f(P) = Q \in \{\alpha_1, \ldots, \alpha_6\}$, then $f^*P = \sum_{P \in f^{-1}(Q} e_P \cdot P = 2P$. Thus, $f^*\mathcal{O}\mathbb{P}^1(Q) = f^*\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_X(2P)$.

On the other hand, using Riemann-Roch, $h^0(X, \mathcal{O}_X(2P)) - h^0(X, \mathcal{O}_X(K_X - 2P)) = \deg(2P) - g(X) + 1 = 1$. Since $H^0(X, \mathcal{O}_X(2P)) = H^0(X, f^*\mathcal{O}_{\mathbb{P}^1}(1)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \cong k^2$, $h^0(X, \mathcal{O}_X(K_X - 2P)) = 2$. However, $\deg(K_X - 2P) = 2g(x) - 2 - 2 = 0$. Thus, $K_X \sim 2P$. Hence, the map $f : X \to \mathbb{P}^1$ is the same as the one determined by $|K_X|$.

(c) If $P_i \neq \infty \in \mathbb{P}^1$, for $i = 1, 2, 3$, then let $P_1 = [a : 1]$, $P_2 = [b : 1]$, $P_3 = [c : 1]$, then the Möbius transform $\phi(z) = \frac{z-a}{z-c}\frac{b-c}{b-a}$ maps P_1 to 0, P_2 to 1, and P_3 to ∞. If $P_1 = \infty$, $P_2 = [b : 1]$, $P_3 = [c : 1]$, then we take $\phi(z) = \frac{b-c}{z-c}$. Since $\mathrm{Aut}(\mathbb{P}^1) = PGL(2)$, such ϕ is unique.

(d) The symmetric group S_3 acts on distinct element $\beta_1, \beta_2, \beta_3 \in k\backslash\{0, 1\}$ by permuting $\{0, 1, \infty, \beta_1, \beta_2, \beta_3)$, then sending the first three element to 0, 1, ∞ by Möbius transform again, then call them $\beta_1', \beta_2', \beta_3'$. Then we define $[\beta_1, \beta_2, \beta_3]$ to be the equivalence class of $(\beta_1\beta_2, \beta_3)$ modulo such S_3- action.

(e) Given any genus 2 curve X, $|K_X|$ gives $f : X \to \mathbb{P}^1$ with six distinct brached points $P_1, \ldots, P_6$. Then using Möbius transform, we sends $P_1 \mapsto 0$, $P_2 \mapsto 1$, $P_3 \mapsto \infty$, $P_i \mapsto \beta_{i-3}$, for $i = 4, 5, 6$. We then get an equivalence class $[\beta_1, \beta_2, \beta_3]$ modulo S_3-action described in (d). Now, if $\phi : X \xrightarrow{\sim} X'$ be an isomorphism, then $\phi^* K_{X'} \sim K_x$. Thus, $|\phi^* K_{X'}|$ gives a morphism to $\mathbb{P}^1$ which differ to the one from $|K_X|$ by an $\psi \in \mathrm{Aut}(\mathbb{P}^1) = PGL(2)$. Then as in (d), the tuple $(\beta_1', \beta_2', \beta_3')$ differ by $(\beta_1, \beta_2, \beta_3)$ by an S_3-action. Thus, $[\beta_1', \beta_2', \beta_3'] = [\beta_1, \beta_2, \beta_3]$.

Also, (b) implies that starting from six points of $\mathbb{P}^1$, one can construct a genus two curve X whose $\phi_{|K_X|}$ is branched exactly at the given six points. Thus, we established the isomorphism class $[X]$ with the tuple $[\beta_1, \beta_2, \beta_3]$ modulo S_3-action.

Exercise 4 (by Yi-Tsung Wang)**.**

Proof. Let $f(x, y, z) = x^3y + y^3z + z^3x$. Then $f_x = z^3, f_y = x^3, f_z = y^3 \Rightarrow f$ is non-singular since $(0, 0, 0) \notin \mathbb{P}^2$. Since

$$\begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 3z^2 \\ 3x^2 & 0 & 0 \\ 0 & 3y^2 & 0 \end{pmatrix} = 0$$

every point of X is an inflection point. For $p(a, b, c) \in X$, the tangent line at p is

$$c^3(x - a) + a^3(y - b) + b^3(z - x) = 0$$

that is, $c^3x + a^3y + b^3z = 0$. Then the natural map $X \to X^*$ is defined by $(a, b, c) \mapsto (c^3, a^3, b^3)$, which is a Frobenius morphism, hence is isomorphic and purely inseparable. □

Exercise 5 (by Yu-Ting Huang)**.**

(a) Let G act on X, then $f^{-1}(f(P))$ is an orbit of the group action, then $|f^{-1}f(P)| = \frac{n}{r}$ and each element in $f^{-1}f(P)$ are of index r as P. By Hurwitz's theorem, $2g(X) - 2 = n(2g(Y) - 2) + \sum_p(e_p - 1)$. Then $\frac{2g-2}{n} = \frac{1}{n}\sum_P(e_P - 1) = \frac{1}{n}\sum_{i=1}^s \frac{n}{r_i}(r_i - 1) = \sum_{i=1}^s(1 - \frac{1}{r_i})$.

(b) First, note that $2g(Y) - 2 + \sum_{i=1}^s(1 - \frac{1}{r_i}) = \frac{2g(X)-2}{n} > 0$, since $g(X) \geq 2$. If $g(Y) = 0$, $-2 + \sum_{i=1}^s(1 + \frac{1}{r_i}) = \frac{2g(X)-2}{n} \geq \frac{2}{n} \geq 0$. Thus, $\sum_{i=1}^s(1 - \frac{1}{r_i}) \geq \frac{2}{n} + 2$. Consider the minimal possibility of r_i such that $\sum_{i=1}^s(1 - \frac{1}{r_i}) \geq \frac{2}{n} + 2$. We find that $r_i = 2, 3, 7$. Then, $-2 + \frac{1}{2} + \frac{2}{3} + \frac{6}{7} = \frac{1}{42} = \frac{2(g-1)}{n}$. i.e. $n = 84(g - 1)$. In the case $g(Y) = 0$, $n \leq 84(g - 1)$.
As for $g(Y) \geq 1$, $2g(Y) - 2 > 0$, so $\sum_{i=1}^s(1 - \frac{1}{r_i}) > 0$. To find maximal n, we set $s = 1, r_1 = 2, g(Y) = 1$. Then $2 - 2 + (1 - \frac{1}{2}) = \frac{2g-2}{n}$. i.e. $n = 4(g - 1) \leq 84(g - 1)$. Now, we can conclude that $n \leq 84(g - 1)$.

Exercise 6 (by Tzu-Yang Chou).

(a) Let D be effective. We first consider the short exact sequence $0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$ and apply f_*. Since f is finite, so $R^1 f_* \mathcal{O}_X(-D)$ vanishes and hence we have $0 \longrightarrow f_*\mathcal{O}_X(-D) \longrightarrow f_*\mathcal{O}_X \longrightarrow f_*\mathcal{O}_D \longrightarrow 0$ is exact. Taking determinant, we obtain $\det(f_*\mathcal{O}_X) \simeq \det(f_*\mathcal{O}_X(-D)) \otimes \det(f_*\mathcal{O}_D)$. Now we only need that $(\det(f_*\mathcal{O}_D))^{-1} \simeq \mathcal{O}_X(f_*(-D))$ since then for a general divisor, we can write it as a difference of two effective ones and above formula proves the assertion. But this statement follows from Ex(II.6.11)(c).

(b) (a) tells us that $\mathcal{O}_X(f_*D) \simeq \det(f_*\mathcal{O}_X(D)) \otimes (\det(f_*\mathcal{O}_X))^{-1}$ and hence it only depends on the linear equivalence class of D. $f_* \circ f^* = n$ follows from their definitions, where $n = \deg f$.

(c) By Ex(III.7.2) and Ex(III.6.10), we have the following sequence of isomorphisms: $\det(f_*\Omega_X) \simeq \det(f_*\mathscr{H}om_X(\mathcal{O}_X, \Omega_X)) \simeq \det(f_*\mathscr{H}om_X(\mathcal{O}_X, f^!\Omega_Y)) \simeq \det(\mathscr{H}om_Y(f_*\mathcal{O}_X, \Omega_Y)) \simeq \det((f_*\mathcal{O}_X)^{-1} \otimes \Omega_Y) \simeq (\det(f_*\mathcal{O}_X))^{-1} \otimes \Omega_Y^{\otimes n}$

(d) $K_X = f^*K_Y + R \Rightarrow f_*K_X = nK_Y + B \Rightarrow \mathcal{O}_X(-B) \simeq \Omega_Y^{\otimes n} \otimes (\mathcal{O}_X(f_*K_X))^{-1}$, and this is isomorphic to $(det(f_*\mathcal{O}_X))^2$ by (a) and (c).

Exercise 7 (by Po-Sheng Wu).

(a) Since f is finite flat, $f_*\mathcal{O}_X$ is locally free of rank 2. Plus, the injection $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is also injective on residue field, so the kernel $\mathcal{L}$ is locally free of rank 1. By taking det for the short exact sequence we have $\mathcal{L} \cong \det f_*\mathcal{O}_X$, and then by 2.6(d) we have $\mathcal{L}^2 \cong \mathcal{O}_Y$ since f is etale.

(b) On the affine subset $U = \operatorname{Spec}(A) \subset Y$ such that $\mathcal{L}$ is free, the constructed algebra is actually isomorphic to $A[t]/(t^2 - u)$ via $(a, bv) \mapsto a + bt$, where v is a generator of $\mathcal{L}(U)$, and $u = \phi(v \otimes v)$ is a unit of A. Since $A[t]/(t^2 - u)$ is unramified over A, $\operatorname{Spec}(\mathcal{O} \oplus \mathcal{L})$ is etale over Y.

(c) Conversely, if $X \mapsto Y$ is etale of degree 2, then locally $f_*\mathcal{O}_X(U)$ is an unramified algebra of rank 2 over A, which is always able to be written in the form $A[t]/(t^2 - u)$, so the exact sequence in (a) is splitted by $f_*\mathcal{O}_X \to \mathcal{O}_Y$ where the map $A[t]/(t^2 - u) \to A$ is given by taking the constant term. Now we see that (a) and (b) are converse to each other.

3 Embeddings in Projective Space

Exercise 1 (by Shi-Xin Wang).

Since $\deg D \geq 5 = 2g(X) + 1$, by Corollary 3.2, D is very ample. So we only need to show that if D is very ample, then $\deg D \geq 5$. We first show that $\dim |D| \geq 3$. Indeed, if $\dim |D| = 1$, it defines a closed immersion to $\mathbb{P}^1$, which is impossible. Moreover, if $\dim |D| = 2$, it defines a closed immersion from X to $\mathbb{P}^2$ as a plane curve, and hence by Riemann-Roch formula,

$$\deg D = g(X) - 1 + \dim |D| - \dim |K - D| = 4$$

Therefore, $g(X) = \frac{1}{2}(\deg D - 1)(\deg D - 2) = 3 \neq 2$ is a contradiction. On the other hand, by ex.iv.1.5, $\deg D > \dim |D| \geq 3$. Then we may assume $\deg D = 4$. Since $\deg D \geq 2g(X) - 2 = 2$, $\dim |K - D| = -1$. However, there is a contraction

$$\deg D = g(X) - 1 + \dim |D| - \dim |K - D| \geq 5$$

Thus we must have $\deg D \geq 5$.

Exercise 2 (by Yi-Tsung Wang)**.**

(a) Let K be a canonical divisor. Since $\omega_X = \mathcal{O}_X(d - n - 1) = \mathcal{O}_X(1)$, we see that $K = |K|^* L = X.L$ for some line L.

(b) Since X is a plane curve of degree 4, we have $g(X) = 3$. Since $\omega_X = \mathcal{O}_X(1)$ is very ample, so is K. $\ell(K - D) = \ell(K) - 2 = 1$. By Riemann-Roch, $\ell(D) = \deg D + 1 - g + \ell(K - D) = 1$, hence $\dim |D| = 0$.

(c) Suppose not, let $f : X \to \mathbb{P}^1$ be a finite morphism of degree 2, then $D := f^*(\infty)$ is an effective divisor of degree 2. By part (b), $\ell(D) = 1$, and since $f \in \Gamma(X, \mathscr{L}(D))$, f sends all $x \in X$ to $\infty \in \mathbb{P}^1$, contradiction. Hence X is not hyperelliptic.

Exercise 3 (by Tzu-Yang Chou)**.**

By Ex(II.8.4), $\mathcal{O}_X(K) \simeq \mathcal{O}_X(m)$ for some integer m. Moreover, $\deg K = 2g - 20$ so $m > 0$ and hence K is very ample. When $g = 2$, K has degree $2 < 5$ so cannot be very ample by Ex(IV.3.1); thus X must not be a complete intersection.

Exercise 4 (by Yu-Chi Hou)**.**

(a) For $d \geq 1$, let $\nu_d : \mathbb{P}^1 \to \mathbb{P}^d$ be $d-$uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^d$ and let X be its image. Recall the $d-$uple embedding is given by $\nu_d([t_0, t_1]) = [t_0^d : t_0^{d-1}t_1 : \cdots : t_0 t_1^{d-1} : t_1^d]$. From Exercise I.2.12, we know that X is integral, $S(X) = k[x_0, \ldots, x_d]/I(X)$ is integral, and $I(X) = \ker(\theta)$, where $\theta : k[x_0, x_1, \ldots, x_d] \to k[t_0, t_1]$ is given by $x_i \mapsto t_0^{d-i} t_1^i$. In other words, we can write $S(X) = k[t_0^d, t_0^{d-1}t_1, \ldots, t_1^d]$. Given $r \in \mathrm{Frac}(S(X)) = k(t_0, t_1)$ which in integral over $S(X)$. Write $r(t_0, t_1) = \frac{f(t_0,t_1)}{g(t_0,t_1)}$, where $f, g \in k[t_0, t_1]$ and $\gcd(f, g) = 1$, and there exists $a_0, a_1, \ldots, a_{n-1} \in S(X)$ such that

$$r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0.$$

Repeating the proof that UFD are integrally closed (clean out the denominator g and use the relative primeness of g and f), we know that $g \in k^*$ and hence $r = f(t_0, t_1) \in k[t_0, t_1]$. Hence, above equation reads

$$f^n + a_{n-1}f^{n-1} + \cdots + a_1 f + a_0 = 0. \quad (1)$$

By comparing degree, we may assume that $a_0, \ldots, a_{d-1}$ are homogeneous of degree $k_0, \ldots, k_{n-1}$ in degree d monomial of t_0, t_1 and g is homogeneous of degree m in t_0, t_1. Thus, equating the degree of (1) gives $mn = m(n - 1) + dk_{n-1} + \cdots = m + dk_1 = dk_0$. Hence, $m = dk_{n-1} \Rightarrow d \mid m$. Thus,

$f = \sum_i a_i t_0^{dn-1-i} t_1^i$. However, since each monomial $t_0^k t_1^j$ in $a_0, \ldots, a_{d-1}$, these exponents congruent to zero modulo d. As a result, $i \equiv 0 \mod d$. In other words, $f \in S(X)$.

Alternatively, we see that $V := \mathrm{Spec}(k[t_0^d, t_0^{d-1}t_1, \ldots, t_1^d])$ is the affine toric variety associated to the cone $\sigma := \mathbb{R}_{\geq 0}\langle e_1, e_1 + de_2\rangle \subset \mathbb{R}^2$. Since σ is strongly convex polyhedral cone, the affine monoid $S_\sigma = Z^2 \cap \sigma$ is saturated, and hence the variety $V = \mathrm{Spec}(k[S_\sigma])$ is normal. Also, observe that if V is an affine cone over a projective variety X, V is normal if and only if X is projectively normal by definition.

Next, we show that the homogenous ideal $I(X)$ is generated by homogeneous polynomial of degree 2. More precisely, we show that that

$$I(X) = \langle g_{ij} := x_i x_{j+1} - x_{i+1} x_j : 0 \leq i < j \leq d-1\rangle \subset k[x_0, \ldots, x_d].$$

Obviously, $g_{ij} \in \ker(\theta)$ and hence $I(X) \supset \langle g_{ij} : 0 \leq i < j \leq d-1\rangle$. For the converse, given any homogeneous polynomial $f \in I(X)$ of degree n, choose the lexicographic order $x_0 > x_1 > \cdots > x_d$ as monomial ordering and let r be the remainder after division by g_{ij}'s. That is, $r = f - \sum_{0 \leq i < j \leq d-1} a_{ij} g_{ij}$, where $a_{ij} \in k[x_0, \ldots, x_d]$. By equating degree on both sides, we know that r is also a homogeneous polynomial of degree n. We now have two simple observations:

(1) r contains no monomial of the form $-x_i^l$, for $i = 1, \ldots, d-1$. If there were such monomial, then such term can be subtracted by some multiple of $g_{i-1,i} := x_{i-1}x_{i+1} - x_i^2$.

(2) Also, r contains no monomial involving variables x_i, x_j with $j - i \geq 2$. If there were, then again such term can be subtracted by some multiple of $g_{i,j-1} := x_i x_j - x_{i+1} x_{j-1}$.

Following these two observations, r can be decomposed into

$$r = h_0(x_0, x_1) + h_1(x_1, x_2) + \cdots + h_{d-1}(x_{d-1}, x_d),$$

where each h_i is homogeneous of degree n, for all $i = 0, \ldots, d-1$ and contains no term like x_i^n, for $i = 1, \ldots, d-1$.

Finally, for $r = f - \sum_{ij} a_{ij} g_{ij} \in I(X)$, that is to say, $r(t_0^d, t_0^{d-1}, \ldots, t_1^d) = 0$. For each $i = 1, \ldots, d-2$,

$$h_i(x_i, x_{i+1}) = \sum_{k=1}^{n-1} c_k^{(i)} x_i^{n-k} x_{i+1}^k$$

and

$$h_0(x_0, x_1) = c_0^{(0)} x_0^n + \sum_{k=1}^{n-1} c_k^{(0)} x_0^{n-k} x_1^k; h_{d-1}(x_{d-1}, x_d) = \sum_{k=1}^{n-1} c_k^{(d-1)} x_{d-1}^{n-k} x_d^k + c_d^{(d-1)} x_1^d.$$

Thus, for $i = 0, \ldots, d$, plugging x_i by $t_0^{d-i} t_1^i$, we see that:

$$0 = c_0^{(0)} t_0^{nd} + \sum_{k=1}^{n-1} c_k^{(0)} t_0^{nd-k} t_1^k + \sum_{k=1}^{n-1} c_k^{(1)} t_0^{n(d-1)-k} t_1^{r+k} + \cdots + \sum_{k=1}^{n-1} c_k^{(d-1)} t_0^{n-k} t_1^{n(d-1)+k} + c_d^{(d-1)} t_1^{nd}.$$

Therefore, $c_k^{(i)} = 0$ for all i, k. That is, $r = 0$.

(b) Let X be a curve of degree d in $\mathbb{P}^n$ with $d \leq n$ and $X \nsubseteq H$, for any hyplerplane H in $\mathbb{P}^n$. Take any hyperplane H, let $D = X.H$ be the very ample divisor on X. Thus, $\deg(D) = \deg(X.H) = \deg(X) = d$ and $\dim |D| = n$ (otherwise, there exists a proper subspace $V \subset h^0(X, \mathcal{O}_X(D))$ such that $X \subset \mathbb{P}(V^*) \subsetneq \mathbb{P}^n$). Now, since $X \nsubseteq H$, there exists $P \notin \mathrm{Bs}|D|$, then $\dim |D-P| = \dim |D|-1 = n-1$ and $\deg(D-P) = d-1$.

If $n > d$, then pick $P_1, \ldots, P_d \notin \mathrm{Bs}|D|$, inductive on aboce arguement gives $\dim |D - \sum_{i=1}^d P_i| = n-d > 0$ yet $\deg(D - \sum_{i=1}^d P_i) = 0$. Therefore, $D - \sum_{i=1}^d P_i \sim 0$. If so, then $h^0(X, \mathcal{O}_X(D - \sum_{i=1}^n P_i) = 1$, contradiction. Hence, $n = d$. By Exercise IV.1.5, $\deg(d) = \dim |D|$ if and only if $D \sim 0$ or $g(X) = 0$. By assumption, $\deg(D) > 0$, we then must have $g(X) = 0$ and $\mathcal{O}_X(H) = \mathcal{O}_{\mathbb{P}^1}(dH)$. Therefore, $X \cong \nu_d(\mathbb{P}^1)$ up to $\mathrm{Aut}(\mathbb{P}^n)$.

(c) If X is of degree 2 in $\mathbb{P}^n$. If X is not contained in any hyperplane, then $n = 2$ by (b). If there exists a hyperplane $H \cong \mathbb{P}^{n-1}$ such that $X \subseteq H$, then replacing n by $n-1$ and repeating the previous argument, we still get $n = 2$. Hence, X is a plane conic.

(d) Let X be a curve of degree 3. The same argument in (c) shows that $X \subseteq \mathbb{P}^3$. We now have two cases. If X is not contained in any plane $\mathbb{P}^2$, then $X \cong \nu_3(\mathbb{P}^1)$ by (b). It is indeed the twisted cubic curve up to a projective transform. If X falls into some plane, then it is a plane cubic.

Exercise 6 (by Tzu-Yang Chou)**.**

(a) Let n be the smallest integer such that $X \subseteq \mathbb{P}^n$. First, Ex(IV.3.4)(b) implies that the case $n > 3$ is contained in (1). Also, for the case $n = 2$, we have $g = \frac{(4-1)(4-2)}{2} = 3$. For $n = 3$, we have $g < 3$ by Ex(IV.3.5)(b), so it remains to show that the genus cannot be 2 in this case. But X embed into $\mathbb{P}^3$ as a degree 4 curve, so there's a degree 4 very ample divisor D, which contradicts to Ex(IV.3.1).

(b) Now we assume that $X \subseteq \mathbb{P}^3$ with $g = 1$. We consider the cohomology sequence of $0 \longrightarrow \mathscr{I}_X(2) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(2) \longrightarrow \mathcal{O}_X(2) \longrightarrow 0$, which is a four-term one. We see that $h^0(\mathbb{P}^3, \mathscr{I}_X(2)) = 10 - 8 + h^1(\mathbb{P}^3, \mathscr{I}_X(2)) \geq 2$. Then the assertion follows from Bezout's theorem.

Exercise 7 (by Yi-Heng Tsai)**.**

Since $\mathrm{char}\, k \neq 2$, the curve has only one node at $(x, y) = (0, 0)$. Suppose there is a non-singular curve C which projects to it, then $deg(C) = 4$ and $g(C) = 2$ (contradicts to Ex3.6).

Exercise 9 (by Pei-Hsuan Chang)**.**

Let H be a plane in $\mathbb{P}^3$. We have: H intersect X least then d distinct point $\Leftrightarrow$ H contain a tangent line of X. Also, there are 3 intersection point of H and X are collinear $\Leftrightarrow$ H contain a multisecant of X.

Notice that $T := \{H \in (\mathbb{P}^3)^* \mid H$ contain a tangent line of $X\}$ is locally a subset of $X \times \mathbb{P}^1$; thus, it has at most dimension 2. Consider $S := \{$ mulitsecants of $X\} \subset (X \times X \setminus \triangle)$. It is a proper closed subset of $X \times X$, so S has at most dimension 1. Hence, $\{H \in (\mathbb{P}^3 \mid H$ contains a multisecant of $X\}$ has at most dimension 2. So, $T\cup$ is a proper closed subset of $(\mathbb{P}^3)^*$. Thus, there is an open set $U \subset (\mathbb{P}^3)^*$ as desired.

4 Elliptic Curves

Exercise 1 (by Chi-Kang)**.**

By R-R, we have $h^0(nP) - h^0(K - np) = n$. Note that $K = 0$, so $h^0(K - nP)$ is zero if $n > 0$, and is 1 if $n = 0$. So $h^0(nP) = n$ for $n > 0$, and $h^0(0p) = 1$.
Now embedded X by $|3P|$ into $\mathbb{P}^2$, we say X in $k[z_0, z_1, z_2]$ is defined by $z_1^3 = z_0(z_0 - z_2)(z_0 = \lambda z_2)$.
Now we choose $t_0 = 1$ be a generator of $H^0(P)$, $x_0 \in H^0(2P)$ s,t, $\{t_0, x_0\}$ is a basis of $H^0(2P)$, and similarly choose $y_0 \in H^0(3P)$ s,t, $\{t_0, x_0, y_0\}$ is a basis of $H^0(3P)$. Then R is generated by t_0, x_0, y_0 i,e, $R = k[t_0, x_0, y_0]/(relations)$. As the proof of proposition 4.6, after a change of coordinate we have $y + 0^2 = x_0(x - t_0)(x - \lambda t_0)$. Note that in fact $t_0 = 1 \in H^0(P)$, so $t_0^2 = t_0$, thus we have the relation $y_0^2 = x_0(x_0 - t_0^2)(x_0 - \lambda t_0^2)$. Hence the map

$$k[t, x, y]/(y^2 - x(x - t^2)(x\lambda t^2)) \to R$$

is well-defined and surjective. Now the above 2 rings are intergal domain. Note that for any surjective homomorphism $f : A \to B$ between integral domain, if f is not an isomorphism we must have $\dim A > \dim B$. But for our map both LHS and RHS has Krull dimension 2, hence it must an isomorphism.

Exercise 2 (by Yu-Chi Hou)**.**

Let X be a genus 1 curve and D is a divisor on X with $\deg D \geq 3$. Since $\deg D \geq 3$, D is very ample (cf. Cor. IV.3.2). Hence, the complete linear system $|D|$ gives an embedding $\phi_{|D|} : X \hookrightarrow \mathbb{P}^n$, where $n = \dim |D| = \deg D + 1$ using Riemann-Roch.

Lemma 1. *X is projectively normal if and only if for any $m \geq 0$, the natural map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) \to H^0(X, \mathcal{O}_X(m))$ is a surjection.*

The lemma is really a special case of Ex. II.5.14.

To check the condition of the lemma, we proceeds inductively on m. For $m = 1$, this follows directly from $\phi_{|D|}^* \mathcal{O}_{\mathbb{P}^n}(1) = O_X(D)$. Assume the induction hypothesis holds for $m-1$, then we consider the following diagram

$$\begin{array}{ccc} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) & \longrightarrow & H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m+1)) \\ \downarrow\!\!\downarrow & & \downarrow \\ H^0(X, \mathcal{O}_X(mD)) \otimes H^0(X, \mathcal{O}_X(D)) & \longrightarrow & H^0(X, \mathcal{O}_X((m+1)D)), \end{array}$$

where the horizontal maps are given by multiplication map and the vertical arrow is the natural map coming from $X \hookrightarrow \mathbb{P}^n$. By induction hypothesis, the left arrow is surjective. If we can prove the surjectivity of the bottom horizaontal arrow, then the surjectivity of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m+1)) \to H^0(X, \mathcal{O}_X((m+1)D))$ will follows.

Starting from here, we use the assumption that X is an elliptic curve. First of all, we can pick $P \in X$ such that $dP \sim D$, where $d = \deg D$ and from Riemann–Roch,

$$h^0(X, \mathcal{O}_X(nP)) = \begin{cases} 1 & , n = 0, 1 \\ n & , n \geq 2. \end{cases}$$

Hence, for $k \geq 1$, we have a sequence of strict inclusion

$$H^0(X, \mathcal{O}_X(kP)) \subsetneq H^0(X, \mathcal{O}_X((k+1)P)).$$

Namely, there exists unique $f \in K(X)$ which is regular outside P and $\text{ord}_P(f) = k+1$, for each $k \geq 1$. Thus, for any $f \in H^0(X, \mathcal{O}_X((n+m)P))$ there exists $g \in H^0(X, \mathcal{O}_X(nP)), h \in H^0(X, \mathcal{O}_X(mP))$ such that $gh = f$, for any $n, m \geq 3$.

As a result, we see that the multiplication map

$$H^0(X, \mathcal{O}_X(mdP)) \otimes H^0(X, \mathcal{O}_X(dP)) \to H^0(X, \mathcal{O}_X((m+1)dP))$$

is surjective since $d \geq 3$.

Exercise 3 (by Pei-Hsuan Chang)**.**

Let $f = y^2 - x(x-1)(x-\lambda)$. Then regular functions on X except P_0 is $k[x,y]/ < f > + : R$. Thus, $K(X) = \text{Frac}(R) = \{a(x) + b(x)y \mid a(x), b(x) \in k(x)\}$. Now, for each $\varphi \in \text{Aut}(X)$, we can assume $\varphi(x,y) = (x', y') = (u_1(x) + v_1(x), u_2(x) + v_2(x)y)$. Notice that $\forall P = (x,y) \in X$, $0 = \varphi(0) = \varphi(P + (-P)) = \varphi(P) + \varphi(-P)$ in the group law. So

$$P_0 = \varphi(x,y) + \varphi(x,-y) = (u_1(x) + v_1(x), u_2(x) + v_2(x)y) + (u_1(x) - v_1(x), u_2(x) - v_2(x)y),$$

then $u_1(x) + v_1(x) = u_1(x) - v_1(x)$ and $u_2(x) + v_2(x)y = -(u_2(x) - v_2(x)y)$. Hence, $v_1(x) = u_2(x) = 0$, so $\varphi(x,y) = (u_1(x), v_2(x)y)$.

Now, we homogenizes φ to get

$$\tilde{\varphi}(x,y,z) = (u_1\left(\frac{x}{z}\right), v_2\left(\frac{x}{z}\right)\frac{y}{z}, 1) = (\tilde{u_1}(x,z), \tilde{v_2}(x,z)y, z^n),$$

where $\tilde{u_1}, \tilde{v_2}$ are homogeneous rational functions of degree n and $n-1$ respectively. Since $\tilde{\varphi}(P_0) = P_0$, $\tilde{\varphi}(0,1,0) = (\tilde{u_1}(0,0), \tilde{v}_2(0,0) \cdot 1, 0) = (0, t, 0)$ for some $t \neq 0$. Thus, $\tilde{v_2}(0,0) \neq 0 \Rightarrow \tilde{v_2}(x,z)$ is constant, say $\tilde{v_2}(x,z) = c$. Hence $n = 1 \Rightarrow \tilde{u_1}$ is linear. Now, de-homogenize $\tilde{\varphi}$ and get $\varphi(x,y) = (x', y') = (ax + b, cy)$ for some constant $a, b, c \in k$ on the affine piece.

Exercise 4 (by Tzu-Yang Tsai)**.**

The equation equivalent to $(y + \frac{a_1}{2}x + \frac{a_3}{2})^2 = x^3 + (a_2 + \frac{a_1^2}{4}) + (a_4 + \frac{a_1 a_3}{2})x + a_6 + \frac{a_3^2}{4}$, so by a linear transformation, we get $Y^3 = x^3 + Ax^2 + Bx + C$, where $A, B, C \in k_0$.
Let the roots of $x^3 + Ax^2 + Bx + C = 0$ be α, β, γ, we map $\{\begin{smallmatrix}\alpha \mapsto 0 \\ \beta \mapsto 1\end{smallmatrix}$ by a linear transformation, then $\gamma \mapsto \frac{\gamma - \alpha}{\beta - \alpha} = \lambda$. Thus

$$\begin{aligned} j(\lambda) &= 2^8 \frac{(1 - \lambda + \lambda^2)^3}{\lambda^2(1-\lambda)^2} \\ &= 2^8 \frac{(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha)^3}{(\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2} \end{aligned}$$

, where both numerator and denominator are symmetric polynomial, which can be represented by elementary symmetric polynomial A, B, C. As a result, j is a rational function of $\{a_i\}$, furthermore, $j \in k_0$.
For $j \neq 0, 1728$, take $A = 0, C = tB$,

$$j = 2^8 \frac{B^3}{4B^3 + 27C^2} \Rightarrow B = \frac{-27jt^2}{4(j-1728)}$$

so simply take $t = 1$, notice that $B \in k$, we get an elliptic curve in k with j as j-invariant.
For $j = 0$, $y^2 + y = x^3$ is the curve; for $j = 1728$, $y^2 = x^3 + x$ is the curve.

Exercise 5 (by Shuang-Yen Lee)**.**

(a) By Hurwitz formula, f has no ramification points. Let $P_0 + Q = f^*P_0$, then $P_0 \neq Q$. Since $\ell(P_0 + Q) = \ell(2P_0) = \ell(2Q) = 2$ (by R-R), there exist $h_1 \in L(P_0 + Q)$, $h_2 \in L(2P_0)$ and $h_3 \in L(2Q)$ which are not constant. Since $\ell(P_0) = \ell(Q) = 1$, $h_1^2 \neq L(2P_0) \cup L(2Q)$. So $L(2P_0+2Q) = \langle 1, h_1^2, h_2, h_3 \rangle_k$. Note that

$$(\pi \circ f)^*(\infty) = f^*\pi^*(\infty) = f^*(2P_0) = 2P_0 + 2Q,$$

$\pi \circ f \in k^\times h_1^2$, say $\pi \circ f = a^2 h_1^2 = (ah_1)^2$ for some $a \in k^\times$. Let $\pi' = ah_1$, $g = [x \mapsto x^2]$, then $\pi \circ f = g \circ \pi'$ and $\deg g = 2$, so we get $\deg \pi' = 2$.

(b) By (a).

(c) The branch points of g are $0, \infty$. ∞ is a branch point of π since $\pi^*(\infty) = 2P_0$. 0 is a branch point of π since $f^*\pi^*(0) = \pi'^* g^*(0) = 2\pi'^*(0)$ and note that f has no ramification points. Suppose that other two branch points of π are 1, λ. Then

$$\pi'^*((1) + (-1)) = 2f^*(2Q_1), \quad \pi'^*((\lambda^{1/2}) + (-\lambda^{1/2})) = f^*(2Q_2)$$

for some Q_1, $Q_2 \in X$, so 1, -1, $\lambda^{1/2}$, $-\lambda^{1/2}$ are branch points of π'.

Now we have two ways to count j. By the map π, we have

$$j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(1-\lambda)^2}.$$

By the map π', since the cross ratio

$$\lambda' := (1, -1; \lambda^{1/2}, -\lambda^{1/2}) = \left(\frac{1 - \lambda^{1/2}}{1 + \lambda^{1/2}}\right)^2,$$

we have

$$j = 2^8 \frac{(\lambda'^2 - \lambda' + 1)^3}{\lambda'^2(1-\lambda')^2} = 2^8 \frac{(\lambda^2 + 14\lambda + 1)^3}{16\lambda(1-\lambda)^4}.$$

So $16(\lambda^2 - \lambda + 1)^3(1-\lambda)^2 = (\lambda^2 + 14\lambda + 1)^3\lambda$.

(d) By solving the equation above, we have $\lambda = -1,\ 3 \pm 2\sqrt{2},\ \frac{1}{32}(1 \pm 3\sqrt{7}i),\ \frac{1}{2}(1 \pm 3\sqrt{7})$ and

$$j = 2^6 \cdot 3^3, 2^6 \cdot 5^3, -3^3 \cdot 5^3, -3^3 \cdot 5^3,$$

respectively.

Exercise 9 (by Chi-Kang)**.**

(a) The identity map is an isogenus, and the composition of 2 finite morphism is finite, so we only need to show if $f : X \to X'$ is a finite morphism of degree n, then there exists $X' \to X$ be another finite morphism. By exercise IV.4.7 we have a dual morphism $\hat{f} : X' \to X$ s,t, $\hat{f} \circ f = n_X$ is a finite morphism with degree n^2, hence $\hat{f}$ is also finite morphism with degree n, and thus isogenus is an equivalent realtion.
(b) Suppose $f : X \to X'$, $g : X \to X''$ are 2 finite morphism with the same (group theoritic) kernel, then $X' \cong X''$ as abelian group. So there is a natural group isomorphism $g \circ f^{-1} : X' \cong X/(\ker f) \cong X''$, and this is a morphism between curves since both f, g is. Thus $g \circ f^{-1}$ is a bijective morphism between curves, hence it is an isomorphism since X', X'' are smooth.

Now since $\hat{f} \circ f = n_X$ so $\ker f \subset \ker n_X$. And by exercise 4.7 we have $f \circ \hat{f} = n_{X'}$, so both $f, \hat{f}$ has degree n, thus $\deg n_X = n^2$, so X has n^2 element of order n, hence X has at most countably many subgroups G which is a subgroup of some $\ker n_X$. Hence X has at most countablley many isogenus classes.

Exercise 10 (by Shi-Xin Wang)**.**

To construct the map $\phi : Pic(X \times X) \to R := End(X, P_0)$, we let $M \in Pic(X \times X)$ and p_1, p_2 be two projections from $X \times X$ to X. We may guess M should be sent to $M \otimes (p_1^*(M|_{X\times\{P_0\}}) \otimes p_2^*(M|_{\{P_0\}\times X}))^{-1}$, denoted by N_M. However, N_M does not lie in R. Remark that we have an isomorphism $\varphi : Pic^0 X \to X$. Therefore, we may consider

$$\phi(M) := [P \mapsto \varphi(N_M|_{X\times P})].$$

This is well defined since $N_M|_{X\times P}$ has the same degree with $N_M|_{X\times P_0}$, i.e. they are both in $Pic^0 X$. Clearly, $p_1^* Pic X \oplus p_2^* Pic X \subset ker\phi$. Now let $M \in ker\phi$. Since $N_M|_{X\times P} \cong \mathcal{O}_{X\times P}$, by seesaw theorem, $N_M \cong p_2^* L$ for some $L \in Pic X$. Therefore, $M = p_1^*(M|_{X\times\{P_0\}}) \otimes p_2^*(L \otimes M|_{\{P_0\}\times X})$, and hence $p_1^* Pic X \oplus p_2^* Pic = ker\phi$. On the other hand, for any $\alpha in R$, consider the line bundle $M \in Pic(X \times X)$ corresponding to the divisor

$$D = (\alpha, id_X)(X) - \{P_0\} \times X$$

where $(\alpha, id_X) : X \to X \times X$ is the morphism given by $P \mapsto (\alpha(P), P)$. Then N_M still corresponds to the divisor D and

$$\varphi(N_M|_{X\times P}) \cong \varphi(\mathcal{O}_X(\alpha(P) - P_0)) = \alpha(P)$$

Exercise 11 (by Pei-Hsuan Chang)**.**

(a) Let L be the parallelogram, A be the area of L. Then area of $f(L)$ is $|\alpha^2|A$. Now,

$$\deg f = [L : \alpha L] = \frac{|\alpha^2|A}{A} = |\alpha|^2.$$

(b) By exercise 4.4.7(c), we have $\hat{f} \circ f$ is an endomorphism corresponding to $\deg f = |\alpha|^2$. Thus, $\hat{f}$ is an endomorphism corresponding to $|\alpha|^2 \cdot \alpha^{-1} = \bar{\alpha}$.

(c) Let L be the lattice $\mathbb{Z} \oplus \tau\mathbb{Z}$. Now, if $\tau \in \mathbb{Q}(\sqrt{-d})$ and integral over $\mathbb{Z}$, then τ^2 can be written as integral linear combination of τ and 1. Thus, $\mathbb{Z}[\tau] = \mathbb{Z} \oplus \tau\mathbb{Z}$. Also, for $a, b \in \mathbb{Z}$, $(a + b\tau)\tau = a\tau + b\tau^2 \in L$. Hence, $\forall a + b\tau \in \mathbb{Z}[\tau]$, $(a + b\tau)L \subset L$, which means $\mathbb{Z}[\tau] \subset R$.

For any $f \in R$, say f corresponding to $\alpha \in \mathbb{C}$. Since $\alpha L \subset L$ and $1 \in L \Rightarrow \mathbb{Z} \oplus \tau\mathbb{Z} \Rightarrow R \subset \mathbb{Z}[\tau]$. To sum up, $R = \mathbb{Z}[\tau]$.

Exercise 12 (by Po-Sheng Wu)**.**

(a)(b) Suppose the complex multiplication was given by α, then $|\alpha|^2 = 1$ for (a), 2 for (b) respectively. Since α is imaginary quadratic and integral, we can assume that $\alpha = (a + b\sqrt{-d})/2, b, d > 0$, d squarefree, then $a^2 + db^2 = 4($ or 8, respectively). So $(a, b, d) = (0, 2, 1), (\pm 1, 1, 3)$ for (a), $(a, b, d) = (0, 2, 2), (\pm 1, 1, 7), (\pm 2, 2, 1)$ for (b), and we get $\tau = i, \omega$ for (a), $\tau = \sqrt{-2}, (1+\sqrt{-7})/2, i$ for (b), respectively. Moreover, we have $j(\sqrt{-2}) = 8000, j((1 + \sqrt{-7})/2) = -3375, j(i) = 1728$ comparing with 4.5(d), using the fact that if $\mathrm{Re}(\tau) = 0$ then $j(\tau) > 0$.

Exercise 13 (by Yi-Heng Tsai)**.**

Hasse invariant $= 0$ i.e. $h_p(\lambda) = 0$. $\Rightarrow j = \frac{2^8(\lambda^6 - 3\lambda^5 + 6\lambda^4 + 6\lambda^3 + 6\lambda^2 - 3\lambda + 1)}{(\lambda^2 - 2\lambda + 1)\lambda^2} = \frac{2^8(2\lambda^4 - 4\lambda^3 + 2\lambda^2)}{(\lambda^2 - 2\lambda + 1)\lambda^2} = 2^9 = 5$.

Exercise 14 (by Tzu-Yang Tsai)**.**

By 4.21, Hasse invariant of X is 0 if and only if the coefficient of $(xyz)^{p-1}$ in f^{p-1} is 0. Now $f(x, y, z) = x^3 + y^3 - z^3$, thus it's clear that $p \in \mathcal{B}$ if and only if $3 \mid p - 2$, thus by Dirichlet's theorem the density of $\mathcal{B}$ in prime is $\frac{1}{2}$.

Exercise 17 (by Ping-Hsun Chuang)**.**

Proof. X is the curve $y^2 + y = x^3 - x$ in $\mathbb{P}^2$ with $P_0 = [0 : 1 : 0]$.

(a) Write $Q = [a : b : 1] \in X$. If $a = 0$, then we have $y^2 + y = 0$ and thus $Q = [0 : 0 : 1]$ or $[0 : -1 : 1]$.
Case 1: $Q = [0 : 0 : 1] = P$. The tangent line at $P\,[0 : 0 : 1]$ of X is $x = -y$ by the implicit function theorem. Solve $\begin{cases} x = -y \\ y^2 + y = x^3 - x \end{cases}$ and get $(x, y) = (0, 0)$ and $(1, -1)$. Note that the solution $(0, 0)$ has multiplicity 2. Then, we have $2P + R \sim 0$, where $R = [1 : -1 : 1]$. Now, the hyperplane $x - z = 0$ passing through P_0 and R. Solve $\begin{cases} x - z = 0 \\ y^2 z + yz^2 = x^3 - xz^2 \end{cases}$ and get $[x, y, z] = [0 : 1 : 0]$, $[1 : -1 : 1]$ and $[1 : 0 : 1]$. In consequence, we have $R + R' \sim 0$, where $R' = [0 : 1 : 0]$ and thus $2P \sim -R \sim R' = [1 : 0 : 1]$.

Case 2: $Q = [0 : 0 : 1] = P$. The hyperplane $x = 0$ passing through $P\,[0 : 0 : 1]$, $Q\,[0 : -1 : 1]$, and $P_0\,[0 : 1 : 0]$. Then, we have $P + Q + P_0 \sim 0$ and thus $P + Q \sim 0$.

Case 3: $a \neq 0$. The hyperplane $bx - ay = 0$ passing through $Q\,[a:b:1]$ and $P\,[0:0:1]$. Solve $\begin{cases} bx - ay = 0 \\ y^2 + y = x^3 - x \end{cases}$ and get $(x,y) = (0,0)$, (a,b), and $\left(\frac{b^2}{a^2} - a, \frac{b^3}{a^3} - b\right)$. Then, $P + Q + R \sim 0$, where $R = \left(\frac{b^2}{a^2} - a, \frac{b^3}{a^3} - b\right)$. Now, the hyperplane $x - \left(\frac{b^2}{a^2} - a\right) z = 0$ passing through P_0 and R. Solve $\begin{cases} x - \left(\frac{b^2}{a^2} - a\right) z = 0 \\ y^2 z + yz^2 = x^3 - xz^2 \end{cases}$ and get $[x:y:z] = P_0,\ R,\ R' = \left[\frac{b^2}{a^2} - a, -1 + b - \frac{b^3}{a^3} : 1\right]$. Hence, $R + R' \sim 0$, that is, $P + Q \sim -R \sim R' = \left[\frac{b^2}{a^2} - a, -1 + b - \frac{b^3}{a^3} : 1\right]$.

Finally, we use the above formula to find nP for $n = 1, \cdots, 10$:

P	$2P$	$3P$	$4P$	$5P$	$6P$	$7P$	$8P$	$9P$	$10P$
$(0,0)$	$(1,0)$	$(-1,-1)$	$(2,-3,)$	$\left(\frac{1}{4}, \frac{-5}{8}\right)$	$(6,14)$	$\left(\frac{-5}{9}, \frac{8}{27}\right)$	$\left(\frac{21}{25}, \frac{-69}{125}\right)$	$\left(\frac{-20}{49}, \frac{-435}{343}\right)$	$\left(\frac{161}{16}, \frac{-2065}{64}\right)$

(b) If $p \neq 2$, then the curve become $\left(y + \frac{1}{2}\right)^2 = x^3 - x + \frac{1}{4}$. The discriminant of $x^3 - x + \frac{1}{4}$ is $\frac{37}{16}$. Now, modulo p reduction gives non-zero discriminant if $p \neq 37$. This makes the curve non-singular.

If $p = 37$, the curve is $(y + 19)^2 = (x + 10)\,(x + 32)^2$ which is singular.

If $p = 2$, the partial derivative is given by $\frac{\partial f}{\partial x} = x^2 + 1$ and $\frac{\partial f}{\partial y} = 1 \neq 0$. Thus, the curve is non-singular when $p = 2$.

□

5 The Canonical Embedding

Exercise 1 (by Yu-Chi Hou)**.**

Assume that X is complete intersection in $\mathbb{P}^n$, then there exists hypersurfaces $H_1, \ldots, H_{n-1}$ in $\mathbb{P}^n$ with degree $d_1, \ldots, d_{n-1}$ respectively such that $X = H_1 \cap H_2 \cap \cdots H_{n-1}$. Using adjunction formula repeatly, one has $\omega_X \cong \mathcal{O}_X(\sum_{i=1}^{n-1} d_i - (n+1))$. Let $d := \sum_{i=1}^{n-1} d_i - (n+1)$. Since $g(X) \geq 2$, $\deg(K_X) > 0$. Thus, $d > 0$. We then onsider $d-$uple embedding $\nu_d : \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ with $N = \binom{n+d}{n} - 1$. Therfore, $\omega_X \cong (\nu_d|_X)^* \mathcal{O}_{\mathbb{P}^N}(1)$. Thus, K_X is very ample. However, if X is hyperelliptic, then K_X cannot be very ample, and thus X cannot be complete intersection. In particular, we know that genus 2 curves are hyperelliptic (Ex.IV.1.7) and thus X cannot be complete intersection. This also proves Ex. IV.3.3.

Exercise 2 (by Yu-Chi and Pei-Hsuan Chang)**.**

We first prove a lemma.

Lemma 2 (by Yu-Chi)**.** *Let x be a curve of genus $g \geq 2$, $\tau \in \mathrm{Aut}(X)$ and $\tau \neq 1_X$, then τ fixes at most $(2g+2)-$points.*

Proof. Let s be the number of fixed points of τ. Consider a divisor $D = \sum_{i=1}^{g+1} P_i$ with $P_1, \ldots, P_{g+1}$ are distinct and are not fixed point of τ. Then using Riemann–Roch,

$$h^0(X, D) - h^1(X, D) = \deg(D) + 1 - g = g + 1 + 1 - g = 2.$$

Hence, $h^0(X, D) \geq 2$ implies that there exists a non–constant morphism $f : X \to \mathbb{P}^1$ and $(f) + \sum_{i=1}^{g+1} P_i \geq 0$. In other words, the rational function f has at worst simple pole on $P_1, \ldots, P_{g+1}$. Since $\tau(P_i) \neq P_i$ for all i, $f \circ \tau - f$ is also a non–constant and has simple pole at most on $2g + 2$ points. On the other hand, for any fixed point Q of τ, $Q \in (f \circ \tau - f)$ obviously. Hence, $f \circ \tau - f$ has at least s many zeros. From $\deg(f \circ \tau = f) = 0$,

$$0 = |(f \circ \tau - f)_\infty| - |(f \circ \tau - f)_0| \leq 2g + 2 - s$$

Hence, $s \leq 2g + 2$. □

Solution of exercise 2 (by Pei-Hsuan Chang).
Case 1: X is hyperelliptic $\exists f : X \to \mathbb{P}^1$ of degree 2. Every ramified point is of index 2. By Hurwitz's formula,

$$2 - 2g = 2 \times 2 - \sum_{P \in X} (e_P - 1).$$

So f has $2g + 2$ ramified points. $\forall \varphi \in \operatorname{Aut} X$, $f \circ \varphi$ is also of degree 2, so $f \circ \varphi$ and f are differ by an automorphism of $\mathbb{P}^1$. Hence, if $P \in X$ is a ramified point of f, then $\varphi(P)$ is also a ramified point of f, i.e. φ permute ramified points. Now, if φ is an automorphism of X which fix $2g + 2$ ramified points then by Lemma above, φ is either identity map or switch all the fibres. Hence,

$$|\operatorname{Aut} X| \leq 2 \times |S_{2g+2}| < \infty.$$

Case 2: X is not hyperelliptic Let $f : X \to \mathbb{P}^{g-1}$ be canonical embedding. By Exercise 4.4.6(b), X has $(g-1)^2 g + gd$ hyperosulating points. $\forall \varphi \in \operatorname{Aut} X$, f and $f \circ \varphi$ differ by an automorphism of $\mathbb{P}^{g-1}$. Thus, φ permute hyperosulating points. In this case, g must bigger then 3, so $(g-1)^2 g + gd > 2g + 2$. By the Lemma again, φ is an identity map. Hence,

$$|\operatorname{Aut} X| \leq |S_{(g-1)^2 g + gd}| < \infty.$$

Exercise 3 (by Chi-Kang)**.**

For the hyperelliptic case, let X be a hyperelliptic curve of $g = 4$, then there is a degree 2 morphism $X \to \mathbb{P}^1$. By Hurwitz formula we have the ramification divisor R has degree 10, and since degree is 2 therer are 10 distinct ramafication points. Since up to an automorphism on $\mathbb{P}^1$ we may assume three of them are $0, 1, \infty$, so the moduli space is 7-dimensional.

For non-hyperelliptic case, use the very ample divisor $|K|$ we may assume X is a degree 6 curve in $\mathbb{P}^3$. So by example 5.2.2. X is a complete intersection of a unique quadric and a cubic.

To determine for a given quadric Q, how many complete intersection is, we needto compute $H^0(Q, \mathcal{O}_Q(3))$. By the exact sequence $0 \to \mathcal{O}_{\mathbb{P}^3}(1) \to \mathcal{O}_{\mathbb{P}^3}(3) \to \mathcal{O}_Q(3) \to 0$ and compute the cohomology we have

$h^0(Q, \mathcal{O}_Q(3) = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) - h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) = 16$. Quotient the constant, we know that the dimension of moduli space of degree 3 surface complete intersection with Q is 15. Since the dimension of moduli space of quadric is $h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) - 1 = 9$, we have the dimension of moduli space of genus 4 curves is $9 + 15 - \dim \mathrm{Aut}(\mathbb{P}^3) = 9 + 15 - (16 - 1) = 9$.

Finally, by example 5.5.2, a non-hyperelliptic curve with $g = 4$ has a unique g_3^1 iff Q is singular. Hence the dimension of the moduli space of such curve is $9 - 1 = 8$.

Exercise 4 (by Tzu-Yang Tsai)**.**

Claim $P + Q + R \in g_3^1 \Leftrightarrow P, Q, R$ are colinear under canonical embeding.
Proof: By Riemann-Roch theorem, $\dim |P + Q + R| - \dim |K - P - Q - R| = 3 + 1 - 4 = 0$
, thus $P + Q + R \in g_3^1 \Leftrightarrow \dim |K - P - Q - R| = 1$, but in canonical embedding, $|K - P - Q - R|$ consists of hyperplanes containing P, Q, R, thus $\dim |K - P - Q - R| = 1$ is equivalent to P, Q, R are colinear.

(a) Let σ_1, σ_2 be the two g_3^1, then for any P not a base point of σ_i for $i = 1, 2, !\exists Q_i \neq R_i$ s.t. $P + Q_i + R_i \in \sigma_i \forall i = 1, 2$. Thus we have a projection from $P, \phi : X - P \to \mathbb{P}^2$, which is nonsingular at everywhere except for $\phi(Q_i) = \phi(R_i) = T_i \, \forall i = 1, 2$. Use Riemann extension theorem we get $\bar{\phi} : X \to \mathbb{P}^2$, thus we represent X as a plane curve C with nodes T_1, T_2, and if $degC = r$, $\frac{(r-1)(r-2)}{2} = 4 + 2 = 6 \Rightarrow r = 5$, thereby a quintic curve.

(b)

Exercise 7 (by Po-Sheng Wu)**.**

(a) Let f be the canonical embedding, then since $|K|$ is preserved under Aut$X$, f and $f \circ \sigma$ differ by an automorphism of $\mathbb{P}^2$, $\forall \sigma \in \mathrm{Aut}X$.

(b) Assume char$k \neq 3, 7$. Obviously $\begin{pmatrix} \omega^4 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \in \mathrm{PGL}(k, 2)$ induces automorphism of X of order 3 and 7, and they generate $H \in \mathrm{Aut}X$ of order 21. Since $(2g(X) - 2)/n = 2g(Y) - 2 + \sum_{i=1}^{s}(1 - 1/r_i)$ won't hold for $g(X) = 3, n = 21$ in Ex.2.5., there are automorphisms not in H. Now notice that $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ are hyperosculating points on X with hyperosculating hyperplanes $z = 0, x = 0, y = 0$, and H acts freely on $X \backslash \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, so Aut$X$ acts transitively on the 24 hyperosculating points (Ex 4.6.) since there are extra automorphisms that are not permuting e_i. As a consequence, $24 \mid |\mathrm{Aut}X|$ and $21 = |H| \mid |\mathrm{Aut}X|$, so $168 \leq |\mathrm{Aut}X| \leq 84(g - 1) = 168$.

(c) Since most of the curves of genus 3 are nonhyperelliptic, we may consider only the curves of degree 4 in $\mathbb{P}^2$. Now we show that for each Jordan form J with $J^p = rI$, the family of curves with automorphism induced from some matrix conjugate with J has dimension $\leq \dim|4H| = 14$. J acts on $|4H|$ via $\mathrm{Sym}^4(J)$. Denote $m(J) = \dim(|4H|^{\mathrm{Sym}^4(J)}) = \max_r \mathrm{null}(\mathrm{Sym}^4(J) - rI) - 1$ and $n(J) = \dim\{PJP^{-1} | P \in \mathrm{GL}(k, 3)\} = 9 - \dim\{P \in \mathrm{GL}(k, 3) \mid PJ = JP\}$. The goal is to show that $m(J) + n(J) < 14$. If char$k \neq p$, then by scaling we may assume that $J = \begin{pmatrix} \omega^a & 0 & 0 \\ 0 & \omega^b & 0 \\ 0 & 0 & \omega^c \end{pmatrix}$, where $\omega^p = 1$. Then with some calculation we obtain

$m(J)=8, n(J)=4$ for $p=2$, and $m(J)\le 6, n(J)\le 6$ for $p\ge 3$. If char$k=p$, then again by scaling we may assume $J=\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}$ or $\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}$. For the former case, we calculate that $m(J)\le 8, n(J)=4$, and for the latter case, we have $m(J)\le 4, n(J)=6$ (Note that $p\ne 2$ in this case). As a result, $m(J)+n(J)<14$ holds for every cases, so most of the genus 3 curves has no automorphism by Bertini's theorem.

6 Classification of Curves in $\mathbb{P}^3$

Exercise 1 (by Shi-Xin)**.**

Let X be a rational curve of degree 4 in $\mathbb{P}^3$. First, from the short exact sequence

$$0\to\mathscr{I}_X\to\mathcal{O}_{\mathbb{P}^3}\to\mathcal{O}_X\to 0$$

where $\mathscr{I}_X$ is the ideal sheaf of X, we have a long exact sequence

$$0\to H^0(\mathscr{I}_X(2))\to H^0(\mathcal{O}_{\mathbb{P}^3}(2))\to H^0(\mathcal{O}_X(2))\to\cdots$$

Note that $h^0(\mathcal{O}_{\mathbb{P}^3}(2))=C_3^5=10$. Let D be a hyperplane section of X. Then by Riemann-Roch Theorem, $h^0(2D)=9+h^0(K-2D)=9$ since $\deg K-2D<0$. Therefore, $h^0(\mathscr{I}_X(2))\ge 1$ which means X is contained in a quadric surface Q. If X is containd in two nonsingular quadric surface , by $ex.ii.8.4(g)$, $g(X)=\frac{1}{2}2\cdot 2(2+2-4)+1=1$ which leads to a contradiction. Indeed, since X is rational, it has 4 linearly independent points, and thus Q can not be $x_1^2, x_1^2+x_2^2$. Moreover, by Remark 6.4.1, Q can not be a cone. We conclude that Q is nondegenerate, i.e. nonsingular.

Exercise 2 (by Yu-Chi Hou)**.**

Let X be a degree 5 rational curve in $\mathbb{P}^3$, consider the exact sequence of X twisting by $\mathcal{O}_{\mathbb{P}^3}(3)$,

$$0\to\mathscr{I}_X(3)\to\mathcal{O}_{\mathbb{P}^3}(3)\to\mathcal{O}_X(3)\to 0,$$

where $\mathscr{I}_X$ is the ideal sheaf of X. Taking long exact sequence of cohomology, one has

$$0\to H^0(\mathbb{P}^3,\mathscr{I}_X(3))\to H^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3})\to H^0(X,\mathcal{O}_X(3))\to H^1(X,\mathscr{I}_X(3))\to H^1(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(3))=0.$$

Thus, we have $h^0(\mathbb{P}^3,\mathscr{I}_X(3))-h^1(\mathbb{P}^3,\mathscr{I}_X(3))=h^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3})-h^0(X,\mathcal{O}_X(3))$. Since $\deg X=5=\deg(D.H)$, where $H\subset\mathbb{P}^3$ is a plane. Also, $\deg(\mathcal{O}_X(3))=\deg(3D)=15$, $\deg(K_X)=2g-2=-2<0$. Thus, Riemann–Roch gives $h^0(X,\mathcal{O}_X(3))=15-0+1=16$. On the other hand, $h^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3})=\binom{6}{3}=20$. In conclusion, $h^0(\mathscr{I}_{\mathscr{X}}(3))=h^1(\mathscr{I}_X(3))+4\ge 4$. Thus, X must be contained in a cubic surface.

Now, suppose X is contained in a quadric surface $Q\subset\mathbb{P}^3$. If Q is non-singular, say X has type (a,b) in Q, then $a+b=5$ and $(a-1)(b-1)=0$. This leads a contradiction. If Q is singular, then remark IV.6.4.1 shows that $\deg(X)=2a+1=5$ and $g(X)=a^2-a=2^2-2=2$, a contradiction to the assumption that X is rational.

Exercise 4 (by Yi-Heng Tsai).

Assume there exists such X, then we have a long exact sequence $0 \to H^0(\mathscr{I}_X(2)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_X(2)) \to ...$ with $dimH^0(\mathcal{O}_X(2)) < dimH^0(\mathcal{O}_{\mathbb{P}^3}(2)) = 10$. Thus, $dimH^0(\mathscr{I}_X(2)) \geq 1$ which means X lies on some quadric surface. However, this contradicts to remark 6.4.1.

Exercise 6 (by Tzu-Yang Chou).

First recall that projectively normal is equivalent to $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l)) \longrightarrow H^0(X, \mathcal{O}_X(l))$ is surjective for any non-negative l.
If $d = 6$, we have $g \leq 4$, so we need that $g \neq 0, 1, 2$. Let D be the hyperplane section (so $\deg D = d = 6$) which is nonspecial in these cases. Riemann-Roch imlies that $l(\mathcal{O}_X(1)) = 6 + 1 - g = 7, 6, 5$ but $h^0(\mathcal{O}_{\mathbb{P}^3}(1)) = 4$, which leads to a contradiction.
If $d = 7$, we have $g \leq 6$. The above argument still works for $g = 0, 1, 2, 3$. For $g = 4$, we use the divisor $2D$. $l(\mathcal{O}_X(2)) = 7 \times 2 + 1 - 4 = 11$ but $h^0(\mathcal{O}_{\mathbb{P}^3}(2)) = 10$.

Exercise 8 (by Shuang-Yen Lee).

If D is a nonspecial divisor of degree d such that $|D|$ has no base point, then by R-R we have $\ell(D) = d + 1 - g$. If $d \leq g$, then $\dim |D| \leq 0$. So $|D| = \{E\}$ or $\varnothing$, this implies $|D|$ has base point or empty.

Conversely, suppose $d \geq g + 1$. Let $S \subseteq X^d$ be the set of divisors $D \in X^d$ such that there exists $P \in X$ with $D - P \sim E \geq 0$ is a special divisor. Note that every $D \in X^d - S$ is a nonspecial base point free divisor of deg d. So we want to show that $S \neq X^d$.

Let $D \in S$ be nonspecial, then there exist $P \in X$ such that $D - P \sim E \geq 0$ is special. We have $D = E + P + (f)$ for some $f \in K(X)$. By R-R, E special implies that

$$\ell(E) = \deg E + 1 - g + \ell(K - E) = d - g + \ell(K - E) \geq d - g + 1.$$

$E+P$ is nonspecial, so $\ell(E+P) = \deg(E+P)+1-g = d-g+1$. Since $L(E+P) \supseteq L(E)$, $L(E+P) = L(E)$. So $f \in L(E + P) = L(E)$, hence $D = (E + (f)) + P$ and $E + (f) \geq 0$ is special. Therefore

$$S \subseteq \{E + P \mid E \geq 0 \text{ special and } P \in X\} \cup \{\text{ special divisors }\}.$$

Since $\dim |K| = g - 1$, the dimension of special divisors as a subset of X^{d-1} and X^d are both $\leq g - 1$. Thus $\dim S \leq g < \dim X$. So $S \neq X$, as desired.