Algebraic Geometry I Homework
Chapter III Cohomology

A course by prof. Chin-Lung Wang

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Exercise 0 (by Kuan-Wen)**.**

This is an example of proof.

Remark. This is an example for how to write in this format.

2 Cohomology of Sheaves

Exercise 1 (by Chi-Kang Chang)**.**

(a) Let $\mathbb{Z}$ be the constant sheaf on X, then $\mathbb{Z}_U$ is a subsheaf of $\mathbb{Z}$ since the map of stalks is obviuosly injective. Then the stalk $(\mathbb{Z}/\mathbb{Z}_U)_x$ is $\mathbb{Z}$ if $x = P, Q$, and is zero otherwise. Hence $\mathbb{Z}/\mathbb{Z}_U$ is the direct sum of skyscraper sheaves $\mathbb{Z}(P) \oplus \mathbb{Z}(Q)$. Then the short exact sequence

$$0 \to \mathbb{Z}_U \to \mathbb{Z} \to \mathbb{Z}(P) \oplus \mathbb{Z}(Q) \to 0$$

induces the long exact sequence

$$0 \to H^0(X, \mathbb{Z}_U) \to H^0(X, \mathbb{Z}) \to H^0(X, \mathbb{Z}(P) \oplus \mathbb{Z}(Q)) \to H^1(X, \mathbb{Z}_U)...$$

Since $H^0(X, \mathbb{Z}) = \mathbb{Z}$, $H^0(X, \mathbb{Z}(P) \oplus \mathbb{Z}(Q)) = \mathbb{Z}^2$, the map $H^0(X, \mathbb{Z}) \to H^0(X, \mathbb{Z}(P) \oplus \mathbb{Z}(Q))$ is not surjective. Hence $H^0(X, \mathbb{Z}(P) \oplus \mathbb{Z}(Q)) \to H^1(X, \mathbb{Z}_U)$ is non-zero. In particular, $H^1(X, \mathbb{Z}_U) \neq 0$.

(b) Write $\mathbb{Z}_Y := \mathbb{Z}/\mathbb{Z}_U$. Then we induction on $n = \dim X$. $n = 1$ is just (a). Now if the consequence holds for dim $< n$, then for dim $= n$, consider the exact sequence

$$0 \to \mathbb{Z}_U \to \mathbb{Z} \to \mathbb{Z}_Y \to 0$$

which will induces the long exact sequence

$$\cdots \to H^i(X,\mathbb{Z}) \to H^i(X,\mathbb{Z}_Y) \to H^{i+1}(X,\mathbb{Z}_U) \to H^{i+1}(X,\mathbb{Z}) \to \cdots$$

Then since $\mathbb{Z}$ is constant, hence flasque, so we have $H^i(X,\mathbb{Z}) = 0$ if $i > 0$, hence $H^i(X,\mathbb{Z}_Y) \cong H^{i+1}(X,\mathbb{Z}_U)$, so we can change to show that $H^{n-1}(X,\mathbb{Z}_Y) \neq 0$.

Now we write $Y := \cup_{i=0}^n H_i$ be the union of $n+1$ general hyperplanes. Define $V := Y - \cup_{i=1}^n H_i = H_0 - \cup_{i=1}^n (H_0 \cap H_i)$, then V is $\mathbb{P}^{n-1}$ removes n general hyperplane. Now we have the new exact sequence

$$0 \to \mathbb{Z}_{Y-V} \to \mathbb{Z}_Y \to \mathbb{Z}_V \to 0 \tag{1}$$

which will induces the long exact sequence

$$\cdots \to H^{n-1}(X,\mathbb{Z}_{Y-V}) \to H^{n-1}(X,\mathbb{Z}_Y) \to H^{n-1}(X,\mathbb{Z}_V) \to H^n(X,\mathbb{Z}_{Y-V}) \to \cdots$$

Then applying lemma 2.10 we have $H^{n-1}(X,\mathbb{Z}_V) = H^{n-1}(H_0,\mathbb{Z}_V) \neq 0$ by the induction hypothesis, and $H^n(X,\mathbb{Z}_{Y-V}) = H^n(Y,\mathbb{Z}_{Y-V}) = 0$ by Grothendieck vanishing. Hence we have

$$H^{n-1}(X,\mathbb{Z}_Y) \to H^{n-1}(X,\mathbb{Z}_V) \to 0 \tag{2}$$

is exact and so $H^n(X,\mathbb{Z}_U) = H^{n-1}(X,\mathbb{Z}_Y) \neq 0$.

Exercise 2 (by Zi-Li)**.**

$\mathbb{P}^1_k$ is irreducible, hence $\mathscr{K}$ is flasque. By exercise II.1.21(d), $\mathscr{K}/\mathscr{O} \simeq \sum i_p(I_p)$, hence it is also flasque. Morevoer, by exercise II.1.21(e), we have short exact sequence:

$$0 \to \Gamma(X,\mathscr{O}) \to \Gamma(X,\mathscr{K}) \to \Gamma(X,\mathscr{K}/\mathscr{O}) \to 0$$

Hence, $H^i(X,\mathscr{O}) = 0$ for all $i > 0$

Exercise 4 (by Pei-Hsuan Chang)**.**

Take an injective resolution: $0 \to \mathscr{F} \to \mathscr{I}^0 \to \mathscr{I}^1 \to \cdots$. Then for any i, we have a commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_{Y_1 \cap Y_2}(X, \mathscr{I}^i) & \longrightarrow & \Gamma_{Y_1}(X, \mathscr{I}^i) \oplus \Gamma_{Y_2}(X, \mathscr{I}^i) & \longrightarrow & \Gamma_{Y_1 \cup Y_2}(X, \mathscr{I}^i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(X, \mathscr{I}^i) & \longrightarrow & \Gamma(X, \mathscr{I}^i) \oplus \Gamma(X, \mathscr{I}^i) & \longrightarrow & \Gamma(X, \mathscr{I}^i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(X - Y_1 \cap Y_2, \mathscr{I}^i) & \longrightarrow & \Gamma(X - Y_1, \mathscr{I}^i) \oplus \Gamma(X - Y_2, \mathscr{I}^i) & \longrightarrow & \Gamma(X - Y_1 \cup Y_2, \mathscr{I}^i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Since $\mathscr{I}^i$ is flasque, the three columns are exact by Ex. 3.2.3(d). The second and the third row are clearly exact. By nine lemma, the first row is exact, and take cohomology long exact sequence, then get the desired result.

Exercise 5 (by Yu-Ting)**.**

First, we shall prove that $\Gamma_p(X, \mathscr{F}) \cong \Gamma_p(X_p, \mathscr{F}_p)$ for every sheaf $\mathscr{F}$ on X.

$$\Gamma_p(X, \mathscr{F}) = \{s \in \Gamma(X, \mathscr{F}) | s_Q = 0 \ \forall Q \neq P, \in X\}.$$

$$
\begin{aligned}
\Gamma_p(X_p, \mathscr{F}_p) &= \left\{ (U, s) \in \lim_{U \supseteq X_p} \mathscr{F}(U) \middle| S_Q = 0 \ \forall Q \neq P, \in X_p \right\} \\
&= \left\{ (U, s) \in \lim_{p \in U} \mathscr{F}(U) \middle| S_Q = 0 \ \forall Q \neq P, \in X_p \right\}.
\end{aligned}
$$

We have a natural map $\Gamma_p(X, \mathscr{F}) \xrightarrow{f} \Gamma_p(X_p, \mathscr{F}_p)$. Suppose $s \in \ker f$, there exists an open set $U \subseteq X$ and $p \in U$ such that $s|_U = 0$. Since $s \in \Gamma_p(X, \mathscr{F})$, for every $Q \neq P, \in X$, $s_Q = 0$. Then $s_Q = 0$ for every $Q \in X$. This implies $s = 0$ and f is injective.
For any $(U, t) \in \Gamma_p(X_p, \mathscr{F}_p)$, $V := U \mathrm{Supp}(t)$ is open. $W := X \setminus \{p\}$ is also an open set in X, then $\{V, W\}$ is an open cover of X. Define $s_1 = 0$ on W and $s_2 = t_V$ on V. $s_1|_{W \cap V} = s_2|_{W \cap V}$. By glueing s_1 and s_2, we get s such that $f(s) = t$, then f is surjective. We have proved that $\Gamma_p(X, \mathscr{F}) \cong \Gamma_p(X_p, \mathscr{F}_p)$.
Let $0 \to \mathscr{F} \to \mathscr{I}^\cdot$ be any injective resolution of $\mathscr{F}$, then $0 \to \mathscr{F} \to \mathscr{I}^\cdot_p$ is also an injective resolution of $\mathscr{F}_p$. Also for every i, $\Gamma_p(X, \mathscr{I}^i) \cong \Gamma_p(X_p, \mathscr{I}^i_p)$. Hence, $H^i_p(X, \mathscr{F}) = H^i_p(X_p, \mathscr{F}_p)$.

Exercise 6 (by Tzu-Yang Chou)**.**

In accordance with the hint, we procced in the following steps. First $\mathscr{I} \Rightarrow \forall$ open set $U \subseteq X$, subsheaf $\mathscr{R} \subseteq \mathbb{Z}_U$ and $f : \mathscr{R} \longrightarrow \mathscr{I}, \exists$ an extension $\mathbb{Z}_U \longrightarrow \mathscr{I}$: One direction is by definition. Conversely, given $0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G}$ and $\phi : \mathscr{F} \longrightarrow \mathscr{I}$, consider $\{\tilde{\phi} : \tilde{\mathscr{F}} \longrightarrow \mathscr{I} | \mathscr{F} \subseteq \tilde{\mathscr{F}} \subseteq \mathscr{G}$ and $\tilde{\phi}$ extends $\phi\}$ with the partial order given by extension. Zorn's lemma says that $\exists$ a maximal $\psi : \mathscr{F}' \longrightarrow \mathscr{F}$ and we claim that $\mathscr{F}' = \mathscr{G}$.
Assume the contrary, say $\exists U \subseteq X$ an open set with a section $s \in \mathscr{G}(U) \setminus \mathscr{F}'(U)$. This induces a monomorphism $\mathbb{Z}_U \longrightarrow \mathscr{G}$ mapping 1 to s. Now set $\mathscr{R} := \mathbb{Z}_U \cap \mathscr{F}$. By assumption we get an extension $\mathbb{Z}_U \longrightarrow \mathscr{I}$ of $\phi_{\mathscr{R}}$, but now ϕ can be extended to subsheaf of $\mathscr{G}$, generated by $\mathscr{F}'$ and $\mathbb{Z}_U$ (generated by s), which leads to a contradiction to the maximality.
Next, we claim that any subsheaf $\mathscr{R} \subseteq \mathbb{Z}_U$ is finitely generated. $\forall i$, we have $\mathscr{R}(U_i) \subseteq \mathbb{Z}_U(U_i) = \mathbb{Z}$, so we can find one generator x_i, and hence $\{x_1, \cdots, x_n\}$ generate $\mathscr{R}$. Also, $\forall \theta : \mathscr{R} \longrightarrow \varinjlim_{\alpha} \mathscr{I}_\alpha$, we choose some α' such that $\theta(x_i) = y_i$ in $\mathscr{I}_{\alpha'}(U_i)$, then we obtain that θ factors through $\mathscr{I}_{\alpha'}$.
Finally, to show that $\varinjlim_{\alpha} \mathscr{I}_\alpha$ is injective, by the first part, it suffices to prove that given an open set $U \in X$, a subsheaf $\mathscr{R} \subseteq \mathbb{Z}_U$ and $f : \mathscr{R} \longrightarrow \varinjlim_{\alpha} \mathscr{I}_\alpha$, there's an extension. But if we pick some β such that f factors through $\mathscr{I}_\beta$, we obtain an extension $\mathbb{Z} \longrightarrow \mathscr{I}_\beta$ since $\mathscr{I}_\beta$ is injective, so we get the desired extension by composing the natural map $\mathscr{I}_\beta \longrightarrow \varinjlim_{\alpha} \mathscr{I}_\alpha$.

3 Cohomology of a Noetherian Affine Scheme

Exercise 1 (by Pei-Hsuan Chang)**.**

(⇒) Let $X = \operatorname{Spec} A$. Then $X_{red} = \operatorname{Spec} A/\mathfrak{N}$, where $\mathfrak{N}$ is nilradical of A

(⇐) Let $\mathscr{N}$ be nilpotent element on X.
For any coherent sheaf $\mathscr{F}$, define $\mathscr{G}_i = \mathscr{N}^i\mathscr{F}/\mathscr{N}^{i+1}\mathscr{F}$, then $\mathscr{G}_i$ is a coherent $\mathscr{O}_x/\mathscr{N}$-mod. (Notice that $\mathscr{O}_x/\mathscr{N} = \mathscr{O}_{X_{red}}$.)
Thus, we have $H^j(X, \mathscr{G}_i) = H^j(X_{red}, \mathscr{G}_i)$, $\forall i, j$, since $sp(X) = sp(X_{red})$.
By theorem 3.3.5, $H^1(X_{red}, \mathscr{G}_i) = 0$, $\forall i$. ($\mathscr{G}_i$ is coherent sheaf of ideal.) Now, since for all i,

$$0 \longrightarrow \mathscr{N}^{i+1}\mathscr{F} \longrightarrow \mathscr{N}^i\mathscr{F} \longrightarrow \mathscr{G}_i \longrightarrow 0$$

is exact,

$$\cdots \longrightarrow H^1(X, \mathscr{N}^{i+1}\mathscr{F}) \longrightarrow H^1(X, \mathscr{N}^i\mathscr{F}) \longrightarrow H^1(X, \mathscr{G}_i) = 0 \longrightarrow \cdots$$

is exact. Thus, $H^1(X, \mathscr{N}^{i+1}\mathscr{F}) \longrightarrow H^1(X, \mathscr{N}^i\mathscr{F})$ is surjective, for all i.
But $\mathscr{N}^d = 0$, for some large $d \Rightarrow H^1(X, \mathscr{N}^d\mathscr{F}) = 0 \Rightarrow H^1(X, \mathscr{N}^{d-1}\mathscr{F}) = 0$.
Inductively, $H^1(X, \mathscr{F}) = 0$. By Theorem 3.3.7, X is affine.

Exercise 2 (by Yi-Tsung Wang)**.**

(⇒) Let $Y \subseteq X$ be an irreducible component, and let $X = \operatorname{Spec} A$. Since Y is a closed subscheme of X, we see that $Y = \operatorname{Spec} {}^{A}/_{I}$ for some $I \trianglelefteq A$. Thus Y is affine.
(⇐) Let $X = X_1 \cup \ldots \cup X_k$ with ideal sheaves $\mathscr{I}_1, \ldots, \mathscr{I}_k$. For any coherent sheaf of ideal $\mathscr{I}$, we have

$$\mathscr{I}_1 \ldots \mathscr{I}_k\mathscr{I} \subseteq \mathscr{I}_1 \ldots \mathscr{I}_{k-1}\mathscr{I} \subseteq \ldots \subseteq \mathscr{I}_1\mathscr{I} \subseteq \mathscr{I}$$

Note that $\mathscr{I}_1 \ldots \mathscr{I}_k = 0$ since X is reduced. Consider the exact sequence

$$0 \to \mathscr{I}_1 \ldots \mathscr{I}_r\mathscr{I} \to \mathscr{I}_1 \ldots \mathscr{I}_{r-1}\mathscr{I} \to {}^{\mathscr{I}_1 \ldots \mathscr{I}_{r-1}\mathscr{I}}/_{\mathscr{I}_1 \ldots \mathscr{I}_r\mathscr{I}} \to 0$$

and ${}^{\mathscr{I}_1 \ldots \mathscr{I}_{r-1}\mathscr{I}}/_{\mathscr{I}_1 \ldots \mathscr{I}_r\mathscr{I}}$ is a quasi-coherent sheaf and X_r is affine. By Serre criterion,

$$H^1\left(X, {}^{\mathscr{I}_1 \ldots \mathscr{I}_{r-1}\mathscr{I}}/_{\mathscr{I}_1 \ldots \mathscr{I}_r\mathscr{I}}\right) = H^1\left(X_r, {}^{\mathscr{I}_1 \ldots \mathscr{I}_{r-1}\mathscr{I}}/_{\mathscr{I}_1 \ldots \mathscr{I}_r\mathscr{I}}\right) = 0$$

Thus we have

$$H^1\left(X, \mathscr{I}_1 \ldots \mathscr{I}_r\mathscr{I}\right) \twoheadrightarrow H^1\left(X, \mathscr{I}_1 \ldots \mathscr{I}_{r-1}\mathscr{I}\right)$$

In particular we have

$$0 = H^1\left(X, \mathscr{I}_1 \ldots \mathscr{I}_k \mathscr{I}\right) \twoheadrightarrow H^1\left(X, \mathscr{I}\right)$$

which implies that $H^1\left(X \mathscr{I}\right) = 0$. By Serre criterion, X is affine.

Exercise 3 (by Wei-Ping)**.**

(a) Check directly by definition.

(b) By $\Gamma_Y(X, \widetilde{I}) = \Gamma_\alpha(I)$, and $I^\cdot$ injective then $\widetilde{I}^\cdot$ is flasque resolution, trivial.

(c) Every $H^i_\alpha(M)$ is a quotient of a group in $\Gamma_\alpha(I^\cdot)$ so clearly every thing is annihilate by multiplying high power of α.

Exercise 6 (by Shi-Xin)**.**

(a) Recall the notation $X = \bigcup_{i=1}^n U_i = \bigcup_{i=1}^n SpecA_i$ and $\mathcal{F}|_{U_i} = \tilde{M}_i$ for some A_i-mod M_i. Denote the map $U_i \to X$ by f_i and embed M_i into an injective A_i-mod I_i, and then we can define $G := \oplus f_{i*}(\tilde{I}_i)$. Note that we have a natural inclusion $\mathcal{F} \to \mathcal{G}$ defined by $\mathcal{F} \to f_{i*}$.

Now, for any $0 \to \mathcal{F}' \to \mathcal{F}''$ and a morhphism $\mathcal{F}' \to \mathcal{G}$ where $\mathcal{F}', \mathcal{F}''$ are quasi-coherent sheaves of modules, we can get for any i, a morphism $\mathcal{F}'|_{U_i} \to \tilde{I}_i$ which is corresponding to $\mathcal{F}' \to f_{i*}(\tilde{I}_i)$. Since I_i is injective, $\tilde{I}_i$ is injective, and hence for any i, it induces a morphism $\mathcal{F}''|_{U_i} \to \tilde{I}_i$. Thus we get a morphism $\mathcal{F}'' \to \mathcal{G}$ which must make the diagram commute.

(b) We use a generalization of exercise 3.7.

Proposition 1. *X is a noetherian scheme, and $\mathcal{F}$ is a quasi-coherent sheaf of modules on X. Let $\mathcal{I}$ be a quasi-coherent sheaf of ideals corresponding to a closed subscheme $Z = X - U$ for some open subset U. Then we have*

$$\varinjlim_n Hom_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) \cong \Gamma(U, \mathcal{F}).$$

Back to the problem, let $\mathscr{I}$ be an injective sheaf in the category of quasi-coherent sheaves of modules on X. We apply the proposition to $\mathcal{F} = \mathscr{I}$, then for any open subset U, we have a commutative diagram

$$\begin{array}{ccc}
\Gamma(U, \mathscr{I}) & \xrightarrow{\phi} & \varinjlim_n \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathscr{I}) \\
{\scriptstyle\alpha}\uparrow & & {\scriptstyle\beta}\uparrow \\
\Gamma(X, \mathscr{I}) & \xrightarrow{\psi} & \varinjlim_n \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^n, \mathscr{I})
\end{array}$$

where ϕ, ψ are isomorphism. Moreover, since $0 \to \mathcal{I} \to \mathcal{O}_X$ is exact, β is surjective, and hence α is surjective. Thus $\mathscr{I}$ is flasque.

(c) By (b), we know that any injectives is flasque and hence acyclic. Consequently, calculating the derived functors by taking injective resolutions gives the usual cohomology functors.

Exercise 7 (by Shuang-Yen)**.**

This is a sentence.

(a) Let $\mathfrak{a} = \langle f_1, \ldots, f_r \rangle$. Define $\alpha_n : \mathrm{Hom}_A(\mathfrak{a}^n, M) \to \Gamma(U, \widetilde{M})$ by $\varphi \mapsto s$ where

$$s(\mathfrak{p}) = \frac{\varphi(f^n)}{f^n} \in M_{\mathfrak{p}} \text{ for } f \in \mathfrak{a} \setminus \mathfrak{p}.$$

It's clearly well-defined. Since the maps α_n are compatible with each other, we have a map

$$\alpha : \varinjlim_n \mathrm{Hom}_A(\mathfrak{a}^n, M) \to \Gamma(U, \widetilde{M}).$$

To show that α is injective, suppose $\alpha(\varphi) = 0$ for some $\varphi \in \mathrm{Hom}_A(\mathfrak{a}^n, M)$. Then $\varphi(f_i^n)/f_i^n = 0$ in M_f, so we have $\varphi(f_i^{n+n_i}) = 0$ for some large n_i. Since we may take n_i to be the same, $\varphi(f_i^N) = 0$ for all i for some N, hence $\varphi(\mathfrak{a}^{rN}) = 0$, or $\varphi = 0$. To show that α is surjective, let $s \in \Gamma(U, \widetilde{M})$, then

$$s|_{D(f_i)} = \frac{b_i}{f_i^{n_i}} \text{ for some } b_i \in M, n_i \geq 0 \text{ and } \frac{b_i}{f_i^{n_i}} = \frac{b_j}{f_j^{n_j}} \text{ in } D(f_i f_j).$$

Similar to the proof of $\Gamma(D(f), \widetilde{M}) = M_f$, we can take N large enough such that we can define a map $\varphi : \mathfrak{a}^N \to M$ such that

$$f_1^{m_1} \cdots f_r^{m_r} \mapsto f_1^{m_1} \cdots f_i^{m_i - n_i} \cdots f_r^{m_r} b_i \text{ if } m_i \geq n_i,$$

this is well-defined by the condition above. Note that $\alpha_N(\varphi) = s$, so α is surjective.

(b) It suffices to show that $\Gamma(X, \widetilde{I}) \twoheadrightarrow \Gamma(U, \widetilde{I})$ for all $U = X \setminus Z(\mathfrak{a})$, then it's equivalent to show that $\mathrm{Hom}_A(A, I) = I \twoheadrightarrow \varinjlim \mathrm{Hom}_A(\mathfrak{a}^n, I)$. Note that $\mathrm{Hom}_A(A, I) \twoheadrightarrow \mathrm{Hom}_A(\mathfrak{a}^n, I)$ for all n since I is injective, so the map above is surjective. Hence $\widetilde{I}$ is flasque.

Exercise 8 (by Tzu-Yang Tsai)**.**

Suppose $\theta : I \to I_f, z \mapsto \frac{1}{x_0}$, then $\frac{z}{1} = \frac{1}{x_0} \Rightarrow x_0^k(x_0 z - 1) = 0$ for some $k \in \mathbb{N}_0$. But this leads to $x_0^{k+1} z = x_0^k \Rightarrow x_0^{k+1} z a_{k+1} = 0 = x_0^k a_{k+1} \neq 0 \rightarrow\!\leftarrow$
Thus $\frac{1}{x_0}$ is not in the range, therefore θ is not surjective.

4 Čech Cohomology

Exercise 1 (by Zi-Li)**.**

Note that $f_*\mathscr{F}$ is quasi coherent on Y, we may compute cohomology via Čech complex. Take an affine cover $\mathscr{U} = \{U_i\}$ of Y, then $f^{-1}\mathscr{U} = \{f^{-1}(U_i)\}$ is an affine cover of X. We see that $C^p(f^{-1}\mathscr{U}, \mathscr{F}) = C^p(\mathscr{U}, f_*\mathscr{F})$, hence, $H^i(X, \mathscr{F}) \simeq H^i(Y, f_*\mathscr{F})$

Exercise 3 (by Yi-Heng)**.**

Take the open cover $U = D(x) \cup D(y)$,then $0 \longrightarrow C^0 \xrightarrow{d} C^1 \longrightarrow 0$ where $d : C^0 = k[x,y]_x \times k[x,y]_y \to C^1 = k[x,y,x^{-1},y^{-1}], (f,g) \mapsto f-g$. Thus,$H^1(U, \mathscr{O}_U)$ is generated by $\{x^i y^j | i,j \in \mathbb{Z}_{<0}\}$ as a k-vector space

Exercise 4 (by Yu-Ting)**.**

(a) For every p, $C^p(\mathfrak{U}, \mathscr{F}) \xrightarrow{\tilde{\lambda}^p} C^p(\mathfrak{B}, \mathscr{F})$ can be induced from $\mathscr{F}(U_{\lambda(j)}) \to \mathscr{F}(V_j)$. For $\alpha \in C^p(\mathfrak{U}, \mathscr{F})$, p, and $j_0 < j_1 < \cdots < j_{p+1}$,

$$\begin{aligned}(\tilde{\lambda}(d\alpha))_{j_0 j_1 \cdots j_{p+1}} &= (d\alpha)_{\lambda(j_0)\lambda(j_1)\cdots\lambda(j_{p+1})}\big|_{V_{j_0 j_1 \cdots j_{p+1}}} \\ &= \sum_{k=0}^{p+1} (-1)^k \alpha_{\lambda(j_0)\cdots\lambda(\hat{j}_k)\cdots\lambda(j_{p+1})}\big|_{V_{j_0 j_1 \cdots j_{p+1}}} \\ &= \sum_{k=0}^{p+1} (-1)^k (\tilde{\lambda}\alpha)_{j_0 \cdots \hat{j}_k \cdots j_{p+1}}\big|_{V_{j_0 j_1 \cdots j_{p+1}}} = (d(\tilde{\lambda}\alpha))_{j_0 j_1 \cdots j_{p+1}}\end{aligned}$$

The map $\tilde{\lambda}$ and d commute. Then we have the map $\check{H}^p(\mathfrak{U}, \mathscr{F}) \to \check{H}^p(\mathfrak{B}, \mathscr{F})$.

(b) Let $0 \to \mathscr{F} \to \mathscr{I}^\cdot$ be an injective resolution. Then there is a map $C^\cdot(\mathfrak{U}, \mathscr{F}) \xrightarrow{f_\mathfrak{U}}$, which is unique up to homotopy. Also, we have $C^\cdot(\mathfrak{B}, \mathscr{F}) \xrightarrow{f_\mathfrak{B}} \mathscr{I}^\cdot$. Then $\lambda \circ f_\mathfrak{B}$ is homotopic to $f_\mathfrak{U}$ and the map is compatible.

(c) Embed $\mathscr{F}$ in a flasque sheaf $\mathscr{G}$, and let $\mathscr{R} = \mathscr{G}/\mathscr{F}$. Then $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{R} \to 0$ is exact. Define a complex $D^\cdot(\mathfrak{U})$ such that $0 \to C^\cdot(\mathfrak{U}, \mathscr{F}) \to C^\cdot(\mathfrak{U}, \mathscr{G}) \to D^\cdot(\mathfrak{U})$ is exact. We have the following long exact sequences:

$$\cdots \to \Gamma(X, \mathscr{G}) = \check{H}^0(\mathfrak{U}, \mathscr{G}) \to H^0(D^\cdot(\mathfrak{U})) \to \check{H}^1(\mathfrak{U}, \mathscr{F}) \to \check{H}^1(\mathfrak{U}, \mathscr{G}) = 0$$

$$\cdots \to \Gamma(X, \mathscr{G}) \to \Gamma(X, \mathscr{R}) \to H^1(X, \mathscr{F}) \to H^1(X, \mathscr{G}) = 0$$

After taking the direct limit of the first sequence, we have

$$\cdots \to \Gamma(X, \mathscr{G}) \to \varinjlim H^0(D^\cdot(\mathfrak{U})) \to \varinjlim \check{H}^1(\mathfrak{U}, \mathscr{F}) \to 0.$$

To show $\lim_{\rightarrow} \check{H}^1(\mathfrak{U}, \mathscr{F}) \cong H^1(X, \mathscr{F})$, it suffices to show $\lim\limits_{\rightarrow} H^0(D^\cdot(\mathfrak{U})) \cong \Gamma(X, \mathscr{R})$. But by the definition of $D^\cdot(\mathfrak{U})$, injectivity is clear. For $\alpha \in \Gamma(X, \mathscr{R})$, by the surjectivity of stalks, there exists a refinement $\mathfrak{B}$ such that α lies in the image of $C^0(\mathfrak{B}, \mathscr{G})$.

Exercise 6 (by Tzu-Yang Chou).

At $P \in X$, we have a short exact sequence $0 \longrightarrow \mathscr{I}_P \longrightarrow \mathscr{O}^*_{X,P} \longrightarrow (\mathscr{O}_{X,P}/\mathscr{I}_P)^* \longrightarrow 0$, which gives the sequence $0 \longrightarrow \mathscr{I} \longrightarrow \mathscr{O}^*_X \longrightarrow \mathscr{O}^*_{X_0} \longrightarrow 0$ since $(\mathscr{I}_P)^2 = (\mathscr{I}^2)_P = 0$. Taking the long exact sequence of cohomology and use Ex(III.4.5) we proves the assertion.

Exercise 7 (by Wei-Ping).

Two affines are $k[\frac{x_0}{x_2}, \frac{x_1}{x_2}]/(f), k[\frac{x_0}{x_1}, \frac{x_2}{x_1}]/(f)$, Consider Čech complex $0 \to \Gamma(U, O_X) \oplus \Gamma(V, O_X) \to \Gamma(U \cap V, O_X) \to 0$. $H^0(X, O_X)$ is pair of $(\frac{g}{x_1^n}, \frac{h}{x_2^m})$ such that $fq = x_2^m g - x_1^n h$, where f, q, g, h are polynomials. The statement $f(1,0,0) \neq 0$ says that f has pure degree of x_0, then q can't have term with pure zero degree since $x_2^m g - x_1^n h$ doesn't.
Now we want to write $fq = x_2^m g' - x_1^n h'$ then comparing to original equation we can substitute fq with 0 then conclude that $\frac{g}{x_1^n} = \frac{h}{x_2^m} = c$ for some constant c. Observe that every term in $x_2^m g - x_1^n h$ at least have degree m of x_2 or degree n of x_1, so when we consider highest degree of x_0 we see that every term in q also have this property, hence we can write $fq = x_2^m g' - x_1^n h'$ for some g', h', as we desired. Therefore $H^0(X, O_X) = k$ and dimension is 1. We can also use the fact that this group is global section so it is constant k.
$H^1(X, O_X)$ is calculated by quotient all elements in form $\frac{g}{x_1^n}, \frac{h}{x_2^m}$ from degree zero part of $k[x_0, x_1, x_2]_{(x_1, x_2)}$. Since every thing is written uniquely into k-linear combination of $\frac{x_0^{i+j}}{x_1^i x_2^j}$, and we can reduce if $i + j \geqslant d = \deg f$, so the basis of $H^1(X, O_X)$ is those with $0 \leqslant i + j < d$, which has $\frac{1}{2}(d-1)(d-2)$. The reduction of degree comes from $f = x_0^{\deg f} + f'$, where f' consists of term with at least one of degree of x_1 or x_2 isn't zero.

Exercise 8 (by Yi-Tsung Wang).

(a) By exercise 2.5.15(e), we may write $\mathscr{F} = \bigcup\limits_{\alpha} \mathscr{F}_\alpha$, where $\mathscr{F}_\alpha$ can be taken to be all coherent subsheaves, hence $\{\mathscr{F}_\alpha\}_\alpha$ is a direct system and $\mathscr{F} = \varinjlim\limits_{\alpha} \mathscr{F}_\alpha$. By proposition 3.2.9, we have $\varinjlim\limits_{\alpha} H^i(X, \mathscr{F}_\alpha) = H^i(X, \mathscr{F})$. If $H^r(X, \mathscr{G}) = 0$ for any coherent sheaf $\mathscr{G}$, then $H^r(X, \mathscr{F}) = 0$ in this case. Therefore it is sufficient to consider only coherent sheaves on X.

(b) Take a locally free coherent sheaf $\mathscr{E}$ such that $\mathscr{E} \to \mathscr{F} \to 0$ and take $\mathscr{R}$ be the kernel. If $H^{r'}(X, \mathscr{E}') = 0$ for any locally free coherent sheaf $\mathscr{E}'$ and any $r' \geq r$ for some r, then $H^{r'+1}(X, \mathscr{R}) = H^{r'}(X, \mathscr{F})$ for any $r' \geq r$. Now since $\mathscr{R}$ is also coherent and $H^{\dim X+1}(X, \mathscr{R}) = 0$, by the induction we see that $H^{r'}(X, \mathscr{F}) = 0$ for any $r' \geq r$. Therefore it is even sufficient to consider only locally free coherernt sheaves on X.

(c) Let $\mathscr{U} = \{U_0, \ldots, U_r\}$ be the open affine cover, then for any quasi-coherent sheaf $\mathscr{F}$ and $n \geq r+1$, since $C^n(\mathscr{U}, \mathscr{F}) = 0$, we have $\check{H}^n(\mathscr{U}, \mathscr{F}) = 0$. By theorem 3.4.5, for any $n \geq r+1$, $H^n(X, \mathscr{F}) = 0$. Thus $\operatorname{cd}(X) \leq r$.

(d)

(e) Write $Y = \bigcap_{i=1}^{r} H_i$ with H_i hypersurfaces, let $U = X \backslash Y = \bigcup_{i=1}^{r} X \backslash H_i$, where $X \backslash H_i$ is affine. By part (d) we see that $\operatorname{cd}(X) \leq r-1$.

5 The Cohohomology of Projective Spaces

Exercise 1 (by Zi-Li).

We have long exact sequence

$$\ldots \to H^i(X, \mathscr{F}') \to H^i(X, \mathscr{F}) \to H^i(X, \mathscr{F}'') \to \ldots$$

Note that cohomology groups vanish for large i, and taking dimension as k-vector space is additive, we have $\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'')$

Exercise 2 (by Chi-Kang Chang).

(a) For a fixed closed immersion $i : X \hookrightarrow \mathbb{P}^n$, we have i is a closed immersion, hence an affine morphism of Noetherian separated schemes, so by exercise 4.1 we have $H^i(X, \mathscr{F}) \cong H^i(\mathbb{P}^N, i_*\mathscr{F})$, and $\mathscr{O}_X(1) = i^*\mathscr{O}_{\mathbb{P}^N}(1)$. Hence $H^i(X, \mathscr{F}(n)) \cong H^i(\mathbb{P}^N, (i_*\mathscr{F})(n)$, so replace $\mathscr{F}$ by $i_*\mathscr{F}$, we can assume $X = \mathbb{P}^N$.

Now we induction on N, when $N = 0$, $\chi(\mathscr{F}(n))$ is a constant, so there is nothing to do. Suppose the consequence holds for dimension less than N, then for $\mathbb{P}^N$, we consider the map $\varphi : \mathscr{F}(-1) \to \mathscr{F}$ obtained by multiple with a general linear polynomial f, and the induced exact sequence

$$0 \to \ker\varphi \to \mathscr{F}(-1) \to \mathscr{F} \to \text{coker}\varphi \to 0.$$

This induces the two exact sequences

$$0 \to \ker\varphi \to \mathscr{F}(-1) \to \text{im}\varphi \to 0$$
$$0 \to \text{im}\varphi \to \mathscr{F} \to \text{coker}\varphi \to 0$$

So by exercise 5.1 we have

$$\chi(\mathscr{F}(n)) = \chi(\text{coker}\varphi(n)) + \chi(\text{im}\varphi(n)) = \chi(\mathscr{F}(n-1)) + \chi(\text{coker}\varphi(n)) - \chi(\ker\varphi(n)).$$

Then since both cokerφ and kerφ is supported on $V(f) \cong \mathbb{P}^{N-1}$, by induction hypothesis both $\chi(\text{coker}\varphi(n))$ and $\chi(\ker\varphi(n))$ are numerical polynomials, so $\chi(\mathscr{F}(n)) - \chi(\mathscr{F}(n-1)) = \chi(\text{coker}\varphi(n)) - \chi(\ker\varphi(n))$ is also numerical polynomial, hence by I.7.3 $(\chi(\mathscr{F}(n))$ is a numerical polynomial.

(b) Set $M = \Gamma_*\mathscr{F} = \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathscr{F}(n))$, then in I.7 we define $\varphi_M(n) := \dim_k H^0(X, \mathscr{F}(n))$. Then by Serre's vanishing, for $n >> 0$ we have

$$\chi(\mathscr{F}(n)) - \dim_k H^0(X, \mathscr{F}(n)) = \dim_k \Gamma(X, \mathscr{F}(n)) = \varphi_M(n) = P_M(n)$$

Hence $\chi(\mathscr{F}(n)) = P_M(n))$.

Exercise 3 (by Chi-Kang Chang).

(a) Since X is integral, we have X is a projective variety, hence $H^0(X, \mathscr{O}_X) \cong \Gamma(X, \mathscr{O}_X) = k$. Hence we have

$$\begin{aligned} p_a(X) &= (-1)^r(\chi(\mathscr{O}_X) - 1) \\ &= (-1)^r(\sum_{i=1}^{r}(-1)^i H^i(X, \mathscr{O}_X)) \\ &= \sum_{i=1}^{r}(-1)^{r+i} H^i(X, \mathscr{O}_X) \\ &= \sum_{i=0}^{r-1}(-1)^i H^{r-i}(X, \mathscr{O}_X). \end{aligned}$$

(b) When X is a closed subvariety of $\mathbb{P}_k^r$, we have $\chi(\mathscr{O}_X) = P_M(0)$ with $M := \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathscr{O}_X(n))$ by exercise 5.2. And using the definition of I.7, we have$P_X(n) = \dim_k S(X)_n$. Since $\mathscr{O}_X = \mathscr{O}_{\mathbb{P}_k^r}/\mathscr{I}_X$, $S(X) = k[x_0, ..., x_r]/I(X)$, and $\mathscr{O}_X(n) = S(X)(n)^{\sim}$. We have $\Gamma(X, \mathscr{O}_X(n)) = S(X)_n$, and so $P_X(n) = P_M(n)$ for $n >> 0$, therefore $P_X = P_M$, and hence $p_a(X)$(in chapter III)$=(-1)^r(\chi(\mathscr{O}_X)-1) = (-1)^r(P_M(0) - 1) = (-1)^r(P_X(0) - 1) = p_a(X)$(in chapter I).

(c) Let $f : X \dashrightarrow X'$ be a birational map between nonsingular projective curves. Then since a birational map between non-singular projective varieties has indeterminacy codimension not less than 2, f is in fact a morphism. And similarly the birational map f^{-1} can be extend to a morphism. So in fact f is an isomorphism, hence $X \cong X'$ and then $p_a(X) = p_a(X')$.

In particular $p_a(\mathbb{P}^1) = 0$ by theorem III.5.1. And let X be a non-singular plane curve of degree ≥ 3, then we have $p_a(X) = \frac{1}{2}(d-1)(d-2) \geq 1$, so X is not rational.

Exercise 7 (by Shi-Xin).

(a) For any coherent $\mathcal{O}_Y$-modules $\mathcal{F}$, we have

$$H^i(Y, \mathcal{F} \otimes (i^*\mathcal{L})^n) = H^i(X, i_*(\mathcal{F} \otimes (i^*\mathcal{L})^n)) = H^i(X, i_*\mathcal{F} \otimes \mathcal{L}^n))$$

Then since $i_*\mathcal{F}$ is a coherent $\mathcal{O}_X$-modules, the ampleness of $\mathcal{L}$ implies the ampleness of $i^*\mathcal{L}$.

(b) ($\Rightarrow$) Since X_{red} is a closed subscheme of X, it follows from (a).

($\Leftarrow$) Let N be the sheaf of nilradical ideals of $\mathcal{O}_X$. Then the Noetherian condition implies $N^r = 0$ for some r. Now, for any coherent $\mathcal{O}_Y$-modules $\mathcal{F}$, consider the filtration

$$\mathcal{F} \supset N\mathcal{F} \supset \ldots \supset N^r\mathcal{F} = 0$$

Therefore, for each $i = 1, \ldots, r$ and any n, we have

$$0 \to N^i\mathcal{F} \otimes \mathcal{L}^n \to N^{i-1}\mathcal{F} \otimes \mathcal{L}^n \to (N^i\mathcal{F}/N^{i-1}\mathcal{F}) \otimes \mathcal{L}^n \to 0$$

We might the long exact sequences induced by the above short exact sequences. Moreover, for any i, $N^i\mathcal{F}/N^{i-1}\mathcal{F}$ is a coherent $\mathcal{O}_{X_{red}}$-module where allow us to apply the ampleness of $\mathcal{L}_{red}$ such that the cohomology group vanishes. Thus by the descending induction on i and $N^r\mathcal{F} = 0$, we can deduce that $H^p(X, i_*\mathcal{F} \otimes \mathcal{L}^n)) = 0$ for any $p > 0, n >> 0$.

(c) ($\Rightarrow$) Since each irreducible component is obviously a closed subscheme of X, it just follows from (a).

($\Leftarrow$) By (b), we might assume X is reduced. Write $X = \bigcup_{i=1}^r X_i$ with the decomposition of irreducible components, and let $\mathcal{I}_i$ be the corresponding sheaf of ideals of X_i. Consider the induction on r. Clearly, $r = 1$ is a trivial case. Now, if $r > 1$, since we

$$0 \to \mathcal{I}_1\mathcal{F} \to \mathcal{F} \to \mathcal{F}/\mathcal{I}_1\mathcal{F} \to 0$$

Moreover, $\mathcal{F}/\mathcal{I}_1\mathcal{F}$ is a coherent $\mathcal{O}_{X_1}$-module, so for $p > 0, n >> 0$, we have

$$H^p(X, ((\mathcal{F}/\mathcal{I}_1\mathcal{F})) \otimes \mathcal{L}^n)) = H^p(X_1, ((\mathcal{F}/\mathcal{I}_1\mathcal{F})) \otimes (\mathcal{L} \otimes \mathcal{O}_{X_1})^n)) = 0$$

Note that $Supp(\mathcal{I}_1\mathcal{F}) \subset \bigcup_{i=2}^r X_i$. Thus by the induction hypothesis, $H^p(X, \mathcal{I}_1\mathcal{F} \otimes \mathcal{L}^n) = 0$, and hence by the long exact sequences induced by the above short exact sequences, we deduce that $H^p(X, i_*\mathcal{F} \otimes \mathcal{L}^n)) = 0$ for any $p > 0, n >> 0$.

(d) ($\Rightarrow$) We only need finiteness, which preserves the coherent sheaves via pushforward. Then by the same argument in (a), one can prove the result.

($\Leftarrow$) By (b),(c), we might assume X, Y are integral. Let $\mathcal{F}$ be a coherent $\mathcal{O}_Y$-module. By exercise 4.2(a)(b), there is a coherent $\mathcal{O}_X$-module $\mathcal{G}$ and a morphism $\beta : f_*\mathcal{G} \to \mathcal{F}^{\oplus m}$ for some m such that β is an isomorphism at the generic point of Y. Then define $K := ker(\beta), C := coker(\beta)$ and consider the noetherian induction on $Supp(\mathcal{F})$. We have two short exact sequence

$$0 \to K \to f_*\mathcal{G} \to Im\beta \to 0$$
$$0 \to Im\beta \to \mathcal{F}^{\oplus m} \to C \to 0$$

Then $Supp(K) \subsetneq Supp(\mathcal{F}), Supp(C) \subsetneq Supp(\mathcal{F})$. Furthermore, since K, C are both coherent $\mathcal{O}_Y$-modules, by the induction hypothesis, we have $H^p(Y, K \otimes \mathcal{L}^n) = 0, H^p(Y, C \otimes \mathcal{L}^n) = 0$ for $p > 0, n >> 0$. Thus by the exact sequences and projection formula, we conclude that

$$H^p(Y, \mathcal{F} \otimes \mathcal{L}^n)^{\oplus m} = H^p(Y, f_*\mathcal{G} \otimes \mathcal{L}^n) = H^p(X, \mathcal{G} \otimes (f^*\mathcal{L})^n) = 0$$

for $p > 0, n >> 0$.

Exercise 10 (by Pei-Hsuan Chang).

Let $\mathscr{F}^1 \xrightarrow{f_1} \mathscr{F}^2 \xrightarrow{f_2} \mathscr{F}^3 \xrightarrow{f_3} \cdots \xrightarrow{f_{r-1}} \mathscr{F}^r$ be exact seqence of coherent sheaves on X. Since X is noetherian, $\ker f_i$ is also coherent, $\forall i \in \{1, \dots r-1\}$. Apply Theorem 3.5.2(b) to $\ker f_i$, then $\exists n_i$ such that $\forall n \geq n_i$, $H^1(X, (\ker f_k)(n)) = 0$. Since $0 \longrightarrow \ker f_i \longrightarrow \mathscr{F}^i \longrightarrow \operatorname{Im} f_i \longrightarrow 0$ is exact, for all i. Take cohomology long exact sequence, then get

$$0 \longrightarrow \Gamma(X, (\ker f_i)(n)) \longrightarrow \Gamma(X, \mathscr{F}^i(n)) \longrightarrow \Gamma(X, (\operatorname{Im} f_i)(n)) = \Gamma(X, (\ker f_{i+1}(n)) \longrightarrow 0\ ,$$

if $n \geq n_i$. Now, let $N = \max_i n_i$, then for all $n \geq N$,

$$\Gamma(X, \mathscr{F}^1(n)) \longrightarrow \Gamma(X, \mathscr{F}^2(n)) \longrightarrow \cdots \longrightarrow \Gamma(X, \mathscr{F}^r(n))$$

is exact.

6 Ext Groups and Sheaves

Exercise 3 (by Zi-Li)**.**

(b) Let $U = SpecA \subset X$, then $\mathscr{E}xt^i_X(\mathscr{F}, \mathscr{G})|_U \simeq \mathscr{E}xt^i_U(\mathscr{F}|_U, \mathscr{G}|_U)$, hence we may assume $X = SpecA, \mathscr{F} = \tilde{M}, \mathscr{G} = \tilde{N}$, where M is finitely generated. By exercise 6.7, $\mathscr{E}xt^i_X(\tilde{M}, \tilde{N}) \simeq Ext^i_A(M, N)\tilde{}$, hence, $\mathscr{E}xt^i_X(\mathscr{F}, \mathscr{G})$ is quasi-coherent.

(a) By 6.3(b), we remain to show that $Ext^i_A(M, N)$ is finitely generated A module when A is noetherian and M, N are finitely generated A module.
Take free projective resolution of M:

$$... \to A^{k_1} \to A^{k_0} \to M \to 0$$

Then, we have

$$0 \to Hom_A(A^{k_0}, N) \to Hom_A(A^{k_1}, N) \to ...$$

Note that $Hom_A(A^k, N) \simeq N^k$, hence $Ext^i_A(M, N)$ is finitely generated.

Exercise 4 (by Chun-Yi)**.**

Since we suppose $\mathscr{C}oh(X)$ has enough locally frees, $\forall \mathscr{F} \in \mathscr{C}oh(X)$, $\mathscr{F} = \mathscr{F}'/\mathscr{F}''$ for some locally free $\mathscr{F}'$. Consider the quotient map $u : \mathscr{F}' \longrightarrow \mathscr{F} = \mathscr{F}'/\mathscr{F}''$, then $F(u) : \mathscr{E}xt^i(\mathscr{F}', \mathscr{G}) \longrightarrow \mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ is zero map, since $\mathscr{E}xt^i_X(\mathscr{F}', \mathscr{G}) \mid_U = \mathscr{E}xt^i_U(\mathscr{F}' \mid_U, \mathscr{G} \mid_U) = \mathscr{E}xt^i_U(\mathscr{O}^n_x, \mathscr{G} \mid_U) = 0 \Rightarrow \mathscr{E}xt^i(\cdot, \mathscr{G})$ is coerasable contravariant δ-functor, hence universal.

Exercise 5 (by Wei-Ping)**.**

This is similar to the version of Ext.

(a) $0 \to \mathscr{F} \to \mathscr{F} \to 0$ so one direction is easy. Conversely, we get $\mathrm{Ext}^1(\mathscr{F}_x, \mathscr{G}_x) = 0$ for arbitrary $\mathscr{G}_x$, so $\mathscr{F}_x$ is projective over a local ring so is free. This holds for all $x \in X$, so $\mathscr{F}$ is locally free.

(b) By definition just take the resolution with length less or equal to n, then one direction is easy. For the converse direction, use the induction by "cutting" technique, base case is just (a). Given a resolution $\cdots \to \mathscr{L}_2 \to \mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F} \to 0$, cut down to $\cdots \to \mathscr{L}_2 \to \mathscr{L}_1 \to \mathrm{Im}\, \mathscr{L}_1 \to 0$ to get $\mathrm{hd}(\mathrm{Im}\, \mathscr{L}_1) \leqslant n-1$, and by long exact sequence from $0 \to \mathrm{Im}\, \mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F} \to 0$ we have $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) \simeq \mathscr{E}xt^{i-1}(\mathrm{Im}\, \mathscr{L}_1, \mathscr{G})$ for all $i \geqslant 2$, and the latter is zero by induction when $i > n$.

(c) Taking stalk gives resolution again so we have "$\geqslant$". Let $n = \sup_x \mathrm{pd}_{O_x} \mathscr{F}_x$, then stalk is zero when $i > n$, by(b) we conclude "$\leqslant$".

Exercise 6 (by Jung-Tao).

(a) If M is projective, we are done.

Following the hint, we want to prove the claim $\text{Ext}^i(M, N) = 0$ for all $i > 0$ and all finitely generated A module N by applying descending induction on i. (6.11.A) implies that we do have an induction basis.

$\text{Ext}^i(M, A) = 0 \implies \text{Ext}^i(M, A^n) = 0$, and $\forall$ finitely generated A module N, there is a short exact sequence

$$0 \to K \to A^n \to N \to 0$$

For some finitely generated module K. The long exact sequence induced by above shows that there is an isomorphism $\text{Ext}^i(M, N) \cong \text{Ext}^{i+1}(M, K) = 0$, so we have proved the claim.

Because M is also finitely generated, there is a short exact sequence

$$0 \to K \to A^n \to M \to 0$$

, $\text{Ext}^1(M, K) = 0$ implies the exact sequence above splits, M is a direct summand and is projective.

(b) Followed by induction. (cf. exercise 3.6.5)

Exercise 7 (by Pei-Hsuan Chang).

Take a free resolution of M, $A^{n_\bullet} \to M \to 0$. Then we get a locally free resolution of $\tilde{M}$, $\mathscr{O}_X^{n_\bullet} \to \tilde{M} \to 0$. Now,

$\text{Ext}^i_X(\tilde{M}, \tilde{N}) = h^i(\text{Hom}_X(\mathscr{O}_X^{n_\bullet}, \tilde{N})) = h^i(\text{Hom}_A(A^{n_\bullet}, N))$ (By Ex. 2.5.3) $= \text{Ext}^i_A(M, N)$.

Also,

$\mathscr{E}^i(\tilde{M}, \tilde{N}) = h^i(\mathscr{H}(\mathscr{O}_X^{n_\bullet}, \tilde{N})) = h^i((\text{Hom}(A^{n_\bullet}, N))^\sim) = (h^i(\text{Hom}(A^{n_\bullet}, N)))^\sim = (\text{Ext}^i_A(M, N))^\sim$.

For the third equality, we can check on every stalk: for any $\mathfrak{p} \in \text{Spec}\, X$,

$$\begin{aligned} h^i((\text{Hom}(A^{n_\bullet}, N))^\sim)_\mathfrak{p} = h^i(((\text{Hom}(A^{n_\bullet}, N))^\sim)_\mathfrak{p}) &= h^i(\text{Hom}(A^{n_\bullet}, N)_\mathfrak{p}) \\ &= h^i(\text{Hom}(A^{n_\bullet}, N))_\mathfrak{p} = ((h^i(\text{Hom}(A^{n_\bullet}, N)))^\sim)_\mathfrak{p} \end{aligned}$$

Exercise 9 (by Tzu-Yang Chou).

(a) Recall that regular local rings are UFDs. Thus Ex(III.6.8) says that the category $Coh(X)$ has enough locally free and hence $\exists$ locally free resolution of $\mathscr{F}$. On the other hand, since $\mathscr{O}_{X,x}$ is regular, by (III.6.11A) we have $\text{pd}\mathscr{F}_x \leq \dim \mathscr{O}_{X,x}, \forall x \in X \Rightarrow \text{hd}\mathscr{F} = \sup_x(\text{pd}\mathscr{F}_x) \leq \dim \mathscr{O}_{X,x}$ by Ex(III.6.5) and hence the homological dimension of $\mathscr{F}$ is finite.

(b) Similar to Ex(II.6.11). To show the independence of the choice of resolution, we only need to consider the case that one resolution surjects to another. Then we just need to look at the kernels and note that for an exact sequence $0 \longrightarrow \mathscr{L}_n \longrightarrow \cdots \mathscr{L}_1 \longrightarrow 0$, we have that $\sum_i (-1)^i[\mathscr{L}_i] = 0$ in $K(X)$. $(*)$

δ is a group homomorphism: for $0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$, we first choose two locally free resolutions for $\mathscr{F}'$ and $\mathscr{F}''$ respectively. Then the horseshoe lemma gives a locally free resolution for $\mathscr{F}$.
ϵ and δ are mutually inverse: $\epsilon \circ \delta = 1$ since we have $(*)$; $\delta \circ \epsilon = 1$ since for any locally free sheaf $\mathscr{E}$, $0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{E} \longrightarrow 0$ is a locally free resolution.

Exercise 10 (by Shuang-Yen)**.**

(a) Let $U = \mathrm{Spec} A \subseteq Y$ be open affine, then

$$\mathscr{H}om_Y(f_*\mathcal{O}_X, \mathscr{G})|_U = \mathscr{H}om_U(f_*\mathcal{O}_X|_U, \mathscr{G}|U) = \mathscr{H}om_U((f|_V)_*\mathcal{O}_V, \mathscr{G}|_U),$$

where $V = f^{-1}(U) = \mathrm{Spec} B$ is also affine. Let $\varphi : A \to B$ be the map defined by $f|_V$ and let $\mathscr{G}|U = \widetilde{M}$, then

$$\mathscr{H}om_U((f|_V)_*\mathcal{O}_V, \mathscr{G}|_U) = \mathscr{H}om_U(\widetilde{{}_BA}, \widetilde{M}) = \mathrm{Hom}_A({}_AB, M)^\sim$$

by (Ex 6.7) since B is finitely generated A-module. So $\mathscr{H}om_Y(f_*\mathcal{O}_X, \mathscr{G})$ is a quasi-coherent $f_*\mathcal{O}_X$-module since $\mathrm{Hom}_A({}_AB, M)$ is a B-module.

(b) Note that

$$f_*\mathscr{H}om_X(\mathscr{F}, f^!\mathscr{G}) = \mathscr{H}om_{f_*\mathcal{O}_X}(f_*\mathscr{F}, \mathscr{H}om_Y(f_*\mathcal{O}_X, \mathscr{G})) = \mathscr{H}om_Y(f_*\mathscr{F}, \mathscr{G}).$$

(c) Let $0 \to f^!\mathscr{G} \to \mathscr{I}^\cdot$ be an injective resolution, then

$$\begin{aligned}\mathrm{Ext}^i_X(\mathscr{F}, f^!\mathscr{G}) &= h^i(\mathrm{Hom}_X(\mathscr{F}, \mathscr{I}^\cdot)) \cong h(\mathrm{Hom}_{f_*\mathcal{O}_X}(f_*\mathscr{F}, f_*\mathscr{I}^\cdot)) \\ &\to h^i(\mathrm{Hom}_Y(f_*\mathscr{F}, f_*\mathscr{I}^\cdot)) \to h^i(\mathrm{Hom}_Y(f_*\mathscr{F}, \mathscr{J}^\cdot)).\end{aligned}$$

The last map is by the chain map $f_*\mathscr{I}^\cdot \to \mathscr{J}^\cdot$ lifting from $f_*f^!\mathscr{G} \xrightarrow{\mathrm{id}} f_*f^!\mathscr{G}$. So we have the map $\mathrm{Ext}^i_X(\mathscr{F}, f^!\mathscr{G}) \to \mathrm{Ext}^i_Y(f_*\mathscr{F}, f_*f^!\mathscr{G})$. Note that we have the map $f_*f^!\mathscr{G} = \mathscr{H}om_Y(f_*\mathcal{O}_X, \mathscr{G}) \xrightarrow{(f^\#)^*} \mathscr{H}om_X(f_*\mathcal{O}_X, \mathscr{G}) = \mathscr{G}$. Hence, there's a natural map $\mathrm{Ext}^i_X(\mathscr{F}, f^!\mathscr{G}) \to \mathrm{Ext}^i_Y(f_*\mathscr{F}, \mathscr{G})$.

(d) Taking global sections on both side of the isomorphism in (b), we have

$$\mathrm{Hom}_X(\mathscr{F}, f^!\mathscr{G}) \cong \mathrm{Hom}_Y(f_*\mathscr{F}, \mathscr{G}),$$

which proves the case $i = 0$. Assume that $\mathscr{F}$ is locally free, then

$$\operatorname{Ext}^i_X(\mathscr{F}, f^!\mathscr{G}) \cong \operatorname{Ext}^i_X(\mathcal{O}_X, \mathscr{F}^\vee \otimes f^!\mathscr{G}) \cong H^i(X, \mathscr{H}om_X(\mathscr{F}, f^!\mathscr{G})).$$

Since $f_*\mathcal{O}_X$ is locally free, $f_*\mathscr{F}$ is also locally free, so

$$\operatorname{Ext}^i_Y(f_*\mathscr{F}, \mathscr{G}) \cong \operatorname{Ext}^i_Y(\mathcal{O}_Y, (f_*\mathscr{F})^\vee \otimes \mathscr{G}) \cong H^i(Y, \mathscr{H}om_Y(f_*\mathscr{F}, \mathscr{G})),$$

then by (b) and (Ex 4.1),

$$H^i(X, \mathscr{H}om_X(\mathscr{F}, f^!\mathscr{G})) \cong H^i(Y, f_*\mathscr{H}om_X(\mathscr{F}, f^!\mathscr{G})) \cong H^i(Y, \mathscr{H}om_Y(f_*\mathscr{F}, \mathscr{G})).$$

For general $\mathscr{F}$, we induction on i. Define F^i, G^i be the functors that map $\mathscr{F}$ to $\operatorname{Ext}^i_X(\mathscr{F}, f!\mathscr{G})$ and $\operatorname{Ext}^i_Y(f_*\mathscr{F}, \mathscr{G})$ respectively. Let $0 \to \mathscr{R} \to \mathscr{E} \to \mathscr{F} \to 0$ be exact where $\mathscr{E}$ is a locally free sheaf, then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
F^{i-1}(\mathscr{E}) & \longrightarrow & F^{i-1}(\mathscr{R}) & \longrightarrow & F^i(\mathscr{F}) & \longrightarrow & F^i(\mathscr{E}) & \longrightarrow & F^i(\mathscr{R}) \\
\downarrow \wr & & \downarrow \wr & & \downarrow \varphi_i & & \downarrow \wr & & \downarrow \varphi_i' \\
G^{i-1}(\mathscr{E}) & \longrightarrow & G^{i-1}(\mathscr{R}) & \longrightarrow & G^i(\mathscr{F}) & \longrightarrow & G^i(\mathscr{E}) & \longrightarrow & G^i(\mathscr{R})
\end{array}$$

The first row is clearly exact. The second row is exact since $0 \to f_*\mathscr{R} \to f_*\mathscr{E} \to f_*\mathscr{F} \to 0$ is exact by checking on open affine subsets of Y and the fact that f is affine. By five lemma, φ_i is injective. Since $\mathscr{F}$ is arbitrary, φ_i' is also injective. By five lemma again, φ_i is surjective, hence φ_i is an isomorphism.

7 The Serre Duality Theorem

Exercise 1 (by Shi-Xin)**.**

Since X is projective and hence of finite type, and $\mathcal{L}$ is ample, by Theorem 2.7.6, $\mathcal{L}^m$ is very ample for some large m, i.e. it induces an closed immersion $X \to \mathbb{P}^n$ for some n Therefore, $dim_k(\Gamma(X, \mathcal{L}^m)) = n + 1 > 1$. Now if $H^0(X, \mathcal{L}^{-1}) \neq 0$, there must be some nonzero section $s \in \Gamma(X, \mathcal{L}^{-m})$. Then it defines a morphism $\phi : \Gamma(X, \mathcal{L}^m) \to \Gamma(X, \mathcal{O}_X)$ by multiplying s. However, we must have $dim(Im(\phi)) > 1 = \Gamma(X, \mathcal{O}_X)$, which leads to a contradiction.

Exercise 2 (by Yi-Tsung Wang)**.**

(a) Let $t_Y : H^n(Y, \omega_Y^\circ) \to k$ be the trace map. Then the trace map $t_X : H^n\left(X, f^!\omega_Y^\circ\right) \to k$ is given by

$$H^n\left(X, f^!\omega_Y^\circ\right) \cong \operatorname{Ext}^n_X\left(\mathcal{O}_X, f^!\omega_Y^\circ\right) \xrightarrow{\text{Ex.3.6.10(c)}} \operatorname{Ext}^n_Y\left(f_*\mathcal{O}_X, \omega_Y^\circ\right)$$

$$\to \operatorname{Ext}^n_Y\left(\mathcal{O}_Y, \omega_Y^\circ\right) \cong H^n\left(Y, \omega_Y^\circ\right) \xrightarrow{t_Y} k$$

Now for any coherent sheaf $\mathscr{F}$ on X, $f_*\mathscr{F}$ is coherent on Y, hence $\operatorname{Hom}_Y\left(f_*\mathscr{F}, \omega_Y^\circ\right) \cong H^n\left(Y, f_*\mathscr{F}\right)'$. Thus $\operatorname{Hom}_Y\left(\mathscr{F}, f^!\omega_Y^\circ\right) \cong \operatorname{Hom}_Y\left(f_*\mathscr{F}, \omega_Y^\circ\right) \cong H^n\left(Y, f_*\mathscr{F}\right)'$, and by ex 3.4.1, $H^n\left(Y, f_*\mathscr{F}\right) \cong H^n\left(X, \mathscr{F}\right)$. Therefore we get that $\operatorname{Hom}_X\left(\mathscr{F}, f^!\omega_Y^\circ\right) \cong H^n\left(X, \mathscr{F}\right)'$, i.e., $f^!\omega_Y^\circ$ is a dualizing sheaf.

(b) By cor 3.7.12, $\omega_Y = \omega_Y^\circ$ and $\omega_X = \omega_X^\circ = f^!\omega_Y^\circ = f^!\omega_Y$, and we have a natural map $f_*f^!\omega_Y \to \omega_Y$, thus a natural trace map $t : f_*\omega_X \to \omega_Y$.

Exercise 3 (by Shi-Xin)**.**

By exercise 2.5.16(b)(c), since $0 \to \Omega_X \to \mathcal{O}_X(-1)^{n+1} \to \mathcal{O}_X \to 0$, we have a filtration

$$\bigwedge^r(\mathcal{O}_X(-1)^{n+1}) = \mathcal{F}^0 \supset \ldots \supset \mathcal{F}^r \supset \mathcal{F}^{r+1} = 0$$

such that $\mathcal{F}^p/\mathcal{F}^{p+1} \cong \Omega_X^p \otimes \bigwedge^{r-p}\mathcal{O}_X$. Note that $\bigwedge^k\mathcal{O}_X$ is $\mathcal{O}_X$ for $k = 0, 1$, is 0 for $k \neq 0$. Therefore, the filtration is just $\mathcal{F}^0 \supset \mathcal{F}^r \supset \mathcal{F}^{r+1} = 0$ where $\mathcal{F}^r \cong \Omega_X^r$. Then we have

$$\bigwedge^r(\mathcal{O}_X(-1)^{n+1})/\Omega_X^r \cong \mathcal{F}^0/\mathcal{F}^r \cong \Omega_X^{r-1} \otimes \bigwedge^1\mathcal{O}_X \cong \Omega_X^{r-1},$$

which gives the following short exact sequence

$$0 \to \Omega_X^r \to \bigwedge^r(\mathcal{O}_X(-1)^{n+1}) \to \Omega_X^{r-1} \to 0$$

By Serre duality theorem, we can deduce that $H^i(X, \mathcal{O}_X(-r)) = 0$ whenever $i < n$ or $r < n+1$, and hence the long exact sequence induced by the above short exact sequence implies $H^i(X, \Omega_X^r) \cong H^{i-1}(X, \Omega_X^{r-1})$ whenever $1 \le i, r < n+1$ or $1 \le i < n, r \ge n+1$. Moreover, since $H^0(X, \Omega_X^0) \cong \Gamma(X, \mathcal{O}_X) \cong k$, we have $H^i(X, \Omega_X^i) \cong k$. Also, since for $i < n$, $H^i(X, \Omega_X^n) \cong H^i(X, \mathcal{O}_X(-n-1)) = 0$, it follows that $H^i(X, \Omega_X^r) = 0$ whenever $i < r$, $0 \le r \le n$. Again by Serre duality theorem, we can show that $H^i(X, \Omega_X^r) = 0$ whenever $i > r$, $0 \le r \le n$. Thus we conclude that $H^i(X, \Omega_X^r)$ is k if $i = r$, is 0 if $i \ne r$

Exercise 5 (by Shuang-Yen)**.**

Let $y_i = \sum_j a_{ij} x_j$ for some $a_{ij} \in A$. Define $\epsilon : E = \bigoplus_{i=1}^{n+r} \to A$ by $e_i \mapsto x_i$ for $i = 1, \ldots, n$ and $e_{n+i} \mapsto y_i$ for $i = 1, \ldots, r$. Let $f_i = e_{n+i} - \sum_j a_{ij} e_j$ for $i = 1, \ldots, r$, then $\epsilon(f_i) = 0$, and we have

$$
\begin{aligned}
d(e_{i_1} \wedge \cdots \wedge e_{i_s} \wedge f_{j_1} \wedge \cdots \wedge f_{j_t}) \\
&= \sum_{p=1}^{s} \epsilon(e_{i_p}) e_{i_1} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_s} \wedge f_{j_1} \wedge \cdots \wedge f_{j_t} \\
&\qquad + \sum_{q=1}^{t} \epsilon(f_{j_q}) e_{i_1} \wedge \cdots \wedge e_{i_s} \wedge f_{j_1} \wedge \cdots \wedge \widehat{f_{j_q}} \wedge \cdots \wedge f_{j_t} \\
&= \sum_{p=1}^{s} \epsilon(e_{i_p}) e_{i_1} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_s} \wedge f_{j_1} \wedge \cdots \wedge f_{j_t}
\end{aligned}
$$

Let $K_{s,t} \subseteq K_{s+t}(\epsilon)$ be the submodule generated by $e_{i_1} \wedge \cdots \wedge e_{i_s} \wedge f_{j_1} \wedge \cdots \wedge f_{j_t}$, then $d(K_{s,t}) \subseteq K_{s-1,t}$. So

$$
\begin{aligned}
H_k(\mathbf{x}, \mathbf{y}; M) = h_k(K. \otimes M) &= \frac{\ker(K_k \otimes M \to K_{k-1} \otimes M)}{\operatorname{Im}(K_{k+1} \otimes M \to K_k \otimes M)} \\
&\cong \frac{\bigoplus_{s+t=k} \ker(K_{s,t} \otimes M \to K_{s-1,t} \otimes M)}{\bigoplus_{s+t=k} \operatorname{Im}(K_{s+1,t} \otimes M \to K_{s,t} \otimes M)} \\
&\cong \bigoplus_{s+t=k} \frac{\ker(K_{s,t} \otimes M \to K_{s-1,t} \otimes M)}{\operatorname{Im}(K_{s+1,t} \otimes M \to K_{s,t} \otimes M)} \\
&\cong \bigoplus_{s+t=k} H_s(\mathbf{x}; M) \otimes \bigwedge^t A.
\end{aligned}
$$

Exercise 7 (by Chi-Kang Chang)**.**

(ONLY PARTIAL SOLUTION NOW) Let M be a f,g, A-module, then M is free iff there is a minimal generating set of M is linearly independent. Let $m_1, \ldots, m_k$ be a minimal generation set.

($\Leftarrow$): If M is free, let $x_1, ..., x_n \in A$, then we have the folloeing are equivalent:
(1) x_n is a zero divisor in $M/\sum_{i=1}^{n-1} x_i M$
(2) there exist $a_1, ..., a_k \in A$ s,t, $x_n(a_1m_1 + ... + a_km_k) \in \sum_{i=0}^{n-1} x_i M$ and $a_1m_1 + ... + a_km_k \notin \sum_{i=0}^{n-1} x_i M$
(3) there exist $a_1, ..., a_k \in A - (x_1, ..., x_{n-1})$ s,t, $x_n a_i \in (x_1, ..., x_{n-1})$ for all i
(4) x_n is a zero divisor in $A/(x_1, ..., x_{n-1})$.
(where (2) equivalent to (3) need the freeness, the others are just by definition.)

So by the above equivalent we see that $x_1, ..., x_n$ is a M regular sequence iff it is a A regular sequence. Hence depth$M$=depth$A$=dim$A$=dim(Supp$M$) since A is a regular local ring.

8 Higher Direct Images of Sheaves

Exercise 2 (by Zi-Li).

For any open affine $U \subset Y$, we have $R^i f_*(\mathscr{F})|_U = R^i f_*(\mathscr{F}|_{f^{-1}(U)})$, hence, we may assume X, Y are affine. Then, $R^i f_*(\mathscr{F}) \simeq H^i(X, \mathscr{F})\tilde{} = 0$. Hence, the hypotheses of 8.1 are satisfied.

Exercise 3 (by Tzu-Yang Chou).

We first pick an injective resolution for $\mathscr{F}$, say $0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I}^\cdot$. Then, $0 \longrightarrow \mathscr{F} \otimes f^*\mathscr{E} \longrightarrow \mathscr{I}^\cdot \otimes f^*\mathscr{E}$ is also an injective resolution. On the other hand, we note that $f^*\mathscr{E}$ is still locally free and hence we can use the original projection formula to yield that $f_*(\mathscr{I} \otimes f^*\mathscr{E}) \simeq f_*\mathscr{I} \otimes \mathscr{E}$. Now $R^i f_*(\mathscr{F} \otimes f^*\mathscr{E}) \simeq H^i(f_*(\mathscr{I} \otimes f^*\mathscr{E})) \simeq H^i(f_*\mathscr{I} \otimes \mathscr{E}) \simeq H^i(f_*\mathscr{I}) \otimes \mathscr{E} \simeq R^i f_*(\mathscr{F}) \otimes \mathscr{E}$.

9 Flat Morphisms

Exercise 1 (by Yi-Heng).

By exercise II.3.18(c) and exercise II.3.19, it suffices to show that $f(U)$ is stable under generization for every open subset U in X. If $y \in f(U)$ and $y \in \overline{\{y_1\}}$, we want to prove $y_1 \in f(U)$. Let $SpecA$ be an open affine neighborhood of y and y_1. Since f is flat, the local homomorphism $A_y = \mathscr{O}_y \longrightarrow B_x = \mathscr{O}_x$ is flat where $y = f(x)$ and $x \in SpecB \subset f^{-1}(SpecA)$. Note that $y_1 \in SpecA_y$ since $y \in \overline{\{y_1\}}$. Moreover, the extension of the maximal ideal of A_y is contained in the maximal ideal of B_x, which implies the homomorphism is faithfully flat. Thus, there exists $x_1 \in SpecB$ such that $f(x_1) = y_1$.

Exercise 2 (by Yi-Tsung Wang).

Let $X : \{(x, y, z) = (t^3, t^2, t)\} \subseteq \mathbb{A}^3$, then for $a \neq 0$, X_a is given by $\{(x, y, z) = (t^3, t^2, at)\}$. Take $I \trianglelefteq k[x, y, z, a]$ by eliminating t such that a is not a zero-divisor in $k[x, y, z, a]/I$, we get $I = (a^3x - z^3, a^2y - z^2, x^2 - y^3, ax - yz, xz - ay^2)$. Then for $a = 0$, we have

$$I_0 = \left(z^3, z^2, x^2 - y^3, yz, xz\right) = \left(z^2, x^2 - y^3, yz, xz\right)$$

At the origin O, there is an element z such that $z^2 = 0$, hence we get an embedding point at the cusp.

10 Smooth Morphisms

Exercise 1 (by Tzu-Yang Tsai)**.**

Let $\mathscr{I}$ be the corresponding ideal sheaf of X, $A = \mathcal{A}_k^2$, then we have an exact sequence

$$\mathscr{I}/\mathscr{I}^2 \to \Omega_{A/k} \otimes \mathcal{O}_X \to \Omega_{X/k} \to 0$$

Consequently, $\Omega_{X/k} = \widetilde{\mathcal{O}_X dx \oplus \mathcal{O}_X} dy/(2ydy)$. If we take a maximal ideal in $k[x,y]/(\mathscr{I}, 2y)$ corresponding to a point p, $\Omega_{X/Y} \otimes k(p)$ has dimension 2, but its relative dimension is 1, thus not smooth.

Exercise 3 (by Tzu-Yang Tsai)**.**

(i)⇔(ii) By the definition of smooth.

(iii)⇒(ii) We have

$$m_y/m_y^2 \otimes_{k(y)} k(x) = (m_y \otimes_{\mathcal{O}_y} k(y)) \otimes_{k(y)} k(x) = m_y \otimes_{\mathcal{O}_y} (\mathcal{O}_x/m_y\mathcal{O}_x)$$

Also, $m_y \otimes_A \mathcal{O}_x \cong m_y\mathcal{O}_x = m_x$, thus

$$m_y \otimes (\mathcal{O}_x/m_y\mathcal{O}_x) = (m_y \otimes \mathcal{O}_x)/m_y^2 = m_x/m_x^2$$

Thus the map T_f, x is an isomorphism, i.e. it's smooth of relative dimension 0.

(ii)⇒(iii) Fact: Let $\mathcal{O}_x = B, \mathcal{O}_y = A$, if $\hat{A} \to \hat{B}$ is an isomorphism then f is unramified at x.

Proof. We know $m_y^n\mathcal{O}_x = m_x^n \, \forall \, n \in \mathbb{N}$, and the composition

$$A/m_y^n \to B/m_y^n B \to \hat{A}/m_y^n\hat{A}$$

is an isomorphism, it's left to show $\hat{A} \to \hat{B}$ is injective, in other words,

$$m_y^n\hat{A} \cap B = m_y^n B \, \forall \, n \in \mathbb{N}$$

Notice that $B = A + m_y^n B$ and $m_y^n B \subseteq m_y^n\hat{A}$, so $\forall \, b \in B$, it can be represented as $a + \epsilon, a \in A, b \in m_y^n B$ $m_y^n\hat{A} \cap B \subseteq m_y^n B$.

Conversely, if $b \in m_y^n\hat{A}, a \in m_y\hat{A} \cap A = m_y^n$, thus $b \in m_y^n B$. Thus we only need to prove $\hat{A} \to \hat{B}$ is an isomorphism, but $\mathcal{O}_x = \mathcal{O}_y + m_x^n, m_x^n = m_x^{n+1} + m_y^n \forall n \in \mathbb{N}$, thus $\hat{A} \to \hat{B}$ is an isomorphism. Also, by (II 8.6A), $\dim_{k(x)} \Omega_{X/Y} \otimes k(x) \geq$tr.deg $k(x)/k(y)$, equality holds if and only if $k(x)$ is separately extended over $k(y)$. Thus in this case, $k(x)$ is separately extended over $k(y)$ of transcendental degree 0, i.e. a separable algebraic extension.

11 The Theorem on Formal Functions

Exercise 1 (by Pe-Hsuan Chang)**.**

Corollary 11.2 is false without the projective hypothesis. Let $X = \mathbb{A}_k^n$, $P = (0, \dots 0)$, $U = X - P$, and $f : U \hookrightarrow X$ be the inclusion. Notice that the fibres of f all have dimension 0. (So $r = 0$.) But we can show that $R^{n-1} f_* \mathcal{O}_U \neq 0$. Since $R^{n-1} f_* \mathcal{O}_U$ is associated to $V \mapsto H^{n-1}(f^{-1}(V), \mathcal{O}_U|_{f^{-1}(V)})$, we just compute the $n-1$th Čech cohomology to show it is not zero. Take $U_i = \operatorname{Spec} k[x_1, \dots, x_n, x_i^{-1}]$ be an open cover of U. Then the Čech complex will be

$$\dots \longrightarrow \bigoplus_{i=1}^{n} k[x_1, \dots, x_n, x_1^{-1}, \dots, \hat{x_i}^{-1}, \dots x_n^{-1}] \longrightarrow k[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}] \longrightarrow 0$$

$$\bigoplus_{i=1}^{n} f_i \longmapsto \sum_{i=1}^{n} (-1)^{i-1} f_i$$

Thus, to show $n-1$th cohomology is not zero, just need to show the map above is not surjective. But $x_1^{-1} x_2^{-1} \dots x_n^{-1}$ is clearly not in the image. Hence, $R^{n-1} f_* \mathcal{O}_U \neq 0$.

Exercise 2 (by Pei-Hsuan Chang)**.**

Use Stein factorization then we have a projective morphism g with connected fibres and a finite morphism h such that the diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle h} & \\ & Y' & \end{array}$$

Our goal is to show that g is an isomorphism so that f and h just deffer by an isomorphism, and thus, f is a finite morphism. Since we can replace Y' by $\operatorname{Im} g$, so we may assume g is surjective. Notice that $f^{-1}(h(y'))$ is finite and $g^{-1}(y') \subseteq f^{-1}(h(y'))$, but $g^{-1}(y')$ should be connected, and hence, $g^{-1}(y')$ is a single point. Also, g is projective and thus, is proper, so g give a homeomorphism on the under lying spaces. In the proof of Stein factorization, we see that $g_* \mathcal{O}_X = \mathcal{O}_{Y'}$. Thus, g is an isomorphism. This prove the statement.

Exercise 8 (by Pei-Hsuan Chang)**.**

Our goal is to show that $R^i f_*(\mathscr{F})\widehat{_y} = 0$. Since $R^i f_*(\mathscr{F})$ is coherent, This imply that $R^i f_*(\mathscr{F})_y$ is zero. (Since for M: finitely generated A-module, A: noetherian local ring, $\hat{M} = M \otimes_A \hat{A}$, and notice $\hat{A}$ is faithfully flat A-module.) So there must exists some open set of y such that $R^i f_*(\mathscr{F}) = 0$. Also, by theorem on formal function, this is equivalence to show that $H^i(X_y, \mathscr{O}_y/\mathfrak{m}_y^k \otimes \mathscr{F}) = 0$, for all k.

To show this, we use induction on k. Notice that we already have $H^i(X_y, \mathscr{O}_y/\mathfrak{m}_y \otimes \mathscr{F}) = 0$. Then for each k, we consider the exact sequance:

$$0 \to \mathfrak{m}_y^k/\mathfrak{m}_y^{k+1} \to \mathscr{O}_y/\mathfrak{m}_y^{k+1} \to \mathscr{O}_y/\mathfrak{m}_y^k \to 0.$$

Since $\mathscr{F}$ is flat over Y, so we can tensor $\mathscr{F}$ and take cohomology to get:

$$\cdots \to H^i(X_y, \mathfrak{m}_y^k/\mathfrak{m}_y^{k+1} \otimes \mathscr{F}) \to H^i(X_y, \mathscr{O}_y/\mathfrak{m}_y^{k+1} \otimes \mathscr{F}) \to H^i(X_y, \mathscr{O}_y/\mathfrak{m}_y^k \otimes \mathscr{F}) \to \cdots$$

By induction hypothesis, $H^i(X_y, \mathscr{O}_y/\mathfrak{m}_y^k \otimes \mathscr{F}) = 0$. Also, $\mathfrak{m}_y^k/\mathfrak{m}_y^{k+1}$ is just the direct sum of copies of $\mathscr{O}_y/\mathfrak{m}_y$ since it is a vector space over $\mathscr{O}_y/\mathfrak{m}_y$. Thus, $H^i(X_y, \mathfrak{m}_y^k/\mathfrak{m}_y^{k+1} \otimes \mathscr{F})$ is also zero. Hence, $H^i(X_y, \mathscr{O}_y/\mathfrak{m}_y^k \otimes \mathscr{F}) = 0$, for all k. This proof the statement.

12 The Semicontinuity Theorem

Exercise 1 (by Yu-Ting Huang)**.**

We may assume Y is affine. Since Y is finite type over k, we may further assume that $Y = \operatorname{Spec} k[x_1, x_2, \ldots, x_n]/I$, where I is generated by $f_1, \ldots, f_m$. Then $\dim m_y/m_y^2 = n - \operatorname{rk} J$, where J is the Jacobian matrix of f_i at y. Note that the determinant is a continuous function. For a submatrix J' of J, if $\det(J') \neq 0$ at y, then there exists a neighborhood U such that $\det(J') \neq 0$ at y' for $y' \in U$. This implies φ is upper semicontinuous.

Exercise 4 (by Yu-Ting Huang)**.**

Note that

$$h^0(y, \mathscr{L} \otimes \mathscr{M}^-) = \dim H^0(X_y, \mathscr{L}_y \otimes \mathscr{M}_y^{-1}) = \dim H^0(X_y, \mathscr{O}_{X_y}) = 1.$$

By Grauert, $f_*(\mathscr{L} \otimes \mathscr{M}^{-1})$ is a locally free sheaf of rank 1. Since $\mathscr{L} \otimes \mathscr{M}^{-1}$ and $f_*(\mathscr{L} \otimes \mathscr{M}^{-1})$ are both invertible sheaf, we have the isomorphism

$$f^* f_*(\mathscr{L} \otimes \mathscr{M}^{-1}) \xrightarrow{\sim} \mathscr{L} \otimes \mathscr{M}^{-1}.$$

Then $\mathscr{N} := f_*(\mathscr{L} \otimes \mathscr{M}^{-1})$ is what we want.

Exercise 5 (by Yu-Ting Huang)**.**

Let $\pi : X \to Y$ and $\mathscr{E}$ is of rank r. Define the map $\phi : \operatorname{Pic} Y \times \mathbf{Z} \to \operatorname{Pic} X$ by $\phi(\mathscr{L}, n) = \pi^* \mathscr{L} \otimes \mathscr{O}_X(n)$. We will show that ϕ is bijective.
The injectivity part is same as II Ex.7.9 (i.e. the assumption that Y is regular is not required.) Suppose that $\pi^*\mathscr{L} \otimes \mathscr{O}_X(n) \simeq \mathscr{O}_X$. By the projection formula, $\mathscr{O}_Y \simeq \pi_*\mathscr{O}_X \simeq \pi_*(\pi^*\mathscr{L} \otimes \mathscr{O}_X(n)) \simeq \mathscr{L} \otimes \pi_*\mathscr{O}_X(n)$. Then $\pi_*\mathscr{O}_X(n) \simeq \mathscr{L}^{-1}$. By II.7.11, if $n = 0$, $\mathscr{L} = \mathscr{O}_X$, and if $n < 0$, $\mathscr{L}^{-1} = 0$ (contradiction). For $n > 0$, $\pi_*\mathscr{O}_X(n) \simeq S^n(\mathscr{E})$, where the rank of $S^n(\mathscr{E}) > 1$. This contradicts that $\mathscr{L}$ is an invertible sheaf. Hence, the injectivity is verified.
For $\mathscr{M} \in \operatorname{Pic}(X)$, we consider sheaf $\mathscr{M}_y$ on the fiber X_y. Note that $\mathscr{M}_y$ is an invertible sheaf. By II.9.9, since X is flat over Y, the Hilbert polynomial of $\mathscr{M}_y$ is independent of y. Then we can write $\mathscr{M}_y = \mathscr{O}_{\mathbb{P}^r}(n)$ for all $y \in Y$. Now, we have for each $y \in Y$, $(\mathscr{M} \otimes \mathscr{O}_X(-n))_y \simeq \mathscr{O}_{X_y}$. By Exercise 4, there exists $N \in \operatorname{Pic} Y$ such that $\pi^*\mathscr{N} \simeq \mathscr{M} \otimes \mathscr{O}_X(-n)$, then $\mathscr{M} \simeq \pi^*\mathscr{N} \otimes \mathscr{O}_X(-n)$.
Now, we have $\phi : \operatorname{Pic} Y \times \mathbf{Z} \xrightarrow{\sim} \operatorname{Pic} X$.