Algebraic Geometry I Homework
Chapter II Schemes

A course by prof. Chin-Lung Wang

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Exercise 0 (by Kuan-Wen)**.**

This is an example of proof.

Remark. This is an example for how to write in this format.

1 Sheaves

Exercise 1 (by Chun-Yi)**.**

Let $\mathcal{F}$ be the constant presheaf associated to A on X, $\mathcal{F}^+$ be the constant sheaf associated to A on X. Define θ: $\mathcal{F} \to \mathcal{F}^+$ by $\theta(U)(a) = (a, 1, ..., 1) \in A^r$, where r is the number of components of U, then θ is a sheaf morphism.
Now if φ: $\mathcal{F} \to \mathcal{G}$ is a morphism, then ψ: $\mathcal{F}^+ \to \mathcal{G}$ defined by $\psi(U)(a_1, ..., a_r) = \varphi(U)(a_1)$ is the unique morphism such that $\varphi = \psi \circ \theta \implies \mathcal{F}^+$ is the sheaf associated to the presheaf $\mathcal{F}$.

Exercise 2 (by Chun-Yi)**.**

(a) Consider the commutative diagram, where $p \in U$

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathcal{F}_p & \xrightarrow{\quad\varphi_p\quad} & \mathcal{G}_p \end{array}$$

Let $s \in \ker \varphi(U)$, $s' = f(s) \in (\ker \varphi)_p$, then $\varphi_p(s') = \varphi_p(f(s)) = g(\varphi(U))(s) = g(0) = 0 \implies (\ker \varphi)_p \subseteq \ker \varphi_p$
Conversely, given $s' \in \ker \varphi_p$, we can pull back to $s \in \ker \varphi(U)$ such that $f(s) = s'$. Let $t = \varphi(U)(s)$, then $g(t) = 0$, that is $\exists V$ such that

$t\,|_V = 0 \implies \varphi(U)(s) = 0 \implies \bar{s} = \langle s, U\cap V\rangle \in (\ker\varphi)_p$
Similarly, let $t \in \mathrm{im}\varphi(U)$, $t' = g(t) \in (\mathrm{im}\varphi)_p$, then $\exists s \in \mathcal{F}(U)$ such that $\varphi(U)(s) = t$. Now $t' = g(\varphi(U)(s)) = \varphi_p(f(s)) \implies t' \in (\mathrm{im}\varphi)_p$.
Conversely, if $t' \in \mathrm{im}\varphi_p$, then $\exists s' \in \mathcal{F}_p$ such that $\varphi_p(s') = t'$, pull back to $s \in \mathcal{F}(U)$ such that $f(s) = s'$, then $g(\varphi(U)(s)) = t' \implies t' \in (\mathrm{im}\varphi)_p$.

(b) φ is injective $\iff \ker\varphi = 0 \iff \ker\varphi_p = 0\ \forall p \iff \varphi_p$ is injective $\forall p$
φ is surjective $\iff \mathrm{im}\varphi = \mathcal{G} \iff \mathrm{im}\varphi_p = \mathcal{G}_p\ \forall p \iff \varphi_p$ is surjective $\forall p$

(c) A sequence of sheaves and morphisms is exact $\iff \ker\varphi^i = \mathrm{im}\varphi^{i-1} \iff \ker\varphi^i_p = \mathrm{im}\varphi^{i-1}_p\ \forall p \iff$ The corresponding sequence of stalks is exact $\forall p$

Exercise 3 (by Pei-Hsuan)**.**

(a) Suppose $\varphi : \mathscr{F} \to \mathscr{G}$ is surjective, then $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$ is surjective, $\forall p \in X$. $\forall U \subseteq X$, $\forall s \in \mathscr{G}(U)$, we have commutative diagram:

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\varphi(U)} & \mathscr{G}(U) \\ \downarrow & & \downarrow \\ \mathscr{F}_p & \xrightarrow{\varphi_p} & \mathscr{G}_p \end{array}$$

Thus, $\exists t_p \in \mathscr{F}_p$ such that $\varphi_p(t_p) = s_p$, $\forall p \in X$. This means $\exists V_p$: neighborhood of p, $\exists t \in \mathscr{F}(U)$ such that $\varphi(t|_{V_p}) = s|_{V_p}$. So, $\{V_p\}_{p\in U}$ is what we desired.

Conversely, if $\forall U \subseteq X$, $\forall s \in \mathscr{G}(U)$, there exists an open cover $\{U_i\}_i$ of U such that $\forall i$, $\exists t_i \in \mathscr{F}(U_i)$, $\varphi(t_i) = s|_{U_i}$. Then $\forall p \in X$, $\forall s_p \in \mathscr{G}_p$, say $p \in U_i$. Then pull s_p back to $s \in \mathscr{G}(U)$. Thus, $\exists t_i \in \mathscr{F}(U_i)$ such that $\varphi(t_i) = s|_{U_i}$. Push this t_i forward to $\bar{t_i} \in \mathscr{F}_p$, then $\varphi_p(\bar{t_i} = s_p$. Thus, $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$ is surjective, $\forall p \in X \Rightarrow \varphi : \mathscr{F} \to \mathscr{G}$ is surjective.

(b) Let $X = U_1 \sqcup U_2$, where U_1, U_2 are open and connected. Let $\mathscr{F}$ be a constant sheaf defined by $\mathcal{A}$. Consider $\mathscr{G}$ to be a constant presheaf defined by $\mathcal{A}$. Define $\varphi : \mathscr{F} \to \mathscr{G}$ to be $\begin{array}{ccc} \mathscr{F}(U) & \to & \mathscr{G}(U) \\ f & \mapsto & f(U\cap U_1) + f(U \cap U_2) \end{array}$. It is a surjective presheaf morphism. Now, consider $\varphi^+ : \mathscr{F} \to \mathscr{G}^+$ which is defined by $\begin{array}{ccc} \mathscr{F}(U) & \to & \mathscr{G}^+(U) \\ f & \mapsto & (\varphi(f), 0) \end{array}$, if U is disconnected. (Notice that $\mathscr{G}(U) = \mathscr{G}^+(U)$, if $U \subseteq U_i$, for some i.) Thus, $\mathrm{Im}(\varphi^+) = (\mathrm{Im}\,\varphi)^+ = \mathscr{G}^+$, so φ^+ is surjective sheaf morphism. But, φ^+_U is not surjective whenever U is disconnected.

Exercise 4 (by Jung-Tao)**.**

(a) $\ker\varphi = 0 \implies (\ker\varphi)_P = 0 \implies \ker\varphi_P = 0$, for all points P.
the map $\varphi_P \colon \mathcal{F}_P \to \mathcal{G}_P$ induces a map $\varphi_P^+ \colon \mathcal{F}_P^+ \to \mathcal{G}_P^+$,
since $\mathcal{F}_P = \mathcal{F}_P^+, \mathcal{G}_P = \mathcal{G}_P^+$,

$$\ker\varphi_P = 0, \forall P \implies \ker\varphi_P^+ = 0, \forall P \implies \ker\varphi^+ = 0$$

The last equality is because $\forall t$ an element in the group corresponds to some open set X through the sheaf $\ker\varphi^+$. $t = 0$ locally means for every point P, there is a neighborhood U_P contains P, $t\mid_{U_P} = 0$. $\{U_P\}$ is an open cover of X where $t = 0$ at every component, and $t = 0$ by the definition of sheaf.

(b) the map $\phi \colon \varphi(\mathcal{F}) \to \mathcal{G}$ is injective, where $\varphi(\mathcal{F})$ denote the image of $\mathcal{F}$ as a presheaf, from (a) we get an inclusion map from $\operatorname{im}\varphi = \varphi(\mathcal{F})^+$ to $\mathcal{G}^+ = \mathcal{G}$

Exercise 5 (by Te-Lun)**.**

Let $\varphi \colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves, then:

$$\begin{aligned} \varphi \text{ is an isomorphism} &\iff \varphi_p \text{ is an isomorphism on stalk, for all } p \in X \\ &\iff \varphi_p \text{ is injective and surjective, for all } p \in X \\ &\overset{(*)}{\iff} \varphi \text{ is injective and surjective, for all } p \in X \end{aligned}$$

, where $(*)$ is hold by Exercise 2(b).

Exercise 6 (by Te-Lun)**.**

(a) Let $\mathcal{F}''$ be the presheaf defined by $U \mapsto \mathcal{F}(U)/\mathcal{F}'(U)$, let the natural morphism $\varphi \colon \mathcal{F} \to \mathcal{F}/\mathcal{F}'$ be:

$$\begin{array}{cccc} \varphi(U)\colon & \mathcal{F}(U) & \to & (\mathcal{F}/\mathcal{F}')(U) \\ & t & \mapsto & \left[s\colon U \to \bigcup_{p\in U} \mathcal{F}''_p \text{ by } s(p) = t_p \text{ the germ of } t \text{ in } \mathcal{F}''_p \right] \end{array}$$

, for all $U \overset{\text{open}}{\subset} X$. It is easy to check that this is indeed a morphism of sheaves. Moreover, to show that this morphism is surjective, we consider the induce map of φ on stalk $\varphi_p \colon \mathcal{F}_p \to (\mathcal{F}/\mathcal{F}')_p = \mathcal{F}_p/\mathcal{F}'_p$ for $p \in X$. Let $\langle U, s\rangle \in (\mathcal{F}/\mathcal{F}')_p$, with $s \in (\mathcal{F}/\mathcal{F}')(U)$, $p \in U$. Then there exist a neighborhood $V \subset U$ of p and $t \in \mathcal{F}''(V)$ such that $t_q = s(q)$ for all $q \in V$. Pick a preimage of $t \in \mathcal{F}''(V) = \mathcal{F}(V)/\mathcal{F}'(V)$ in $\mathcal{F}(V)$, say t'. Then $\varphi_p(\langle V, t'\rangle) = \langle V, \varphi(t')\rangle = \langle V, s|_V\rangle = \langle U, s\rangle$. Hence, φ_p is surjective for all $p \in X$, by Exercise 2(b), φ is surjective.
Last, note that obviously, $(\ker\varphi)_p = \ker\varphi_p = \mathcal{F}_p$ for all $p \in X$, so $\ker\varphi = \mathcal{F}$.

(b) (i) Let $\varphi' \colon \mathcal{F} \to \mathcal{F}$ be injective, then $\mathcal{F}' \simeq \mathrm{Im}\varphi$ as presheaves. (Here, $\mathrm{Im}\varphi$ is the presheaf of image, before sheafification), so there is an isomorphism: $\varphi^{-1} \colon \mathrm{Im}\varphi \to \mathcal{F}'$, behold that we have the following:

$$\begin{array}{ccc} \mathrm{Im}\varphi & \xrightarrow{\varphi^{-1}} & \mathcal{F}' \\ \theta\downarrow & & \downarrow \mathrm{Id} \\ (\mathrm{Im}\varphi)^+ & \xrightarrow{(\varphi^{-1})^+} & (\mathcal{F}')^+ = \mathcal{F}' \end{array}$$

$(\varphi^{-1})^+$ is injective since φ is (Exercise 4(a)), and it is sujctive since $\varphi^{-1} = (\varphi^{-1})^+ \circ \theta$ is, so $(\varphi^{-1})^+$ is an isomorphism (Exercise 5). Hence $\mathcal{F}'$ is iormorphic to $\mathrm{Im}\varphi$ regarded as a subsheaf of $\mathcal{F}$. (Exercise 4(b))

(ii) Let $\psi \colon \mathcal{F} \to \mathcal{F}''$ be surjective, define the presheaf $\mathcal{G} \colon U \mapsto \mathcal{F}(U)/\ker\psi(U)$, then we have $\mathcal{F}'' \simeq \mathcal{G}$ as presheaves, doing sheafification as in part (i), then we have $\mathcal{F}'' \simeq \mathcal{G}^+ = \mathcal{F}/\ker\psi \overset{(*)}{=} \mathcal{F}/\mathrm{Im}\psi$. Where (*) holds by exactness.

Exercise 7 (by Shi-Xin)**.**

(a) Since $\mathscr{F}(U)/\ker(\varphi(U)) \cong \mathrm{im}(\varphi(U))$ for any open set U, $\mathscr{F}/\ker\varphi$ and $\mathrm{im}\varphi$ are isomorphic as presheaves, and hence they are isomorphic as sheaves.

(b) For the same reason to (a), since $\mathscr{G}(U)/\mathrm{im}(\varphi(U)) \cong \mathrm{coker}(\varphi(U))$ for any open set U, $\mathscr{G}/\mathrm{im}\varphi$ and $\mathrm{coker}\varphi$ are isomorphic as sheaves.

Exercise 8 (by Yi-Heng)**.**

It suffices to check that $ker(\psi(U)) \subset im(\varphi(U))$ where $\varphi(U) : \mathscr{F}'(U) \to \mathscr{F}(U), \psi(U) : \mathscr{F}(U) \to \mathscr{F}''(U)$. Note that $0 \to \mathscr{F}'_P \to \mathscr{F}_P \to \mathscr{F}''_P$. Thus, for $s \in ker(\psi(U))$, $s_P = \varphi(t_P)$ for some $t_P \in \mathscr{F}'_P$. Let $t^P \in \mathscr{F}'(V^P)$ represents t_P, then $t^P|_{V^P \cap V^Q} = t^Q|_{V^P \cap V^Q}$ since φ is injective. Therefore, there exists $t \in \mathscr{F}'(U)$ such that $t|_{V^P} = t^P$ for all $P \in U$, and $\varphi(U)(t) = s$. Hence, we get $ker(\psi(U)) \subset im(\varphi(U))$.

Exercise 9 (by Pei-Hsuan)**.**

$U \mapsto \mathscr{F}(U) \oplus \mathscr{G}(U)$ is clearly a presheaf. For open set $U \subseteq X$, if $\{V_i\}$ is an open cover of U, then:

1. If $s \oplus t \in \mathscr{F}(U) \oplus \mathscr{G}(U)$, and $s \oplus t|_{V_i} = 0$, for all i. Then $s|_{V_i} = 0$, $t|_{V_i} = 0$, for all i. Thus, $s = 0$, $t = 0$, so $s \oplus t = 0$.

2. If we have $s_i \oplus t_i \in \mathscr{F}(V_i) \oplus \mathscr{G}(V_i)$, and $s_i \oplus t_i|_{V_i \cap V_j} = s_j \oplus t_j|_{V_i \cap V_j}$, for all i, j. Then $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ and $t_i|_{V_i \cap V_j} = t_j|_{V_i \cap V_j}$, so $\exists s \in \mathscr{F}(U)$ and $t \in \mathscr{G}(U)$ such that $s|_{V_i} = s_i$ and $t|_{V_i} = t_i$, for all i. Thus, $s \oplus t|_{V_i} = s_i \oplus t_i$, for all i.

Hence, $U \mapsto \mathscr{F}(U) \oplus \mathscr{G}(U)$ is a sheaf.

Exercise 10 (by Wei)**.**

For a given direct system $\{\mathcal{F}_i\}_{i \in I}$ over a set I in the category of presheaves over some topological space X such that $\varinjlim_{i \in I} \mathcal{F}_i(U)$ exists for each U, define a presheaf $\varinjlim_{i \in I} \mathcal{F}_i$ by

$$\left(\varinjlim_{i \in I} \mathcal{F}_i\right)(U) = \left(\varinjlim_{i \in I} \mathcal{F}_i(U)\right)$$

on the level of open sets, and on the level of inclusions that for each $V \subseteq U$, define

$$\left(\varinjlim_{i \in I} \mathcal{F}_i\right)(U) \to \left(\varinjlim_{i \in I} \mathcal{F}_i\right)(V)$$

by the maps

$$\mathcal{F}_i(U) \to \mathcal{F}_i(V) \to \left(\varinjlim_{i \in I} \mathcal{F}_i\right)(V)$$

There are canonical maps

$$\alpha_{i,U} : \mathcal{F}_i(U) \to \varinjlim_{i \in I} \mathcal{F}_i(U)$$

The data $\{\alpha_{i,U}\}_U$ form a sheaf morphism, and that the diagrams

$$\begin{tikzcd} \mathcal{F}_i \arrow[r, "\mathcal{F}(s)"] \arrow[dr, "\alpha_i"'] & \mathcal{F}_{i'} \arrow[d, "\alpha_{i'}"] \\ & \varinjlim_{i \in I} \mathcal{F}_i \end{tikzcd}$$

commutes for each $s : i \to i'$, by our definition of $\varinjlim_{i \in I} \mathcal{F}_i$.
Now we verify that $\varinjlim_{i \in I} \mathcal{F}_i$ is the direct limit of the direct system $\{\mathcal{F}_i\}_{i \in I}$ in the category of presheaves over a topological space X. For a given presheaf $\mathcal{G}$, and given presheaf morphisms $\rho_i : \mathcal{F} \to \mathcal{G}$ such that for each arrow $s : i \to i'$ in I the diagram commutes :

$$\begin{tikzcd} \mathcal{F}_i \arrow[r, "\rho_i"] \arrow[d, "\mathcal{F}(s)"'] & \mathcal{G} \\ \mathcal{F}_{i'} \arrow[ur, "\rho_{i'}"'] & \end{tikzcd}$$

that is, an element in the set

$$\varprojlim_{i\in I} \mathrm{Hom}_{\mathrm{pSh}_{\mathrm{Ab}}(X)}(\mathcal{F}_i, \mathcal{G})$$

we try to find a unique element ϕ in $\mathrm{Hom}_{\mathrm{pSh}_{\mathrm{Ab}}(X)}(\varinjlim_{i\in I} \mathcal{F}_i, \mathcal{G})$ such that

$$\begin{tikzcd}
\mathcal{F}_i \arrow[r, "\alpha_i"] \arrow[dr, "\rho_i"'] & \varinjlim_{i\in I} \mathcal{F}_i \arrow[d, "\phi"] \\
& \mathcal{G}
\end{tikzcd}$$

commutes for each $i \in I$. Notice that by universal property of $\varinjlim_{i\in I} \mathcal{F}_i(U)$, we can find for each open set U a map ϕ_U such that the diagram commutes :

$$\begin{tikzcd}
\mathcal{F}_i(U) \arrow[r, "\alpha_{i,U}"] \arrow[dr, "\rho_{i,U}"'] & \varinjlim_{i\in I} \mathcal{F}_i(U) \arrow[d, "\phi_U"] \\
& \mathcal{G}(U)
\end{tikzcd}$$

It is easy to check that the data $\{\phi_U\}_U$ defines a sheaf morphism. So far we have proved existence. On the other hand, suppose there is another ϕ', then since on each U and $i \in I$, we have

$$\begin{tikzcd}
\mathcal{F}_i(U) \arrow[r, "\alpha_{i,U}"] \arrow[dr, "\rho_{i,U}"'] & \varinjlim_{i\in I} \mathcal{F}_i(U) \arrow[d, shift right, "\phi'_U"'] \arrow[d, shift left, "\phi_U"] \\
& \mathcal{G}(U)
\end{tikzcd}$$

we have $\phi'_U = \phi_U$, and hence $\phi' = \phi$.

Having shown that $\varinjlim_{i\in I} \mathcal{F}_i$ is the direct limit of the direct system $\{\mathcal{F}_i\}_{i\in I}$ in the category of presheaves, we now consider sheaves. Suppose given a direct system of sheaves $\{\mathcal{F}_i\}_{i\in I}$, we show that $(\varinjlim_{i\in I} \mathcal{F}_i)^+$ is the direct limit in the category of sheaves, where the maps $\mathcal{F}_i \to (\varinjlim_{i\in I} \mathcal{F}_i)^+$ are defined by the composition

$$\mathcal{F}_i \to \varinjlim_{i\in I} \mathcal{F}_i \to (\varinjlim_{i\in I} \mathcal{F}_i)^+$$

Let $\mathcal{G}$ be another sheaf, and suppose given maps from $\mathcal{F}_i$ to $\mathcal{G}$ compatible with the original system, then we have a unique presheaf morphism between $\varinjlim_{i\in I} \mathcal{F}_i$ and $\mathcal{G}$, which corresponds to a unique sheaf morphism between $(\varinjlim_{i\in I} \mathcal{F}_i)^+$ and $\mathcal{G}$. Check that this morphism is then the desired one.

Remark. In categorical language, the existence of direct limits indexed by a small category can often be expressed in terms of representability of the functor

$$X \to \varprojlim_{i \in I} \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}_i, X)$$

with the representing object denoted $\varinjlim_{i\in I} \mathcal{F}_i$. In a more general setting, this exercise expresses the categorical facts :

(1) Let $I, J, \mathcal{C}$ be categories with I being small. Suppose $\mathcal{C}$ admits direct limits indexed by I, then the functor category $[J, \mathcal{C}]$ admits direct limits indexed by I, and the direct limit can be computed pointwisely; namely, is $\mathcal{F} : I \to [J, \mathcal{C}]$ a functor, then we have

$$\left(\varinjlim_{i\in I} \mathcal{F}_i\right)(j) \simeq \left(\varinjlim_{i\in I} \mathcal{F}_i(j)\right)$$

for each $j \in J$. (In this exercise, C is the category of abelian groups, J is the opposite category of open sets associated to the topological space X)

(2) Given a pair of adjoint functors $L \dashv R$ between categories

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "R"] \end{tikzcd}$$

and assuming that $\mathcal{D}$ admits direct limits indexed by I, then by the chain of natural isomorphisms

$$\begin{aligned} \varprojlim_{i\in I} \mathrm{Hom}_{\mathcal{D}}(LF_i, Y) \simeq \varprojlim_{i\in I} \mathrm{Hom}_{\mathcal{C}}(F_i, RY) \simeq \mathrm{Hom}_{\mathcal{C}}(\varinjlim_{i\in I} F_i, RY) \\ \simeq \mathrm{Hom}_{\mathcal{D}}(L \varinjlim_{i\in I} F_i, Y) \end{aligned}$$

we have, by fully faithfulness of the Yoneda embedding, that

$$\varinjlim_{i\in I}(LF_i) \simeq L(\varinjlim_{i\in I} F_i)$$

(In the exercise, the categories are $(\mathcal{C}, \mathcal{D}) = (\mathrm{pSh}_{\mathrm{Ab}}(X), \mathrm{Sh}_{\mathrm{Ab}}(X))$, L is sheafification, R is forgetful functor).

Exercise 12 (by Wei)**.**

For a given inverse system $\{\mathcal{F}_i\}_{i\in I}$ over a set I in the category of presheaves over some topological space X such that $\varprojlim_{i\in I} \mathcal{F}_i(U)$ exists for each U, we define, as in exercise 10, the presheaf $\varprojlim_{i\in I} \mathcal{F}_i$ by the same law, then a routine check shows that $\varprojlim_{i\in I} \mathcal{F}_i$ is the inverse limit of the inverse system of presheaves $\{\mathcal{F}_i\}_{i\in I}$ in the category of presheaves. Now suppose $\{\mathcal{F}_i\}_{i\in I}$ is an inverse system of sheaves, we show that the inverse limit of the system taken in the category of presheaves is actually a sheaf. [TBD]

Remark. In categorical language, the existence of inverse limits indexed by a small category can often be expressed in terms of representability of the functor

$$X \to \varprojlim_{i\in I} \mathrm{Hom}_{\mathcal{C}}(X, \mathcal{F}_i)$$

with the representing object denoted $\varprojlim_{i\in I} \mathcal{F}_i$. We have these statements for inverse limits ((2') is a little different from (2) in remark to exercise 10)

(1') Let $I, J, \mathcal{C}$ be categories with I being small. Suppose $\mathcal{C}$ admits inverse limits indexed by I, then the functor category $[J, \mathcal{C}]$ admits inverse limits indexed by I, and the inverse limit can be computed pointwisely; namely, is $\mathcal{F} : I^{\mathrm{op}} \to [J, \mathcal{C}]$ a functor, then we have

$$\left(\varprojlim_{i\in I} \mathcal{F}_i\right)(j) \simeq \left(\varprojlim_{i\in I} \mathcal{F}_i(j)\right)$$

for each $j \in J$. (In this exercise, C is the category of abelian groups, J is the opposite category of open sets associated to the topological space X)

(2') Given a pair of adjoint functors $L \dashv R$ between categories

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "R"] \end{tikzcd}$$

and assuming that $\mathcal{D}$ admits inverse limits indexed by I, **and that the functor R is fully faithful**, then we have the chain of natural isomorphisms

$$\begin{aligned}\varprojlim_{i\in I} \mathrm{Hom}_{\mathcal{D}}(X, F_i) \simeq \varprojlim_{i\in I} \mathrm{Hom}_{\mathcal{C}}(RX, RF_i) \simeq \mathrm{Hom}_{\mathcal{C}}(RX, \varprojlim_{i\in I} RF_i)\\ \simeq \mathrm{Hom}_{\mathcal{D}}(X, L\varprojlim_{i\in I} RF_i)\end{aligned}$$

we have by fully faithfulness of the Yoneda embedding, that

$$\varprojlim_{i\in I} F_i \simeq L(\varprojlim_{i\in I} RF_i)$$

(In this exercise, the categories are $(\mathcal{C}, \mathcal{D}) = (\mathrm{pSh_{Ab}}(X), \mathrm{Sh_{Ab}}(X))$, L is sheafification, R is forgetful functor; notice that by definition, $\mathrm{Sh_{Ab}}(X)$ is already a full subcategory of $\mathrm{pSh_{Ab}}(X)$)).

Exercise 14 (by Tzu-Yang Tsai)**.**

It's equivalent to show $A := \{P \in U | s_P = 0\}$ is open, but if $s_P = 0, \exists V$ a neighborhood of P s.t. $s|_V = 0 \Rightarrow V \subseteq A \Rightarrow A$ is open.
Take any nonempty open subset $U \subseteq X$, let $j_!(\mathscr{F})$ be the sheaf obtained by extending $\mathscr{F}$ outside of U, as in Ex 1.19 (b) below. Then $\mathrm{Supp}\mathscr{F} = U$ is open, thereby $\mathrm{Supp}\mathscr{F}$ need not to be closed.

Exercise 15 (by Tzu-Yang Tsai)**.**

$\forall f, h \in \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$, define $f+h$ as $(f+h)(s) = f(s)+h(s)\,\forall s \in \Gamma(U, \mathscr{F}|_U)$, then it has a natural structure of abilian group.
To show $\mathscr{H} : U \mapsto \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$, we have to check two conditions:

1. If $\{V_i\}_{i \in I}$ is an open cover of V, and $\phi \in \Gamma(V, \mathscr{H})$ s.t. $\phi|_{V_j} \forall j \in I$, then $\phi(s|_{V_j}) = \phi|_{V_j}(s) = 0 \forall s \in \Gamma(V, \mathscr{F}) \Rightarrow \phi = 0$ in V.

2. If $\{\phi_j\}_{j \in I}$ is a set of morphism s.t. $\phi_i \in \Gamma(V_i, \mathscr{H}|_{V_i})$ and $\phi_i|_{V_i \cap V_j} = \phi_j|_{V_i \cap V_j} \forall i, j \in I$, one can define $\phi \in \Gamma(V, \mathscr{H}) = \mathrm{Hom}(\mathscr{F}|_V, \mathscr{G}|_V)$ s.t. $\phi(s|_{V_i}) = \phi|_{V_i}(s) \forall s \in \Gamma(V, \mathscr{F})$. It's well-defined due to assumption, and we have $\phi|_{V_i} = \phi_i \forall i \in I$.

Conclude above, we have $\mathscr{H}$ is a sheaf.

Exercise 16 (by Tzu-Yang Chou)**.**

(a) Given a constant sheaf for G $\mathscr{F}$ on X where X is irreducible, let $V \subset U$ be open sets in X. For $f \in \mathscr{F}(V)$, we claim that f is a constant map, and hence it is the restriction of the constant map of the same value on U. G has the discrete topology, so for any $g \in G$, both $f^{-1}(g)$ and $V \setminus f^{-1}(g)$ are closed in V. Find closed sets $A, B \subset X$ such that $A \cap V = f^{-1}(g)$ and $B \cap V = V \setminus f^{-1}(g)$, and then $X = (X \setminus V) \cup A \cup B$. Now if $g \in f(V)$, then $f^{-1}(g) \neq \emptyset$, and hence $X = A$ by irreducibility.

(b) It suffices to show that $\mathscr{F}(U) \longrightarrow \mathscr{F}"(U)$ is epic by the left exactness of section functor. Let $s \in \mathscr{F}"(U)$. Recall by exercise 2.1.3 we obtain a covering $U_i (i \in I)$ of U with sections $t_i \in \mathscr{F}(U_i)$ which map to $s|_{U_i}$.
Now let $\mathcal{S} := \{(I', s') | I' \subset I, s' \in \mathscr{F}(\tilde{U})$, where $\tilde{U} := \bigcup_{j \in I'} U_j$ with $s' \mapsto s|_{\tilde{U}}\}$, equipped with a ordering given by $(I', s') \leq (I", s") \Leftrightarrow I' \subseteq I"$ and $s"$

restricts to s'. Then Zorn's lemma gives a maximal element in $\mathcal{S}$, say (I', s'). We claim that $I' = I$: otherwise, there is some $i_0 \notin I'$. Consider $s'|_{\tilde{U}\cap U_{i_0}} - t_{i_0}|_{\tilde{U}\cap U_{i_0}} \mapsto 0$. Then by exactness, $\exists a \in \mathscr{F}'(\tilde{U} \cap U_{i_0})$ which maps to $s'|_{\tilde{U}\cap U_{i_0}} - t_{i_0}|_{\tilde{U}\cap U_{i_0}}$, and since $\mathscr{F}$ is flasque, there is some $b \in \mathscr{F}'(U_{i_0})$ whose restriction is a. Thus, $t_{i_0} + b$ and s' are compatible and glue to give some section mapping to $s|_{\tilde{U}\cup U_{i_0}}$. So we enlarge (I', s') and obtain a contradiction. That is, $I' = I$.

(c) Given open sets $V \subset U$ in X, since $\mathscr{F}'$ is flasque, by (b) we have two short exact sequences and maps between them making the diagram commutes. Now since $\mathscr{F}$ is alsp flasque, we obtain the desired epimorphism $\mathscr{F}"(U) \longrightarrow \mathscr{F}"(V)$, so $\mathscr{F}"$ is flasque.

(d) By definition, $f_*\mathscr{F}(U) \longrightarrow f_*\mathscr{F}(V)$ is just $\mathscr{F}(f^{-1}(U)) \longrightarrow \mathscr{F}(f^{-1}(V))$ which is epic since $\mathscr{F}$ is flasque.

(e) First note that $\mathscr{G}$ is flasque: for any $V \subset U$ in X and $s \in \mathscr{G}(V)$, the section defined to be s on V and 0 elsewhere in $\mathscr{G}(U)$ restricts to s.
$\mathscr{F}$ is a sheaf so $\mathscr{F} \simeq \mathscr{F}^+$ and then by definition of sheafification we can embed $\mathscr{F}^+$ into $\mathscr{G}$.

Exercise 17 (by Yu-Ting)**.**

For $Q \in \{\bar{P}\}$, if $Q \in U$, $i_p(A)(U) = A$, hence $(i_p(A))_Q = A$. For $A \in \operatorname{Ext}\{P\}$, there exists an open set $V \subset \operatorname{Ext}\{P\}$ containing Q such that $P \notin V$ and $i_p(A)(V) = 0$, then $(i_p(A))_Q = 0$.

Exercise 19 (by Pei-Hsuan)**.**

(a) If $p \in Z$,

$$(i_*\mathscr{F})_P = \varinjlim_{P\in V} i_*\mathscr{F}(V) = \varinjlim_{P\in V\cap Z} \mathscr{F}(V \cap Z) = \mathscr{F}_P.$$

If $P \notin Z$,

$$(i_*\mathscr{F})_P = \varinjlim_{P\in V} i_*(\mathscr{F}(V) = \varinjlim_{P\in V\cap Z} \mathscr{F}(\emptyset) = 0.$$

(b) If $p \in U$,

$$(j_!(\mathscr{F}))_P = \varinjlim_{P\in V} j_!(\mathscr{F})(V) = \varinjlim_{P\in V\subseteq U} \mathscr{F}(V) = \mathscr{F}_P.$$

If $P \notin U$,

$$(j_!(\mathscr{F}))_P = \varinjlim_{P\in V} j_!(\mathscr{F})(V) = \varinjlim_{P\in V\subseteq X\setminus U} \mathscr{F}(V) = \varinjlim_{P\in V\subseteq X\setminus U} 0 = \mathscr{F}_P.$$

Since $\varphi : \mathscr{F} \to \mathscr{G}$ is a sheaf isomorphism. $\Leftrightarrow \varphi_P : \mathscr{F}_P \to \mathscr{G}_P$ is isomorphism, $\forall p \in X$. Thus, the uniqueness of $j_!(\mathscr{F})$ follows. Also, for any open set $V \subseteq U$, $j_!(\mathscr{F})|_U(V) = j_!\mathscr{F}(V) = \mathscr{F}(V) \Rightarrow j_!\mathscr{F}|_U \cong \mathscr{F}$.

(c) If $p \in U$, then $0 \to \mathscr{F}_P \to \mathscr{F}_P \to 0 \to 0$ is exact.
$\Rightarrow 0 \to (j_!(\mathscr{F}|_U)_P \to \mathscr{F}_P \to (i_*(\mathscr{F}|_Z))_P \to 0$ is exact.
If $p \in Z$, then $0 \to 0 \to \mathscr{F}_P \to \mathscr{F}_P \to 0$ is exact.
$\Rightarrow 0 \to (j_!(\mathscr{F}|_U))_P \to \mathscr{F}_P \to (i_*(\mathscr{F}|_Z))_P \to 0$ is exact.
Thus, $0 \to (j_!(\mathscr{F}|_U))_P \to \mathscr{F}_P \to (i_*(\mathscr{F}|_Z))_P \to 0$ is exact, since it is exact at every stalk.

Exercise 21 (by Shi-Xin)**.**

(a) Let U be an open subset with a covering $U = \bigcup U_i$. If there are sections $s_i \in \mathscr{I}_Y(U_i) \subset \mathcal{O}_X(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for any i, j, then there is an unique element $s \in \mathcal{O}_X(U)$ such that $s|_{U_i} = s_i$. Since $s_i|_{Y \cap U_i} = 0$ for every i, s must vanish on $\bigcup_i (Y \cap U_i) = Y \cap U$. Thus $s \in \mathscr{I}_Y(U)$.

(b) Note that $i_*\mathcal{O}_Y(U) = \mathcal{O}_Y(i^{-1}(U)) = \mathcal{O}_Y(Y \cap U)$. Since we have a short exact sequence

$$0 \to \mathscr{I}_Y(U) \to \mathcal{O}_X(U) \to \mathcal{O}_Y(Y \cap U) \to 0,$$

$i_*\mathcal{O}_Y(U) \cong \mathcal{O}_X(U)/\mathscr{I}_Y(U)$, which follows that $i_*\mathcal{O}_Y \cong \mathcal{O}_X/\mathscr{I}_Y$ as sheaves.

(c) Since $\mathcal{O}_X(\mathbb{P}^1) \cong k$, the induced map on global sections is

$$0 \to 0 \to k \to k \oplus k$$

It is obvious that the map from k to $k \oplus k$ can't be surjective.

(d) For any open subset $U \subset X$, we have the natural map $\mathcal{O}(U) \to \mathscr{K}(U)$ sending f to $\bar{f} = \frac{f}{1}$. Clearly, if $\bar{f} = 0$, then $f = \bar{f}|_U = 0$, and hence $\mathcal{O} \to \mathscr{K}$ is injective.

Furthermore, for any $q \in X$, $(\mathscr{K}/\mathcal{O})_q = \mathscr{K}_q/\mathcal{O}_q = K/\mathcal{O}_q = I_q = (i_q(I_q))_q = \sum_{p \in X}(i_p(I_p))_q$. Thus $\mathscr{K}/\mathcal{O} \cong \sum_{p \in X}(i_p(I_p))$ as sheaves by Prop1.1.

(e) It suffices to show that $K = \Gamma(X, \mathscr{K}) \to \Gamma(X, \mathscr{K}/\mathcal{O}) = \bigoplus_{p \in X} K/\mathcal{O}_p$ is surjective. Note that every element $[f]$ in $K/\mathcal{O}_p$ lifts to an element f in K. So all we need to show is that for any $f = \frac{f_1}{f_2} \in K$ and $p \in X$, there is a $g \in K$ such that $g \in \mathcal{O}_q$ for every $q \neq p$ and $g - f \in \mathcal{O}_p$.

Write $f_2 = g_2 h_2$ where $Z(g_2) = p$ and $p \notin Z(h_2)$. Then by making into one variable and partial fraction, we can write $f = \frac{g_1}{g_2} + \frac{h_1}{h_2}$ for some $g_1, h_1 \in \mathcal{O}(\mathbb{P}^1)$. Then setting $g = \frac{g_1}{g_2}$ gives the desired result.

Exercise 22 (by Yi-Heng)**.**

For $V' \subset V \subset X$ open, define $\mathscr{F}(V) = \{(s_i) | s_i \in \mathscr{F}_i(V \cap U_i), \varphi_{ij}(s_i|_{V \cap U_i \cap U_j}) = s_j|_{V \cap U_i \cap U_j}\}$ and $\mathscr{F}(V) \to \mathscr{F}(V'), (s_i) \mapsto (s_i|_{V' \cap U_i})$. Thus, $\mathscr{F}$ is a sheaf on X, and $\psi_k(W) : \mathscr{F}|_{U_k}(W) \to \mathscr{F}_k(W), (s_i) \mapsto s_k$ is an isomorphism with inverse $s_k \mapsto (\varphi_{ki}(s_k|_{W \cap U_i}))$ for each open subset W in U_k. Moreover, we have $\varphi_{kj} \circ \psi_k = \psi_j$ on $U_j \cap U_k$ by the definitions.

2 Schemes

Exercise 1 (by Chi-Kang)**.**

We have $V(f) = \{\mathfrak{p} \in \mathrm{Spec}(A) | f \in \mathfrak{p}\}$, so $D(f) = \{\mathfrak{p} \in \mathrm{Spec}(A) | f \notin \mathfrak{p}\}$. Now let $S := \{f^n | n \in N \cup \{0\}\}$, then $A_f = S^{-1}A$, and there is a one-to-one correspondence $\{\mathfrak{p} \in \mathrm{Spec}(A) | \mathfrak{p} \cap S = \emptyset\} \iff \{S^{-1}\mathfrak{p} \in \mathrm{Spec}(A_f)\}$. Now since $\mathfrak{p} \cap S = \emptyset \Leftrightarrow f \in \mathfrak{p}$, we have the underlying topology space of $\mathrm{Spec}(A_f)$ and $D(f)$ has a naturally bijection. Moreover, since $A_f \cong \mathcal{O}_X(D(f))$, we have $\mathcal{O}_{\mathrm{Spec}(A_f)} \cong \mathcal{O}(D(f))$, hence $\mathrm{Spec}(A_f) \cong (D(f)), \mathcal{O}_X|_{D(f)}$.

Exercise 3 (by Shuang-Yen)**.**

(a) Let $X = \bigcup X_\alpha$ where $X_\alpha \cong \operatorname{Spec} A_\alpha$. The only if part is trivial since localization of a reduced ring is also reduced. For the if part, let $U \subseteq X$ and let $U_\alpha = U \cap X_\alpha$, if $a \in \mathcal{O}_X(U)$ is a nilpotent element, then $a_\alpha := a|_{U_\alpha}$ is nilpotent. Since a_α is a map $U_\alpha \to \sqcup (A_\alpha)_\mathfrak{p}$ that is locally constant, for any $\mathfrak{p}$, there is a neighborhood $V_\mathfrak{p}$ such that

$$a_\alpha|_{V_\mathfrak{p}} = \frac{b_p}{s_p} \in \mathcal{O}_{X,\mathfrak{p}},$$

which implies $a_\alpha|_\mathfrak{p} = 0$, then $a_\alpha = 0$, $\forall \alpha \implies a = 0$, hence $\mathcal{O}_X(U)$ is reduced.

(b) May assuem that X is affine, say $\operatorname{Spec} A$. I claim that $X_{\mathrm{red}} \cong \operatorname{Spec} A_{\mathrm{red}}$. For the topological structure, it's clear by the fact that $\mathfrak{N}(A) = \bigcap \mathfrak{p}$. For the sheaf structure, we have morphism induced by $(\mathcal{O}_X(U))_{\mathrm{red}} \to \mathcal{O}_{\operatorname{Spec} A_{\mathrm{red}}}(U)$ of presheaves, then we have a map from $\mathcal{O}_{X_{\mathrm{red}}} \to \mathcal{O}_{\operatorname{Spec} A_{\mathrm{red}}}$ which is an isomrphism since it's an isomorphism on stalk. Then we can define the map $\varphi : X_{\mathrm{red}} \to X$ that is glued by the morphism of schemes induced by the ring homomorphism $A \to A_{\mathrm{red}}$. It's an homeomorphism since $\operatorname{Spec} A \to \operatorname{Spec} A_{\mathrm{red}}$.

(c) Let $i : Y \to Y_{\mathrm{red}}$ be the natural map. To define $g : X \to Y_{\mathrm{red}}$, $(g, g^\#)$ satisfies

$$(f, f^\#) = (i, i^\#) \circ (g, g^\#) = (i \circ g, i_* g^\# \circ i^\#) = (g, g^\# \circ i^\#).$$

So we need $f = g$ and $f^\# = g^\# \circ i^\#$, define $g = f$ and define $g^\# : \mathcal{O}_{Y_{\mathrm{red}}} \to g_*\mathcal{O}_X$ to be the morphism induced by the induced map $(\mathcal{O}_Y(U))_{\mathrm{red}} = \mathcal{O}_Y(U)/\mathfrak{N}(\mathcal{O}_Y(U)) \to \mathcal{O}_X(f^{-1}(U)) = \mathcal{O}_X(g^{-1}(U))$, which is clearly satisfies $f^\# = g^\# \circ i^\#$ and unique. To show that $g^\#_\mathfrak{p}$ is local, note that $f^\#_\mathfrak{p}$ is local then $(g^\#_\mathfrak{p})^{-1} = (f^\#_\mathfrak{p})^{-1}(\mathfrak{m}_{X,\mathfrak{p}})/\mathfrak{N}(\mathcal{O}_{Y,f(\mathfrak{p})})$ is the maximal ideal of $(\mathcal{O}_{Y,f(\mathfrak{p})})_{\mathrm{red}}$.

Exercise 5 (by Zi-Li).

By 2.4, $Hom_{Sch}(X, Spec\mathbb{Z}) = Hom(\mathbb{Z}, \Gamma(X, \mathscr{O}_X))$, however, there is only a unique $\mathbb{Z} \to \Gamma(X, \mathscr{O}_X)$. Hence, $Spec\mathbb{Z}$ is final object in category of schemes.

Exercise 6 (by Jung-Tao).

There is no prime ideal in a zero ring, so $\mathrm{Spec}R = \phi$, and is an initial object for the category of schemes.

Exercise 7 (by Tzu-Yang Chou).

Given $(\phi, \phi^{\#}) : \mathrm{Spec}\, K \longrightarrow X$, let $x := \phi(0_K) \in X$. $\phi^{\#}$ gives a local ring homomorphism $\mathscr{O}_{X,x} \longrightarrow \mathscr{O}_{\mathrm{Spec}\, K,0}$. But $\mathscr{O}_{\mathrm{Spec}\, K,0} = K \Rightarrow m_{\mathrm{Spec}\, K,0} = 0$. This induces $\mathscr{O}_{X,x}/m_{X,x} = k(x) \longrightarrow K$.
Conversely, given $x \in X$ and $k(x)$ included in K, we define $\phi : \mathrm{Spec}\, K \longrightarrow X$ by mapping the only point to x. For the sheaf map, note that for any open set U, $(\phi_* \mathscr{O}_{\mathrm{Spec}\, K}(U) = \mathscr{O}_{\mathrm{Spec}\, K}(\phi^{-1}(U)) = K$ if $x \in U$, 0 otherwise. So we can define $\phi^{\#}(U) :=$ the composition $\mathscr{O}_X(U) \longrightarrow \mathscr{O}_{X.x} \longrightarrow k(x) \longrightarrow K$ if $x \in U$, 0 otherwise.

Exercise 8 (by Shuang-Yen).

Let $f : X \to \mathrm{Spec}\, k$ and let $i : \mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2) \to \mathrm{Spec}\, k$. Let $g : \mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2) \to X$ be a k-morphism, then

$$(f, f^{\#}) \circ (g, g^{\#}) = (i, i^{\#}) \implies (f \circ g, f_* g^{\#} \circ f^{\#}) = (i, i^{\#}).$$

Note that $f \circ g = i$ is always true. If $x = g((\overline{\varepsilon}))$, then $g^{\#}_{(\overline{\varepsilon})}$ is a local homomorphism if and only if $\overline{g^{\#}_{(\overline{\varepsilon})}} : k(x) \to k((\overline{\varepsilon})) = k$ is well-defined and $g_{(\overline{\varepsilon})}(\mathfrak{m}_{X,x}) \subseteq (\overline{\varepsilon})$. This means $k(x) \cong k$ since $g^{\#} \circ f^{\#} = i^{\#}$. Also, $g_{(\overline{\varepsilon})}(\mathfrak{m}_{X,x})$ implies that $g_{(\overline{\varepsilon})}(\mathfrak{m}_{X,x}^2) = (0)$, hence $g_{(\overline{\varepsilon})}$ is uniquely determined by $\overline{g^{\#}_{(\overline{\varepsilon})}|_{\mathfrak{m}_{X,x}}} : \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2 \to k[\varepsilon]/(\varepsilon^2)$, which is equivalent to choose an element in $T_x = \mathrm{Hom}_{k(x)}(\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2, k(x))$.

Exercise 9 (by Yi-Tsung).

Take $U = \mathrm{Spec}A \underset{\text{open}}{\subseteq} X$ such that $U \cap Z \neq \emptyset$. Since $U \cap Z$ is an irreducible closed subset of U, we can write $U \cap Z = V(\mathfrak{p})$ for some $\mathfrak{p} \in U$. Since $\overline{\{\mathfrak{p}\}}^U = V(\mathfrak{p}) = U \cap Z$, we see that $\overline{\{\mathfrak{p}\}}^Z = \overline{\left(\overline{\{\mathfrak{p}\}}^{U \cap Z}\right)}^Z = \overline{U \cap Z}^Z = Z$.
If $Z = \overline{\{\mathfrak{p}_1\}} = \overline{\{\mathfrak{p}_2\}}$, since $V(\mathfrak{p}_i) = \overline{\{\mathfrak{p}_i\}} = Z$, both $\mathfrak{p}_i$ are prime ideals since Z is irreducible, and then $\mathfrak{p}_1 = \sqrt{\mathfrak{p}_1} = \sqrt{\mathfrak{p}_2} = \mathfrak{p}_2$. Hence every irreducible closed subset has a unique generic point.

Exercise 10 (by Jung-Tao).

prime ideals (points) in $\mathbb{R}[x]$ is of the form $(0), (x-a), (x-b)(x-\bar{b})$, where $a \in \mathbb{R}, b \in \mathbb{C}$, and every proper closed set is finite points without (0).

Exercise 11 (by Jung-Tao)**.**

prime ideals in $k[x]$ is (0) or (f), where f is a irreducible polynomial, and every proper closed set is finite points without (0).
The residue field of (0) is $k(x)$.
The residue field of (f) is $k[x]_f/fk[x]_f \cong k[x]/f \cong F_p^d$, where $d = \deg f$
Only one point (0) has the residue field $k(x)$, and the number of points having the residue field F_p^d is the number of irreducible polynomial degree d, denote as $N(d)$.

Notice that $p^d = \sum_{i|d} iN(i), \implies N(d) = \frac{1}{d}\sum_{i|d}\mu(\frac{d}{i})p^i$, where $\mu(x)$ is the Mobius function.

Exercise 12 (by Yi-Heng)**.**

Let $X = \coprod X_i/(x_i \in X_i \sim x_j \in X_j \Leftrightarrow \varphi_{ij}(x_i) = x_j)$ and $\psi_i : X_i \to X$ be the inclusion. Get $\mathscr{O}_X$ by glueing $(\psi_i)_*\mathscr{O}_{X_i}$ via φ_{ij}. Since X_i's are schemes, X is a scheme. By the definition, (1)-(4) can be checked directly.

Exercise 13 (by Yi-Tsung)**.**

(a) ($\Rightarrow$) Any open subsets is noetherian by ex.1.1.7(c) and is quasi-compact by ex.1.1.7(b).
($\Leftarrow$) For any chain $U_1 \subseteq U_2 \subseteq \dots$ of open subsets of X, let $U = \underset{i}{\cup} U_i \underset{\text{open}}{\subseteq} X$. Since U is quasi-compact, there is $n \in \mathbb{N}$ such that $U = \overset{n}{\underset{j=1}{\cup}} U_j$. Thus for any $k \geq n$, $U_n = U_k$, that is, X is noetherian.

(b) Let $X = \operatorname{Spec} A$ and $\{U_i\}_{i \in I}$ be any open cover of X. We may assume $U_i = D(f_i)$ for some $f_i \in A$ since $D(f_i)$ form a base. Then

$$V\left(\sum_{i \in I}(f_i)\right) = \underset{i \in I}{\cap} V((f_i)) = X \backslash \underset{i \in I}{\cup} D((f_i)) = \emptyset$$

gives $1 \in \sum\limits_{i \in I}(f_i)$. Write $1 \sum\limits_{j \in J} a_j f_j$ for some finite set $J \subseteq I$, then we have $1 \in \sum\limits_{j \in J}(f_j)$, and thus

$$\underset{j \in J}{\cup} D((f_j)) = X \backslash \underset{j \in J}{\cap} V((f_j)) = X \backslash V\left(\sum_{j \in J}(f_j)\right) = X$$

This gives a finite subcover, hence X is quasi-compact.
For instance, take $A = k[x_1, x_2, \ldots]$, then $V(x_1) \supsetneq V(x_1, x_2) \subsetneq$ is a chain of closed subsets of $\operatorname{Spec} A$, which will not terminate, hence $\operatorname{Spec} A$ is not noetherian.

(c) For any chain $V_1 \supseteq V_2 \supseteq \ldots$ of closed subsets of $\operatorname{Spec} A$, let $V_i = V(I_i)$. Then $I_1 \subseteq I_2 \subseteq \ldots$ is a chain of ideals in A. Since A is noetherian, there is $n \in \mathbb{N}$ such that for any $k \geq n$, $I_n = I_k$, which implies $V_n = V_k$. Hence $\operatorname{Spec} A$ is noetherian.

(d) Consider $A = k[x_1, x_2, \ldots] \big/ (x_1^2, x_2^2, \ldots)$. For any $\mathfrak{p} \in \operatorname{Spec} A$, $x_i^2 = 0 \in \mathfrak{p}$, thus $x_i \in \mathfrak{p}$, and then $(x_1, x_2, \ldots) \subseteq \mathfrak{p}$. However $A \big/ (x_1, x_2, \ldots) = k$ is a field, thus $(x_1, x_2, \ldots)$ is a maximal ideal. Therefore $\operatorname{Spec} A$ is just a point, which is obviously noetherian. However A is clearly not a noetherian.

Exercise 18 (by Chi-Kang)**.**

(a)
We have f is nilpotent $\Leftrightarrow f^n = 0$ for some $n \in \mathbb{N} \Leftrightarrow f \in \underline{\underline{0}} \cap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p} \Leftrightarrow V(f) = \operatorname{Spec}(A) \Leftrightarrow D(f) = \emptyset$.
(b)
Let $\varphi : A \to B$ be a ring homomorphism, and $f : \operatorname{Spec} B \to \operatorname{Spec} A$ be the induced map. Then the sheaf map $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is given by $\mathcal{O}_X(V) \to f_*\mathcal{O}_Y(V) = \mathcal{O}_Y(f^{-1}(V)$, and the map on the stalk at $\mathfrak{p}$ is $A_{f(\mathfrak{P})} \to B_{(\mathfrak{p}}$, which is injective if φ is. Conversely if $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective, we have $A = \mathcal{O}_X(X) \to f_*\mathcal{O}_Y(X) = \mathcal{O}_Y(f^{-1}(X) = \mathcal{O}_Y(Y) = B$ is also injective.
When the case φ is injective, we have $\overline{f(Y)} = \overline{f(V(0))} = \{\varphi^{-1}(\mathfrak{p}) | 0 \subset \mathfrak{p}\} = \{\mathfrak{p} \in \operatorname{Spec}(A) | \varphi^{-1}(I) \subset \mathfrak{p}\} = V(\varphi^{-1}(0)) = X$, where the last equality since $\varphi^{-1}(0) = 0$ by the injectivity of φ, so f is dominant. (c)
When φ is surjective, we can realize $B = A/I$ for some ideal I, then every prime ideal in B is in the form $\mathfrak{p}/I$ for some $\mathfrak{p} \in V(I) \subset \operatorname{Spec}(A)$. Hence $f(\mathfrak{p}/I) = \mathfrak{p}$ implies the injectivity of f, in particular this induced a bijection $\operatorname{Spec} B \to V(I)$, and $f^{-1}(V(J)) = V(J/I)$ implies the map is continuous, hence $\operatorname{Spec} B \to V(I)$ is a homeomorphism. And similar to (b) the map of the sheaves on the stalk is $A_{f(\mathfrak{P})} \to B_{(\mathfrak{p}}$, since φ is surjective, all the localization maps are surjcetive, hence $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective.
(d)
Now we have $Y \cong f(Y)$ with $f(Y)$ is a closed subset of X. Note that we have $f_*\mathcal{O}_{Y,\mathfrak{p}} = \lim_{\mathfrak{p} \in U} \mathcal{O}_Y(f^{-1}(U)) = \lim_{f^{-1}(\mathfrak{p}) \in U} \mathcal{O}_Y(U) = B_{f^{-1}(\mathfrak{p}}$, where the second equality is since f is a homeomorphism. Now consider the maps on the stalk $A_{\mathfrak{p}} \to B_{f^{-1}\mathfrak{p}}$, which is surjective by assmuption, hence by the local-global principal, $A \to B$ is also surjective.

Exercise 19 (by Yi-Tsung)**.**

((i)⇒(iii)) Let $\operatorname{Spec} A = U \cup V$ with $U \cap V = \emptyset$ and $U, V \underset{\text{clopen}}{\subseteq} \operatorname{Spec} A$. Write $U = V(I)$ and $V = V(J)$ for some ideals I, J in A. Since $V(I + J) = V(I) \cap V(J) = \emptyset$, we have $I + J = A$, and $V(IJ) = V(I) \cup V(J) = \operatorname{Spec} A$, we have $IJ = (0)$. By Chinese remainder theorem, $A = {}^{A}/_{IJ} \cong {}^{A}/_{I} \times {}^{A}/_{J}$, where ${}^{A}/_{I}, {}^{A}/_{J}$ are nonzero.
((iii)⇒(ii)) If $A \cong A_1 \times A_2$, take $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Then it is clear that $e_1 e_2 = 0, e_1^2 = e_1, e_2^2 = e_2$ and $e_1 + e_2 = 1$.
((ii)⇒(i)) For any $\mathfrak{p} \in \operatorname{Spec} A$, if $p \notin V(e_1)$, then $e_1 \notin \mathfrak{p}$. Since $e_1 e_2 = 0 \in \mathfrak{p}$, we have $e_2 \in \mathfrak{p}$, that is, $\mathfrak{p} \in V(e_2)$. Hence $\operatorname{Spec} A = V(e_1) \cup V(e_2)$. If $V(e_1) \cap V(e_2) \not\emptyset$, let $p \in V(e_1) \cap V(e_2)$, then we have $e_1, e_2 \in \mathfrak{p}$, and then $1 = e_1 + e_2 \in \mathfrak{p}$, thus $\mathfrak{p} = R$, contradiction. Therefore $V(e_1) \cap V(e_2) = \emptyset$, and thus $\operatorname{Spec} A$ is disconnected.

3 First Properties of Schemes

Exercise 1 (by Yi-Heng)**.**

One direction is followed by the definition. For the converse, we may assume $Y = SpecB = \cup(V_i = SpecB_g)$ and $f^{-1}(V_i) = \cup(U_{ij} = SpecA_{ij})$ with A_{ij} finitely generated B_i-algebra. Thus, $f^{-1}(V_i) = Spec(A_{ij})_{\overline{g}}$ where $\overline{g}$ is the image of g. Moreover, $(A_{ij})_{\overline{g}}$ is a finitely generated B-algebra since $B_g = (B_i)_g$.

Exercise 2 (by Tzu-Yang Chou)**.**

One direction is trivial. For the converse, we first write Y as the union of some open affines, say $\operatorname{Spec} A_i$, with each $f^{-1}(\operatorname{Spec} A_i)$ quasi-compact. Given some open affine $\operatorname{Spec} R \subseteq Y$, we know that there is a covering of $\operatorname{Spec} R$, consisting of $\operatorname{Spec}(A_i)_{a_i}$, which is finite since $\operatorname{Spec} R$ itself is quasi-compact.
$f^{-1}(\operatorname{Spec} A_i)$ is quasi-compact, so each of them has a finite affine covering by $\operatorname{Spec} B_{ij}$, and hence $f^{-1}(\operatorname{Spec}(A_i)_{a_i})$ also have a finite affine covering. Now $f^{-1}(\operatorname{Spec} R)$ is a finite union of quasi-compact sets; therefore, itself is quasi-compact.

Exercise 3 (by Jung-Tao)**.**

(a) by 3.1, 3.2

(b) by 3.1, 3.2, 3.3(a)

(c) Similar to 3.1, we can assume $Y = \text{Spec } B$, and we can reduce to the case $\text{Spec } A = \cup\text{Spec } A_i$, where $A_i = A_{f_i}$ are finitely generated B module, and $\cup D(f_i)$ is a cover of Spec A, which means A is a finitely generated B module by the same trick in proposition 3.2.

Exercise 4 (by Shi-Xin)**.**

($\Leftarrow$) trivial. ($\Rightarrow$) Let $Y = \bigcup SpecB_i$ be covered by affine open subsets such that for any i, $f^{-1}(SpecB_i) = SpecA_i$ and A_i is a finitely generated B_i-module. Denote ϕ_i be the canonical homomorphism from B_i to A_i. For any affine open $V = SpecB \subset Y$, we have $V \cap SpecB_i = \bigcup_{k=1}^{n_i} Spec(B_i)_{f_{ik}}$ for $f_{ik} \in B_i$. Then since V is quasi-compact, we may assume $V = \bigcup_{i=1}^{n} SpecB_i$ where $f^{-1}(SpecB_i) = SpecA_i$ and A_i is a finitely generated B_i-module. Therefore $U = f^{-1}(V) = \bigcup_{i=1}^{n} SpecA_i$ is affine by the criterion of affineness (exercise 2.17(b)). Moreover, by using the same trick in proposition 3.2., one can show that A is a finitely generated B-module.

Exercise 5 (by Shuang-Yen)**.**

(a) Let $y \in Y$ and let $\operatorname{Spec} B = V \subset Y$ that contains y. f is finite implies that $f^{-1}(V) = \operatorname{Spec} A$ for some A that is finitely generated as a B-module. Then every prime that lies over $y = \mathfrak{p} \in \operatorname{Spec} B$ is finite by some algebraic results. Indeed, we may assume that $\varphi : B \to A$ is injective, then since $B \to A$ is finite integral, localization by $B - \mathfrak{p}$ and quotient $\mathfrak{p}$ we have $K := \operatorname{Quot}(B/\mathfrak{p}) \to (A/\mathfrak{p}A)_{B-\mathfrak{p}} =: R$ is finite. K is a field implies that R has only finitely many prime ideal since it's artinian. So $f^{-1}(y)$ is finite, hence f is quasi-finite.

(b) May assume that Y is affine, then it suffices to show that $f : \operatorname{Spec} A \to \operatorname{Spec} B$ that comes from $\varphi : B \to A$ is a closed map, which is clearly true since

$$f(V(I)) = \{f(\mathfrak{p}) \mid \mathfrak{p} \supseteq I\} = V(\varphi^{-1}(I)).$$

(c) Let X be the affine line with origin doubled and let $Y = \mathbb{A}^1_k$ be the affine line with the natural morphism f. Then it satisfies the condition but not finite since X is not affine.

Exercise 6 (by Yi-Tsung)**.**

For any open affine subset $U = \operatorname{Spec} A \underset{\text{open}}{\subseteq} X$, since X is integral, A is an integral domain. Since X is irreducible, the (unique) generic point ξ must contain in U. Now ξ is corresponding to the minimal prime in A, which is just (0). Thus $\mathcal{O}_\xi = (\mathcal{O}\mid_U)_{(0)} = \operatorname{Frac}(A)$ is a field, and we see that $K(X)$ is isomorphic to the quotient field of A.

Exercise 9 (by Jung-Tao)**.**

(a) Note that in affine case, fiber product is coincide with tensor product, so

$$\operatorname{Spec}(k[x]) \times_{\operatorname{Spec}(k)} \operatorname{Spec}(k[x]) = \operatorname{Spec}(k[x] \otimes_k k[x]) = \operatorname{Spec}(k[x,y])$$

And the corresponding topological space is different from $\operatorname{Spec}(k[x] \times k[x])$, because that prime ideals in $k[x] \times k[x]$ is of the form $(f) \times k[x]$ or $k[x] \times (f)$, does not containing ideals such as $x + y$ or $xy - 1$ in $k[x,y]$

(b) Similarly,

$$\operatorname{Spec}(k(s)) \times_{\operatorname{Spec}(k)} \operatorname{Spec}(k(t)) = \operatorname{Spec}(k(s) \otimes_k k(t))$$

Note that summation of $\frac{f_1(x)}{g_1(x)} \times \frac{f_2(y)}{g_2(y)}$ can be represented as $\frac{h(x,y)}{f(x)g(y)}$, where $h \in k[x,y]$ and $f, g \in k[x]$ by reducing to the common denominator, denote this ring R. $\frac{1}{f(x)}$ is invertible in R, so prime ideals in R is corresponds to prime ideals in $k[x,y]$, and proper prime ideal in R is the prime ideal in $k[x,y]$ which does not touch $f(x)$ or $g(y)$, and is the set of curves in Zariski topology without those axis-parallel line.

Exercise 13 (by Shi-Xin)**.**

(a) If $f : X \to Y$ is a closed immersion for any affine piece $V = \operatorname{Spec} A \subset Y$, $f|_{f^{-1}(V)}$ is still a close immersion, i.e. $X \cap V = SpecA/I$ for some ideal $I \subset A$. Then clearly, f is of finite type since A/I is a finitely generated A-algebra.

(b) If $f : X \to Y$ is a quasi-compact open immersion, then for any affine open $V = SpecA \supset Y$, $f^{-1}(V) = V \cap X$ is open in V; that is, $f^{-1}(V) = \bigcup D(f_i)$ for some $f_i \in A$, and hence by quasi-compactness we might assume $f^{-1}(V) = \bigcup_{i=1}^{n} D(f_i) = \bigcup_{i=1}^{n} SpecA_{f_i}$. Since each A_{f_i} is a finitely generated A-algebra, f is of finite type.

(c) Let $f : X \to Y, g : Y \to Z$ be two of finite type morphisms. We might assume $Y = \bigcup SpecA_i, Z = \bigcup SpecB_j$ are the affine open covers satisfying the definition of finite type.

Suppose $f^{-1}(SpecA_i) = \bigcup_{k=1}^{n_i} \tilde{A_{ik}}, g^{-1}(SpecB_j) = \bigcup_{l=1}^{m_j} \tilde{B_{jl}}$. Then

$$\begin{aligned}(g \circ f)^{-1}(SpecB_j) &= f^{-1}\Big(\bigcup_{l=1}^{m_j} \tilde{B_{jl}}\Big) \\ &= f^{-1}\Big(\bigcup_{l,i}^{finite} (SpecA_i \cap \tilde{B_{jl}})\Big) \\ &= \bigcup_{l,i,k,m}^{finite} Spec(\tilde{A_{ik}})_{f^i_{lm}}\end{aligned}$$

Since each $(\tilde{A_{ik}})_{f^i_{lm}}$ is a finitely generated $\tilde{B_{jl}}$-algebra and hence a finitely generated B_j-algebra

(d) For a morphism $f : X \to S$ of finite type, we need to show that $\tilde{f} : \tilde{X} = X \times_S Y \to Y$ is of finite type.

(1) First, we might assume $S = SpecB, Y = SpecC$ and $X = \bigcup SpecA_i$ is an affine open cover s.t. A_i are a finitely generated B-algebra, and hence $A_i \otimes_B C$ are C-algebra, which shows that $\tilde{f}$ is of finite type.

(2) If $Y = \bigcup SpecC_k = \bigcup Y_i$, then by (1), $f_k : X \times_S SpecC_k \to SpecC_k$ is of finite type. Since $f^{-1}(SpecC_k) = X \times_S SpecC_k$, it follows that f is of finite type.

(3) If $S = \bigcup SpecB_j = \bigcup S_j$, denoting $g : Y \to S$, then gluing the morphisms of finite type defined by $f_i : f^{-1}(Y_i) \times_{S_i} g^{-1}(Y_i) \to g^{-1}(Y_i)$ shows that f is of finite type.

(e) It just follows from $A \otimes_S B$ is a finitely generated S-algebra whenever A, B are finitely generated S-algebra. On the other hand, one can use $(c)+(d)$ to see that $X \times_S Y \to Y \to S$ is a composition of two morphism of finite type and hence is still of finite type.

(f) Since $g \circ f$ is of finite type, there is an affine open cover $Z = \bigcup Z_k = \bigcup SpecC_k$ such that for any k, $(g \circ f)^{-1}(Z_k) = \bigcup_{i=1}^{m_k} SpecA_{k,i}$ where each $A_{k,i}$ is a finitely generated C_k-algebra. Let $Y == \bigcup Y_j = \bigcup SpecB_j$ be an affine open cover. Then we have

$$f^{-1}(Y_j) = \bigcup_k f^{-1}(g^{-1}(Z_k) \cap SpecB_j) = \bigcup_k (\bigcup_{i,l} Spec(A_{k,i})_{f_{jl}})$$

for some $f_{jl} \in A_{k,i}$ Therefore, $f^{-1}(Y_j)$ is covered by $Spec(A_{k,i})_{f_{jl}}$ where each $(A_{k,i})_{f_{jl}}$ is a finitely generated B_j-algebra, and hence f is locally of finite type. Thus by exercise 3.3.(a), f is of finite type since it is quasi-compact.

(g) Let $Y = \bigcup_{i=1}^n Y_i = \bigcup_{i=1}^n SpecB_i$ be a finite affine open cover with each B_i being Noetherian. The $X = \bigcup_{i=1}^n f^{-1}(SpecB_i) = \bigcup_{i,j} SpecA_{ij}$ is covered by finitely many affine open subsets where each A_{ij} is a finitely generated B_i-algebra. Thus $\{A_{ij}\}$ form a finite affine open cover of X with each A_{ij} being Noetherian, and hence it follows that X is Noetherian.

Exercise 14 (by Tzu-Yang Tsai)**.**

Since X is of finite type over field k, we can write $X = \operatorname{Spec} k[x_1, \ldots, x_n]/I = \operatorname{Spec} A$ for some $I \trianglelefteq k[x_1, \ldots, x_n]$. Recall that Jacobson ring is a ring such that its nilradical is equal to its Jacobson radical. Since k is a Jacobson ring, by the property of Jacobson ring, A is also a Jacobson ring. Thus $\bigcap_{\text{maximal ideal in } A} = \mathfrak{Nilrad}(A) = \mathfrak{Jrad}(A) = 0$. Combining the fact that maximal ideals and closed points are one-one correspondence, we get closed points are dense in X.
For the example of closed points not being dense without assumption, let $X = \mathbb{Z}/6\mathbb{Z}, (2), (3)$ are the only two maximal ideals, whose closure is not X.

Exercise 18 (by Wei)**.**

(In the following, X is always Zariski, and that $\mathcal{U}$ will denote the set of open sets of X.) Let $\Omega \subseteq P(P(X))$ be collection of all $\mathfrak{T} \subseteq P(X)$ satisfying

(C1) $\mathfrak{T} \supseteq \mathcal{U}$.

(C2) $\mathfrak{T}$ is closed under taking finite intersections.

(C3) $\mathfrak{T}$ is closed under taking complements.

Consider the set

$$\mathfrak{F} = \bigcap_{\mathfrak{T}\in\Omega} \mathfrak{T}$$

This set lies in Ω and is the smallest element in it. We say a subset of X is constructible if it belongs to $\mathfrak{F}$. This description of $\mathfrak{F}$ is abstract and unusable, so we try to give an explicit description. Define an ascending chain of subsets of $\mathfrak{F}$ by

$$\mathfrak{F}_1 = \mathcal{U}, \quad \mathfrak{F}_{n+1} = \{\bigcap_k^f Y_k : Y_k \in \mathfrak{F}_n\} \cup \{X \setminus Y : Y \in \mathfrak{F}_n\}$$

where the symbol "$\bigcap^f$" means finite intersection. Consider $\bigcup_{k=1}^\infty \mathfrak{F}_k$. This set satisfies the conditions (C1)-(C3), so by minimality of $\mathfrak{F}$, we have $\bigcup_{k=1}^\infty \mathfrak{F}_k = \mathfrak{F}$. Under this description, we see that elements in $\mathfrak{F}$ are really "constructible" in the sense that it can be constructed from open sets in finitely many operations consisting of taking finite intersections or taking complements.

(a) *(A set is constructible iff it is a finite disjoint union of locally closed sets)*
For brevity, we will denote the set of locally closed sets (resp. finite union of locally closed / disjoint finite union of locally closed) as $\mathcal{A}_0$ (resp. $\mathcal{A}$, $\mathcal{A}'$); schematically, we may write

$$\mathcal{A}_0 := \{U \cap C : U \in \mathcal{U}, (X \setminus C) \in \mathcal{U}\}$$
$$\mathcal{A}' := \{\bigsqcup_k^f Y_k : Y_k \in \mathcal{A}_0\}$$
$$\mathcal{A} := \{\bigcup_k^f Y_k : Y_k \in \mathcal{A}_0\}$$

we directly have the following inclusions :

$$\mathcal{A}_0 \subseteq \mathcal{A}' \subseteq \mathcal{A}$$

I claim the following :

(i) $\mathcal{A}$ is closed under taking finite intersections, taking complements.
(ii) Under the condition that X is Noetherian, $\mathcal{A} = \mathcal{A}'$.

Suppose (i) is true, we notice that since $\mathfrak{F}_0 = \mathcal{U} \subseteq \mathcal{A}_0$, we have $\mathfrak{F} \subseteq \mathcal{A}$. (this is by our explicit decomposition of $\mathfrak{F}$)
Suppose (ii) is true, we then have $\mathfrak{F} \subset \mathcal{A}'$.
Let us show $\mathcal{A}' \subseteq \mathfrak{F}$. Let U_k be open, C_k be closed, we have

$$\bigsqcup_k^f \Big(U_k \cap C_k\Big) = X \setminus \bigcap_k^f \Big((X \setminus (U_k \cap (X \setminus (X \setminus C_k)))\Big)$$

this clearly lies in $\mathfrak{F}_6$.
Now we prove the unproved claims (i), (ii) :

Proof of (i). Notice that

$$\Big(\bigcup_k^f Y_k\Big) \cap \Big(\bigcup_{k'}^f Y'_{k'}\Big) = \bigcup_k^f \bigcup_{k'}^f (Y_k \cap Y'_{k'})$$

where $Y_k, Y'_{k'} \in \mathcal{A}_0$, but it is easy to check that $\mathcal{A}_0$ is closed under finite intersections, so $\mathcal{A}$ is closed under finite intersections.
For complements, notice that

$$X \setminus \Big(\bigcup_k^f Y_k\Big) = \bigcap_k^f (X \setminus Y_k) = \bigcap_k^f \Big((X \setminus U_k) \cup (X \setminus C_k)\Big)$$

where $Y_k = U_k \cap C_k$ with U_k open, C_k closed, then we have $(X \setminus Y_k) \in \mathcal{A}$. Since $\mathcal{A}$ is closed under finite intersection, we are done. □

Proof of (ii). Recall that X is Zariski and hence Noetherian. I claim that for two elements in $\mathcal{A}_0$, their union lies in $\mathcal{A}'$. Consider

$$Y = (V \cap A) \cup (W \cap B)$$

Notice that $Y = (V \cap A) \sqcup (W \cap B \setminus V \cap A)$, and that

$$W \cap B \setminus V \cap A = \Big(W \cap (B \setminus V)\Big) \cup \Big((W \setminus A) \cap B\Big)$$

so if we inductively define sets

$$V^{i+1} = W^i \setminus A^i, \quad A^{i+1} = B^i, \quad W^{i+1} = W^i, \quad B^{i+1} = B^i \setminus V^i$$

with initial condition

$$V^0 = V, \quad A^0 = A, \quad W^0 = W, \quad B^0 = B$$

we have

$$W^i \cap B^i \setminus V^i \cap A^i = \Big((W^{i+1} \cap B^{i+1}) \cup (V^{i+1} \cap A^{i+1})\Big) \tag{$*$}$$

and so for each $m \geq 1$ that

$$Y = \Big(\bigsqcup_{i=0}^{m-1} (V^i \cap A^i)\Big) \sqcup \Big((W^m \cap B^m) \cup (V^m \cap A^m)\Big) \tag{$\star$}$$

Notice that by our definition of the W^i, B^i, we have

$$W^i = W^0, \quad B^0 \supseteq B^1 \supseteq B^2 \supset \dots$$

so since X is Noetherian, the chain

$$(W^0 \cap B^0) \supseteq (W^1 \cap B^1) \supseteq (W^2 \cap B^2) \supseteq \dots$$

stabilizes, say $(W^m \cap B^m) = (W^{m+1} \cap B^{m+1})$, but then by $(*)$, we would have $(W^m \cap B^m) \cap (V^m \cap B^m) = \emptyset$, so by $(\star)$, we have decomposed Y into a finite disjoint union of locally closed sets. This proves our claim.
Now we show that given an element in $\mathcal{A}'$ and another element in $\mathcal{A}_0$, their union lies in $\mathcal{A}'$. Write

$$Y = (U \cap C) \cup \Big(\bigsqcup_{i=0}^{n} (V_i \cap D_i) \Big)$$

Notice by above, in each $(V \cap A) \cup (W_i \cap B_i)$, we can break $(W_i \cap B_i)$ into pieces to obtain a disjoint union decomposition of $(V \cap A) \cup (W_i \cap B_i)$, and collecting all the decompositions together, we are done.
We are now ready to show that $\mathcal{A}' = \mathcal{A}$. We do this also by induction. Notice that $\mathcal{A}$ can also be written as union of its subsets $\mathcal{A}_n$ (for $n \geq 1$) defined by

$$\mathcal{A}_n = \{Y_{n-1} \cup Y_0 : Y_n \in \mathcal{A}_{n-1}, Y_1 \in \mathcal{A}_0\}$$

then since $\mathcal{A}_0 \subseteq \mathcal{A}'$ and $\mathcal{A}'$ is closed under union with elements in $\mathcal{A}_0$, we have $\mathcal{A}_n \subseteq \mathcal{A}'$ by induction, and hence $\mathcal{A} \subset \mathcal{A}'$. $\square$

(b) *(Suppose X is irreducible with generic point η and Y is constructible then Y is dense iff $\eta \in Y$.)*
Let η be the generic point of X, and let $P_0 \in Y$. Appealing to the identity :

$$\mathrm{cl}(P_0) \subseteq \mathrm{cl}(Y) \subseteq \bigcup_{P \in Y} \mathrm{cl}(P)$$

and notice that

$$\mathrm{cl}(P) = \begin{cases} \subseteq X \setminus \{\eta\}, & P \neq \eta. \\ = X, & P = \eta. \end{cases}$$

we immediately have the claim (The proof above shows that we needn't assume Y to be constructible nor X to be Zariski).
(If the above happens, it contains an open set)
Notice that for a locally closed set $C \cap U$, if $\eta \in C \cap U$, we immediately have $C = X$; this shows $U = C \cap U$. Apply the finite disjoint union decomposition to the constructible set given in (a), we are done.

(c) *(A set is closed iff constructible and stable under specialization)*
By definition of constructibility and Exercise II.3.17.(d), one of the implication is clear. For the converse, let Y be constructible and stable under specialization. By (a), there exists open sets U_k, closed sets C_k such that

$$Y = \bigsqcup_k^f (U_k \cap C_k)$$

Define the sets :

$$C := \bigcup_k^f C_k, \quad Z^0 := C, \quad Z^{m+1} := \mathrm{cl}_{Z^m}(Y \cap Z^m) = \mathrm{cl}_C(Y \cap Z^m)$$

we get a descending chain of closed subsets

$$Z^0 \supseteq Z^1 \supseteq Z^2 \supseteq \dots$$

so there exists some $l > 0$ such that $Y \cap Z^l$ is dense in Z^l. Decomposing Z^l into finite union of irreducible components, Y will contain all the generic points of these components (if not, then Y isn't dense in some of the component by (b) and will not be dense in Z^l), and since Y is stable under specialization, we have $Y \cap Z^l = Z^l$. By our definition, we get for each $m \geq 1$ that

$$(Y \cap Z^m) = (Y \cap (\mathrm{cl}_C(Y \cap Z^{m-1})) \supseteq (Y \cap Z^{m-1}) \supseteq (Y \cap Z^m)$$

we see that $Y \cap Z^{m-1} = Y \cap Z^m$. By induction, $Y \cap Z^0 = Y \cap Z^l = Z^l$, but recall that $Y \subseteq C = Z^0$, which gives us $Y = Z^l$, so Y is closed.
(A set is open iff constructible and stable under generization)
Since a set is stable under specialization iff its complement is stable under generization, we are done. For a proof of this fact, notice that

$$\begin{aligned} Y \text{ is stable under specialization} \quad &\Leftrightarrow \quad [\eta \in Y \Leftrightarrow \mathrm{cl}(\eta) \subseteq Y] \\ &\Leftrightarrow \quad [\mathrm{cl}(\eta) \subsetneq Y \Leftrightarrow \eta \notin Y] \\ &\Leftrightarrow \quad [\mathrm{cl}(\eta) \cap (X \setminus Y) \neq \emptyset \Leftrightarrow \eta \in X \setminus Y] \\ &\Leftrightarrow \quad X \setminus Y \text{ is stable under generization} \end{aligned}$$

(d) *(Inverse image of a constructible set under a continuous map is constructible)*
By (a), it suffices to show this for locally closed set, which is by

$$f^{-1}(U \cap C) = f^{-1}(U) \cap f^{-1}(C)$$

4 Separated and Proper Morphisms

Exercise 1 (by Zi-Li)**.**

Let $f : X \longrightarrow Y$ be a finite morphism between schemes. First, f is finite type since it is finite. Second, f is separated since it is an affine morphism. Last, we remain to check that f is universally closed. Let $Y \longrightarrow Y'$ and $X' = X \times_Y Y'$, closedness can be checked locally, we may assume that$Y = SpecR, X = SpecA, Y' = SpecB, X' = SpecA \otimes_R B$, where A is finite R module. Hence,$A \otimes_R B$ is finite B module, by exercise $3.5(b)$, a finite morphism is closed, this completes the proof.

Exercise 6 (by Pei-Hsuan)**.**

Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ where A, B are integral domains. Suppose $f : X \to Y$ is proper, Let $K = \operatorname{Frac}(A)$, R be any valuation ring of K containing $f(B)$. Form now on, we abuse of notion with f and the induced map $A \to B$.Then we have the commutative diagram:

$$\begin{tikzcd}
K & A \arrow[l, hook'] \arrow[dl, dashed, "\varphi"] \\
R \arrow[u, hook] & B \arrow[l, "f"] \arrow[u, "f"']
\end{tikzcd}$$

Notice that φ exists since f is proper. Thus, $A \subseteq R$, for every valuation ring R which containing $f(B)$. By Theorem 2.4.11A,

$$\overline{f(B)} = \bigcap_{f(B) \subseteq R, R:\text{valuation ring}} R.$$

Thus, $A \subseteq \overline{f(B)}$, so f is integral. f is both of finite type and integral, so f is finite. This complete the proof.

5 Sheaves of Modules

Exercise 3 (by Zi-Li).

Define $\alpha : Hom_A(M, \Gamma(X, \mathscr{F})) \to Hom_X(\widetilde{M}, \mathscr{F})$ by:
Given $\varphi : M \to \Gamma(X, \mathscr{F})$, define $\psi(D(f)) : \widetilde{M}(D(f)) = M \otimes_A A_f \to \mathscr{F}(D(f)), m \otimes a \mapsto a\varphi(m)$. Glue $\psi(D(f))$ to get $\psi : \widetilde{M} \to \mathscr{F}$.
The inverse of α is taking global section, hence, $\widetilde{\ }$ and Γ are adjoint pair.

Exercise 6 (by Tzu-Yang Tsai).

(a) ($\subseteq$) If $p \in \mathrm{Supp} m$, i.e. $m_p \neq 0$, if $p \notin V(\mathrm{Ann} m)$,
$\exists r \in \mathrm{Ann} m \setminus p$ s.t. $rm = 0 \Rightarrow \frac{m}{1} = \frac{0}{1}$ $\rightarrow\!\leftarrow$

($\supseteq$) If $p \in V(\mathrm{Ann} m)$, that is, $p \supseteq \mathrm{Ann} m \Rightarrow \nexists r \in M \setminus p$ s.t. $rm = 0 \Rightarrow m_p \neq 0$

(b) ($\subseteq$) If $p \in \mathrm{Supp}\mathscr{F}$, i.e. $\mathscr{F}_p = M_p \neq 0$, by finitely generated, we may have $M = A < m_1, \ldots, m_n >$ for some $\{m_i\}_{i=1}^n \Rightarrow p \supseteq \cap_{i=1}^n \mathrm{Ann} m_i \Rightarrow p \in V(\mathrm{Ann} M)$

($\supseteq$) If $p \in V(\mathrm{Ann} M)$, then $p \in \mathrm{Ann} m_i \forall i = 1 \sim n \Rightarrow M_p = \mathscr{F}_p \neq 0 \Rightarrow p \in \mathrm{Supp}\mathscr{F}$

(c) Since $\mathscr{F}$ is coherent, $\mathscr{F}|_U = \tilde{M}$ for some A-module M, where U is an open affine subset $= \operatorname{Spec} A$. Then $\mathrm{Supp}\mathscr{F}|_U = V(\mathrm{Ann} M)$ is closed, thus $\mathrm{Supp}\mathscr{F} = \bigcup_{U \subseteq_{\text{affine}} X}$ is closed.

(d) Recall that $0 \to \mathscr{H}_Z^0 \to \mathscr{F} \to j_*(\mathscr{F}|_U) \to 0$ is an exact sequence, where $U = X \setminus Z, j : U \hookrightarrow X$ is the inclusion map (Ex 1.20). Since j is open immersion, j_* is quasi-compact and separated, and $\mathscr{F}|_U$ is quasi-coherent, these imply $j_*(\mathscr{F}|_U)$ is quasi-coherent. Combining that $\mathscr{F}$ is also quasi-coherent, $\mathscr{H}_Z^0$ is quasi-coherent. By definition $\Gamma_Z(\mathscr{F}) = \{p \in X | \mathrm{Supp} p \subseteq Z\}$ $= \{p \in X | p \in V(\mathrm{Ann} a)\}$, since A is Noetherian $\Rightarrow a$ is finitely generated. Therefore $\Gamma_Z(\mathscr{F}) = \{m \in M | a^n m \text{for some} n \in \mathbb{N}\} \cong \Gamma_a(M)$
$\Rightarrow \Gamma_a\tilde{(M)} \cong \Gamma_Z\tilde{(\mathscr{F})} = \mathscr{H}_Z^0$

(e) The quasi-coherent case has been proved in (d).
For the coherent case, $\Gamma_Z(\mathscr{F})$ is finitely generated, thus $= \mathscr{H}_Z^0 = \Gamma_Z\tilde{(\mathscr{F})}$ is coherent.

Exercise 15 (by Shi-Xin).

(a) Let $X = \operatorname{Spec} A$ be an affine scheme where A is noetherian ring and $\mathcal{F}$ be a quasi-coherent sheaf on X. Then $\mathcal{F} = \tilde{M}$ for some A-module M. Since $M = \lim_{\alpha \in A} M_\alpha$ where $M_\alpha | \alpha \in A$ are all finitely generated A-submodule of M, for any $f \in A$, we must have $M_f = \lim_\alpha (M_\alpha)_f$. It follows that $\mathcal{F}(D(f)) = \lim_\alpha \tilde{M}(D(f))$. Moreover, because $D(f)$ form a basis, $\mathcal{F}(U) = \lim_\alpha \tilde{M}(U)$ for any open set U. Thus $\mathcal{F} = \lim_\alpha \mathcal{F}_\alpha$ where each $\mathcal{F}_\alpha \coloneqq \tilde{M}_\alpha$ is a coherent sheaf.

(b) Let $i : U \to X$ be the inclusion map. Since X is noetherian, U is also noetherian, and hence by Proposition 2.5.8, $i_*(\mathcal{F})$ is quasi-coherent. Then by (a), we can write $i_*(\mathcal{F}) = \lim_\alpha \mathcal{F}_\alpha$. Therefore for any affine open subset $V \subset U$, we must have $\mathcal{F}|_V = \lim_\alpha (\mathcal{F}_\alpha)|_V$. We might assume $\mathcal{F}|_V \cong \tilde{M}, \mathcal{F}_\alpha|_V \cong \tilde{M}_\alpha$ where M, M_α are finitely generated. Then we have $M =\cong M_\alpha$. Since M is finitely generated, there must be some M_α containing all generators of M. Moreover, U can be covered by finitely many affine open subsets, so we can choose $\mathcal{F}' \coloneqq \mathcal{F}_\alpha$ for some α such that $\mathcal{F}'|_U \cong \mathcal{F}$.

(c) Let ρ be the natural map $\mathcal{G} \to i_*(\mathcal{G}|_U)$. Since $\rho^{-1}(i_*\mathcal{F})|_U \cong \mathcal{F}$ and it is quasi-coherent, there is a coherent subsheaf $\mathcal{F}'$ of $\rho^{-1}(i_*\mathcal{F})$ such that $\mathcal{F}'_U \cong \mathcal{F}$. Moreover, $\mathcal{F}' \subset \rho^{-1}(i_*\mathcal{F}) \subset \rho^{-1}(i_*\mathcal{G}|_U) \subset \mathcal{G}$

(d) Let X be a noetherian scheme covered by affine open subsets $\bigcup_{i=1}^n U_i$. We have proved the desired result when $n = 1$. It suffices to show that we can extend over one of them at a time. We might assume $n \geq 2$ and suppose that we have a coherent subsheaf $\mathcal{F}'_1 \subset \mathcal{G}|_{U_1}$ on U_1 such that $\mathcal{F}'_1|_{U \cap U_1} \cong \mathcal{F}|_{U \cap U_1}$. Then we might apply (c) to $\mathcal{F}'_1$ on U_1, so we obtain a coherent sheaf $\mathcal{F}'_2 \subset \mathcal{G}|_{U_1}$ on $U_1 \cup U_2$ such that $\mathcal{F}'_2|_{U \cap (U_1 \cup U_2)} \cong \mathcal{F}|_{U \cap (U_1 \cup U_2)}$. Thus by induction we prove desired result. Moreover, taking $\mathcal{G} = i_*\mathcal{F}$ shows that we can extend a coherent sheaf from an open subset U to X.

(e) Clearly, $\mathcal{F} \supseteq \bigcup \mathcal{F}_\alpha$ for all coherent subsheaves $\mathcal{F}_\alpha$ of $\mathcal{F}$. Conversely, if s is a section in $\mathcal{F}(U)$ where U is an open set of X, let $\mathcal{G}$ be the subsheaf of $\mathcal{F}_U$ generated by s. Then by (d) we can extend $\mathcal{G}$ to a coherent subsheaf $\mathcal{G}'$ of F such that $s \in \mathcal{G}'(U)$. Thus $\mathcal{F} \subset \bigcup \mathcal{F}_\alpha$.

6 Divisors

Exercise 1 (by Wei-Ping)**.**

$X \times \mathbf{P}^n$ is regular and codimension one since locally it is $X \times \mathbf{A}^n$. Irreducible since it is union of two affine pieces with nonempty intersection, and reduced is again local condition, hence $X \times \mathbf{P}^n$ is integral. Also it is separated since $X \times \mathbf{P}^n \to X$ is projective and X is separated.
Now we have $\mathbf{Z} \to \mathrm{Cl}(X \times \mathbf{P}^n) \to \mathrm{Cl}(X \times \mathbf{A}^n) \to 0$, where first map is $1 \mapsto X \times H$, $H = \{x_0 = 0\}$. Since $\mathrm{Cl}(X \times \mathbf{A}^n) \simeq \mathrm{Cl}(X)$, it suffices to show the sequence is exact, then taking closure results a converse map of second map so sequence splits. Let f be element in function field, if $(f) = m \cdot (X \times H)$, then $v_{X \times H}(f) = m, f = \frac{h}{g} x_0^m$. If $m \neq 0$ then h or g involves other divisors, a contradiction. Hence $m = 0$ and first map is injective, done.

Exercise 2 (by Wei-Ping)**.**

(a) To prove well-defined, let η be generic point of divisor Y_i, then consider two choices of covering, say $\{(U_i, f_i)\}, \{(V_j, g_j)\}$. Now choose $\eta \in U_i, \eta \in V_j$, then $\frac{f_i}{f_j}$ has same valuation on open set $U_i \cap V_j$, so $\frac{f_i}{f_j} \in \Gamma(U_i \cap V_j, O^*)$. Therefore $v_{Y_i}(\overline{f_i}) = v_{Y_i}(\overline{g_j})$.

(b) We claim that $(f) \cdot X = (\overline{f})$.
$(f) \cdot X = \sum_i v_{Y_i}(f)(Y_i \cdot X) = \sum_i v_{Y_i}(f)(\sum_j v_{Y_{ij}}(\overline{f_{ij}})Y_{ij})$. Given any generic point of a divisor Z on X, say η, we consider an open neighborhood W of η disjoint with all divisors Y_i occurring in (f) such that $\eta \notin Y_i \cap X$. For every covering from Y_i, choose one open set such that η is in it, say (U_{ik_i}, f_{ik_i}), then consider product $h = \prod_i f_{ik_i}^{v_{Y_i}(f)}$ on their intersection. Then $\frac{h}{f}$ is unit on $(\bigcap_i U_{ik_i}) \cap W$ since valuation is the same on this open set and note that the coefficient is same as those in the previous sum. Therefore $v_{Y_{ij}(\overline{f})}$ is same as coefficient of divisor Y_{ij} in $(f) \cdot X$, as we desired.
Combining (a) and the fact that any divisor on $\mathbf{P}^n$ can be change to some multiple of any hyperplane, in particular, not containing X, we get a homomorphism $\mathrm{Cl}\,\mathbf{P}^n \to X$.

(c) By linearity reduce to the case where $H = V(f)$ is a hypersurface. $i(X, H; Y_i) = \mu_{p_i}(S/I_x + (f))$ where p_i is the prime ideal correspond to Y_i. The valuation ring of Y_i is $(S/I_x)_{(p_i)}$ and $\mu_{p_i}(S/I_x + (f))$ can be viewed as length of $(S/I_x + (f))_{(p_i)}$ over $(S/I_x)_{(p_i)}$. Write $f = ua^{n_i}$ in the valuation ring, then we have filtration

$$(S/I_x + (f))_{(p_i)} = (\overline{a^{n_i}}) \supsetneq (\overline{a^{n_i - 1}}) \supseteq \cdots \supsetneq 0$$

where $\overline{a^{n_i}}$ is image under quotient and localization. Therefore $n_i = i(X, H, Y_i)$, and by Bezout thm we get

$$\deg(D \cdot X) = (\deg D) \cdot (\deg X)$$

(d) $K(X) = S(X)_{((0))}$ so we can find some $f \in K^*$ such that f restricting on X is $\overline{f}$, so by (b) get $(f) \cdot X = (\overline{f})$ Hence for any principal divisor D on X, $\deg(D \cdot X) = \deg((f) \cdot X) = (\deg(f)) \cdot (\deg X) = 0$. The degree function defines a homomorphism to $\mathbf{Z}$ and we get commutative diagram

$$\begin{tikzcd} \mathrm{Cl}\,\mathbf{P}^n \arrow[r] \arrow[d, "\deg"', "\simeq"] & \mathrm{Cl}\,X \arrow[d, "\deg"] \\ \mathbf{Z} \arrow[r, "\cdot(\deg X)"] & \mathbf{Z} \end{tikzcd}$$

Exercise 4 (by Shuang-Yen)**.**

Since $\mathrm{Quot}(A) = k(x_1, \dots, x_n)[z]/(z^2 - f)$, $[\mathrm{Quot}(A) : k(x_1, \dots, x_n)] = 2$. For $\alpha = g + hz \in \mathrm{Quot}(K)$, where $g, h \in k(x_1, \dots, x_n)$, the minimal polynomial of α over $k(x_1, \dots, x_n)$ is

$$\begin{cases} X^2 - 2gX + (g^2 - h^2 f), & \text{if } h \neq 0 \\ X - g, & \text{if } h = 0 \end{cases}$$

When $h = 0$, α is integral over $k[x_1, \dots, x_n]$ if and only if $\alpha = g \in k[x_1, \dots, x_n]$ since $k[x_1, \dots, x_n]$ is a UFD.
When $h \neq 0$, α is integral over $k[x_1, \dots, x_n]$ if and only if $-2g, g^2 - h^2 f \in k[x_1, \dots, x_n]$. Since the characteristic of k is not 2, it's equivalent ti $g, h^2 f \in k[x_1, \dots, x_n]$. Write $h = a/b$ with $a, b \in k[x_1, \dots, x_n]$ and a, b are coprime to each other, then $h^2 f = a^2 f / b^2$, but f is square-free, so $h^2 f \in k[x_1, \dots, x_n]$ if and only if $b \in k^\times$ if and only if $h \in k[x_1, \dots, x_n]$. Hence, α is integral over $k[x_1, \dots, x_n]$ if and only if $g, h \in k[x_1, \dots, x_n]$ if and only if $\alpha \in A$, so A is integrally closed since z is also integral over $k[x_1, \dots, x_n]$.

Exercise 6 (by Chun-Yi)**.**

(a) ($\Rightarrow$) If P, Q, R are collinear. Let L be the line attaching P, Q, R. Since $\deg X = 3$, $L \cap X$ has only three points P, Q, R. Since $L \sim [z = 0]$, $P + Q + R \sim 3P_0$ $\Rightarrow P - P_0 + Q - Q_0 + R - R_0 \sim 0 \Rightarrow P + Q + R = 0$ in the group law of X. ($\Leftarrow$) If $P, Q, R = 0$. Let L be the line passing through P, Q. By Bezout's thm, $L \cap X$ has three points, say P, Q, S, then $P + Q + T = 0$. By ($\Rightarrow$), since the inverse of $P + Q$ is unique, $T = R \Rightarrow R \in L \Rightarrow P, Q, R$ are collinear.

(b) ($\Rightarrow$) Let L be the tangent line passing through P. By Bezout's thm, $L \cap X$ has three points, say P, P, S, then by (a), $P + P + T = 0$. Since the inverse is unique and $P + P = 0 \Rightarrow T = P_0$.
($\Leftarrow$) Let L be the tangent line at P, passing through P_0, then by Bezout's thm, L intersect X at P with multiplicity 2. By (a), $P + P + P_0 = 0 \Rightarrow P + P = 0 \Rightarrow P$ has order 2.

(c) ($\Rightarrow$) Let L be the tangent line at P, then $L \cap X = \{P, PS\}$ for some $S \in X$, and $P + P + S = 0$. Since $3P = 0$, $S = P \Rightarrow L$ intersects X at P with multiplicity 3.
($\Leftarrow$) If P is an inflection point, since $L \cap X$ has only three points, $L \cap X = \{P, P, P\}$, thus $P + P + P = 0$, again by (a).

(d) By Mordell-Weil theorem, the points of X with coordinates in $\mathbb{Q}$ form a subgroup of X. If $z = 0$, the only rational point in (0,1,0). If $z \neq 0$, it suffices to find rational points of $y^2 = x^3 - x$, which is only (1,0),(0,0),(-1,0) $\Rightarrow$ the subgroup is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Exercise 7 (by Yi-Tsung)**.**

In example 6.11.4, we have seen that there is 1-1 correspondence between the set of nonsingular closed points of X and the kernel $\mathrm{CaCl}^0 X$ of the degree map. It suffices to show that the set of nonsingular closed points of X endowed the group structure is isomorphic to $\mathbb{G}_m$. The set of nonsingular closed points of X is just $X \backslash \{(0,0,1)\}$, say $Z = \{(0,0,1)\}$. Consider $\phi : X \backslash Z \to \mathbb{G}_m$, $(x, y, z) \mapsto \dfrac{y - x}{y + x}$. Consider the coordinates change: $(x, y, z) = (4x' - 4y', -4x' - 4y', z')$, then $X \backslash Z = \{x'y'z' = (x' - y')^3\} \backslash \{(0,0,1)\}$, and $\phi(x', y', z') = \dfrac{x'}{y'}$. Now setting $y' = 1$, then $X \backslash Z = \{x'z' = (x' - 1)^3\}$ in $\mathbb{A}^2_k$ and $\phi'(x', z') := \phi(x', 1, z') = x'$, and clearly its inverse map $\mathbb{G}_m \to X \backslash Z$ is defined by $t \mapsto \left(t, \dfrac{(t-1)^3}{t}\right)$. Thus ϕ is bijective as sets. To show that ϕ is a group homomorphism, since for $p \in X \backslash Z$, we have $\phi(-P) = \phi(P)^{-1}$, hence it suffices to show that for $P, Q, R \in X \backslash Z$ collinear, we have $\phi(P)\phi(Q)\phi(R) = 1$, and it is enough to prove the same thing for ϕ', i.e. to prove $\phi'(P)\phi'(Q)\phi'(R) = 1$. Let $L : z' = ax' + b$ be the line passing through P, Q, R, then $\phi'(P), \phi'(Q), \phi'(R)$ are roots of $x'(ax' + b) = (x' - 1)^3$, thus we see that $\phi'(P)\phi'(Q)\phi'(R) = 1$, yielding that ϕ is a group homomophism, and hence an isomorphism. Therefore we see that $\mathrm{CaCl}^0 X \cong \{\text{nonsingular closed points of } X\} \cong \mathbb{G}_m$ as groups.

Exercise 8 (by Chi-Kang)**.**

(a) We need to show that $f^*(\mathscr{L} \otimes_{\mathscr{O}_Y} \mathscr{M}) \cong (f^*\mathscr{L}) \otimes_{\mathscr{O}_X} (f^*\mathscr{M})$. Note that

$$\begin{aligned}
f^*(\mathscr{L} \otimes_{\mathscr{O}_Y} \mathscr{M})(U) &= f^{-1}(\mathscr{L} \otimes_{\mathscr{O}_Y} \mathscr{M}) \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X(U) \\
&= \lim_{V \supset f(U)} (\mathscr{L}(V) \otimes_{\mathscr{O}_Y(V)} \mathscr{M}(V)) \otimes_{\mathscr{O}_Y(V)} \mathscr{O}_X(U) \\
&= \lim_{V \supset f(U)} (\mathscr{L}(V) \otimes_{\mathscr{O}_Y(V)} \mathscr{O}_Y(V)) \otimes_{\mathscr{O}_Y(V)} (\mathscr{M}(V) \otimes_{\mathscr{O}_Y(V)} \mathscr{O}_Y(V)) \otimes_{\mathscr{O}_Y(V)} \mathscr{O}_X(U) \\
&= \lim_{V \supset f(U)} (\mathscr{L}(V) \otimes_{\mathscr{O}_Y(V)} \mathscr{O}_X(U)) \otimes_{\mathscr{O}_Y(V)} (\mathscr{M}(V) \otimes_{\mathscr{O}_Y(V)} \mathscr{O}_X(U)) \\
&= (f^*\mathscr{L}(U)) \otimes_{\mathscr{O}_X(U)} (f^*\mathscr{M}(U))
\end{aligned}$$

so we are done.
($f^*(\mathscr{L} \otimes \mathscr{M}) \cong (f^*\mathscr{L}) \otimes (f^*\mathscr{M})$ holds for any sheaves, not need invertible).

(b) We need to show that $f^*\mathscr{L}(D) \cong \mathscr{L}(f^*(D))$. Since f is finite and sheaf isomorphism can be check locally, we may assume $X = \operatorname{Spec} B, Y = \operatorname{Spec} A$ with B is a finite A module, f is induced by the map $\phi : A \to B$, and both A, B are integral domain of dimension 1.
Note that $\mathscr{L}(D)(U) := \{s \in K(X) = K(U) | (div(s) + D)|_U \geq 0\}$ by construction. Now since X, Y are non-singular affine, every divisor is principal, so there is $s \in K(Y) \cong Q(A)$ s,t, $\operatorname{div}(s) = D$, hence $\mathscr{L}(D) = \widetilde{As}$, and so $\mathscr{L}(f^*D) = \widetilde{B\phi(s)}$. Hence we have

$$\begin{aligned}
f^*\mathscr{L}(D)_P &\cong \mathscr{L}(D)_{f(P)} \otimes_{\mathscr{O}_{Y,f(P)}} \mathscr{O}_{X,P} \\
&\cong As_{\phi^{-1}P} \otimes_{A_{\phi^{-1}P}} B_P \cong B(\phi(s))_P \cong \mathscr{L}(f^*D)_P.
\end{aligned}$$

(c) We need to show $f^*\mathscr{L}(D) = \mathscr{L}(D.X)$ for $D \in \operatorname{Div}(\mathbb{P}^n)$ and $D.X$ defined in 6.2. Since for $U \subset \mathbb{P}^n$ we have $\mathscr{L}(D)(U) := \{s \in K(X) = K(U) | (div(s) + D)|_U \geq 0\}$, we have

$$\begin{aligned}
f^*\mathscr{L}(D)(V) &= (f^{-1}\mathscr{L}(D) \otimes_{f^{-1}\mathscr{O}_{\mathbb{P}^n}} \mathscr{O}_X)(V) \\
&= \lim_{U \supset V} \mathscr{L}(D)(U) \otimes_{\mathscr{O}_{\mathbb{P}^n}(U)} \mathscr{O}_X(V) \\
&= \{s \in K(X) = K(U) | (div(s) + D)|_V \geq 0\} \otimes_{\mathscr{O}_{\mathbb{P}^n}|_V} \mathscr{O}_X(V) \\
&= \{s \in K(X) = K(U) | (div(s) + D.X)|_V \geq 0\} \otimes_{\mathscr{O}_X(V)} \mathscr{O}_X(V) \\
&\cong \mathscr{L}(D.X)(V).
\end{aligned}$$

So we are done.

Exercise 10 (by Tzu-Yang Chou)**.**

(a) $\mathbb{A}^1_k = \operatorname{Spec} k[x]$, so given any $\mathscr{F} \in \operatorname{Proj}(\mathbb{A}^1_k)$, we have $\mathscr{F} \simeq \tilde{M}$ for some finite $k[x]$ module M. Since $k[x]$ is a PID, M is finitely presented. Taking tilde functor we have $0 \longrightarrow \mathscr{O}_X^m \longrightarrow \mathscr{O}_X^n \longrightarrow \mathscr{F} \longrightarrow 0$.

Define $\phi : K(X) \longrightarrow \mathbb{Z}$ by $\mathscr{F} \longmapsto n - m$. Note that this is in fact mapping $\mathscr{F}$ ti the rank of the free part of M. Also, ϕ is epic since $\mathscr{O}_x^n \longmapsto n$. ϕ is mono, since $\mathscr{F} \longmapsto 0 \Leftrightarrow n = m \Leftrightarrow \exists 0 \longrightarrow \mathscr{O}_X^m \longrightarrow \mathscr{O}_X^n \longrightarrow \mathscr{F} \longrightarrow 0$, that is $[\mathscr{F}] = 0$ in $K(X)$. Hence ϕ is the desired isomorphism.

(b) The rank function define a map $K(X) \longrightarrow \mathbb{Z}$ since any short exact sequence of sheaf gives the stalk sequence at the generic point. Surjectivity is similar as in (a).

(c) The exactness at $K(X) \longrightarrow K(X \setminus Y) \longrightarrow 0$ follows from Ex(II.5.15) For the exactness at $K(Y) \longrightarrow K(X) \longrightarrow K(X \setminus Y)$, it's clear that the composition is zero, so it remains to show that if $\mathscr{F}|_{X \setminus Y} = 0$, then $\exists \mathscr{O}_Y$ module $\mathscr{G}$ such that $[i_*\mathscr{G}] = [\mathscr{F}]$ where i is the inclusion $Y \hookrightarrow X$.
Let $\mathscr{F}$ be a coherent sheaf on X only supported on Y, we'll elaborate a finite filtration $0 = \mathscr{F}_n \subseteq \cdots \subseteq \mathscr{F}_0 = \mathscr{F}$ such that $\mathscr{F}_i / \mathscr{F}_{i+1}$ is an $\mathscr{O}_Y$ module. We claim that $\mathscr{F}_i := \ker(\mathscr{F}_{i-1} \longrightarrow i_*i^*\mathscr{F}_{i-1}$ will satisfy the condition, where the map is given by adjunction.
First, $\forall U = \operatorname{Spec} A \subseteq X$, $Y \cap U$ is a closed subset of U and hence $Y \cap U = \operatorname{Spec}(A/I)$ for some ideal $I \subseteq A$. So if $\mathscr{F}|_U \simeq \tilde{M}$, them $i_*i^*\mathscr{F}|_U \simeq \widetilde{M/IM}$. Similarly, we see that $\mathscr{F}_i|_U \simeq \widetilde{I^iM}$.
Now note that $(X \setminus Y) \cap U = \bigcup_i \operatorname{Spec}(A_{x_i})$ where x_i are the generators of I. Since $\mathscr{F}$ is supported on Y, we have $M_{x_i} = 0 \Rightarrow x_i^{n_i}$ annihilate M for some n_i. There are finitely many i, so $\exists$ a uniform N, that is, $I^N M = 0$, amd thus the filtration is locally finite. X is a Noetherian scheme and in particular quasi-compact, again there exists a uniform n such that the filtration terminates at $\mathscr{F}_n$.

Exercise 11 (by Tzu-Yang Chou)**.**

(a) Consider the exact sequence $0 \longrightarrow \mathscr{I}_D \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_D \longrightarrow 0$, then we see that $\mathscr{O}_D$ is isomorphic to the direct sum of the skyscraper sheaf of $\operatorname{coker}(\mathscr{I}_{D,P} \longrightarrow \mathscr{O}_{X,P})$ at P, where P is a point whose coefficient in D is nonzero. More precisely, at P, $\mathscr{O}_D$ is $\mathscr{O}_{X,P}/m_{X,P}^{n_P}$ if we write $D = \sum_P n_P P$.
In particular, in $K(X)$, $[\mathscr{O}_D] = \sum_P [\mathscr{F}_{\mathscr{P}}]$. Now since X is a nonsingular curve, we have $\mathscr{O}_{X,P}/m_{X,P} \simeq m_{X,P}/m_{X,P}^2 \simeq m_{X,P}^i/m_{X,P}^{i+1}, \forall i$. This together with $0 \longrightarrow m_{X,P}^i/m_{X,P}^{i+1} \longrightarrow \mathscr{O}_{X,P}/m_{X,P}^i \mathscr{O}_{X,P}/m_{X,P}^{i+1} \longrightarrow 0$ gives $[\mathscr{F}_P] = n_P[k(P)]$, where $k(P)$ is the skyscraper sheaf of residue field at P. Hence, $[\mathscr{O}_D] = \sum_P n_P[k(P)] = \psi(D)$.
For D, D' : divisors with $D \sim D'$, we need to show $\psi(D) = \psi(D')$. We

may write both of them as differences of effective divisors, so we can assume they are effective. Now, $\psi(D) = [\mathscr{O}_D] = [\mathscr{O}_X] - [\mathscr{I}_D] = [\mathscr{O}_X] - [\mathscr{L}(-D)] = [\mathscr{O}_X] - [\mathscr{L}(-D')] = [\mathscr{O}_X] - [\mathscr{I}_{D'}] = [\mathscr{O}_{D'}] = \psi(D')$.

(b) Find a locally free sheaf $\mathscr{E}_0'$ with epimorphism $\mathscr{E}_0' \longrightarrow \mathscr{F}'$, where $\mathscr{F}'$ is the extension of $\mathscr{F}$ to $\bar{X}$. Restricting on X and taking the kernel, we obtain $0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{E}_0'|_X \longrightarrow \mathscr{F} \longrightarrow 0$. We claim that $\mathscr{K}$ is locally free of finite rank. At P, $\mathscr{K}_P \subseteq (\mathscr{E}_0'|_X)_P = \mathscr{O}_{X,P}^m$ for some m. Since X is nonsingular, $\mathscr{O}_{X,P}$ is PID $\Rightarrow \mathscr{K}_P$ is free of finite rank. This proves the existence of the resolution.
Independence of the choice: Conised two locally free resolutions of $\mathscr{F}$, $0 \longrightarrow \mathscr{E}_1 \longrightarrow \mathscr{E}_0 \longrightarrow \mathscr{F} \longrightarrow 0$ and $0 \longrightarrow \mathscr{E}_1' \longrightarrow \mathscr{E}_0' \longrightarrow \mathscr{F} \longrightarrow 0$ with the maps named after f_1, f_0, f_1', f_0' respectively. We construct a third locally free resolution which maps surjectively to both of them. Let $\mathscr{G}_0 := \{(u.u') \in \mathscr{E}_0 \oplus \mathscr{E}'_0 | f_0(u) = f_0'(u')\}$ and let $\mathscr{E}"_0$ be a locally free sheaf with a epimorphism $\mathscr{E}"_0 \longrightarrow \mathscr{G}_0$. Let $f"_0 : \mathscr{E}"_0 \longrightarrow \mathscr{G}_0 \longrightarrow \mathscr{F}$. Then we have two surjections $\mathscr{E}"_0 \longrightarrow \mathscr{E}_0$ and $\mathscr{E}"_0 \longrightarrow \mathscr{E}_0'$ by natural projections. Next, let $\mathscr{H}_0, \mathscr{H}_0', \mathscr{H}"_0$ be kernels of $f_0, f_0', f"_0$ respectively. Let $\mathscr{G}_1 := \{(u,u') \in \mathscr{E}_1 \oplus \mathscr{E}_1' | \exists u" \in \mathscr{H}"_0$ such that $f_1(u) = p(u"), f_1'(u") = p'(u")\}$ where p, p' are the epimorphisms from $\mathscr{H}"_0$ to $\mathscr{H}_0$ and $\mathscr{H}_0'$ respectively. We also let $\mathscr{E}"_1$ be a locally free sheaf with an epimorphism $\mathscr{E}"_1 \longrightarrow \mathscr{G}$. Then as above, we obtain that $\mathscr{E}"_1$ surjects to all $\mathscr{E}_1, \mathscr{E}_1'$ and $\mathscr{E}"_0$. In fact, this works for locally free resolutions of any length inductively.
Now consider two resolutions with an epic chain map from one to another. Taking kernels in each part and computing determinant we obtain the trivial sheaf, that is these two resolutions have the same determinant (and hecne by symmetry, any two resolutions have the same determinant.) Also, for a short exact sequence $0 \to \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}" \longrightarrow 0$, we have $\det \mathscr{F} = \det \mathscr{F}' \otimes \det \mathscr{F}"$. This gives a map from $K(X)$ to $\operatorname{Pic} X$.
To show that $\det(\psi(D)) = \mathscr{L}(D)$, first consider the effective case. Since $0 \longrightarrow \mathscr{I}_D \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_D \longrightarrow 0$ is a locally free resolution, we have $\det(\psi(D)) = \mathscr{O}_X \otimes \mathscr{I}_D^{-1} = \mathscr{I}_D^{-1} = \mathscr{L}(-D)^{-1} = \mathscr{L}(D)$. For general D, write $D = A - B$ with A, B effective and the same argument works.

(c) Given a coherent sheaf $\mathscr{F}$ on X, let $\mathscr{L} \in \operatorname{Pic}(X)$ be an ample invertible sheaf, $\exists n \in \mathbb{N}$ such that $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ is globally generated, say by $s_1, ..., s_m$. Hence $\exists$ an epimorphism $\mathscr{O}_X^m \longrightarrow \mathscr{F} \otimes \mathscr{L}^{\otimes n}$. At generic point ξ, this gives an epimorphism $K(X)^m \longrightarrow \mathscr{F}_\xi$, so $\exists 0 \leq r \leq m$ such that $K(X)^r \longrightarrow \mathscr{F}_\xi$. Call this isomorphism ϕ, and then there's a dense open set $U \subseteq X$ such that $\phi : \mathscr{O}_U^r \longrightarrow \mathscr{F} \otimes \mathscr{L}^{\otimes n}|_U$ is an isomorphism. Now consider $0 \longrightarrow \ker\phi \longrightarrow \mathscr{O}_X^r \longrightarrow \mathscr{F} \otimes \mathscr{L}^{\otimes n} \longrightarrow cok\ \phi \longrightarrow 0$. Note that $(\ker\phi)_P$ is a submodule of $\mathscr{O}_{X,P}^r$ and hence is free. Also, $(\ker\phi)_P = 0$ on U and

hence $\ker \phi = 0$ on X by Ex(II.5.7)(a). Tensoring $(\mathscr{L}^*)^{\otimes n}$, we obtain that $0 \longrightarrow (\mathscr{L}^*)^{\otimes n} \longrightarrow \mathscr{F} \longrightarrow \mathscr{T} \longrightarrow 0$. It remains to check that $\mathscr{T}$ is a torsion sheaf: looking at the stalk sequence at ξ and we have a exact sequence of vector spaces, so $\mathscr{T}_\xi = 0$.
Now $[\mathscr{F}] - r[\mathscr{O}_X] \in \mathrm{Im}\psi$: $[\mathscr{F}] = r[\mathscr{L}(D)] + [\mathscr{T}]$ by above. Hence $[\mathscr{F}] - r[\mathscr{O}_X] = [\mathscr{T}] + r([\mathscr{L}(D)] - [\mathscr{O}_X])$. The latter term lies in $\mathrm{Im}\psi$ by part (a), so it suffices to check that the class of a torsion sheaf lies in the image of ψ. But since $\mathscr{T}_\xi = 0$, $\mathrm{Supp}\mathscr{T} \subsetneq X$ is closed and hence is finitely many points; thus $\mathscr{T}$ is a direct sum of skyscraper sheaves at each $P \in \mathrm{Supp}\mathscr{T}$, which in $K(X)$ is equal to a multiple of $[k(P)]$, which lies in the image of ψ.

(d) Combining all above, we obtain a split exact sequence $0 \longrightarrow \mathrm{Pic}(X) \longrightarrow K(X) \longrightarrow \mathbb{Z} \longrightarrow 0$.

Exercise 12 (by Tzu-Yang Chou)**.**

Define degree of $\mathscr{F}$ by the degree of the determinant of $\mathscr{F}$, then all properties hold by Ex(II.6.11). For the uniqueness, we induct on $n := \mathrm{rk}\mathscr{F}$. When $n = 0$ we use (2); when $n = 1$ we use (1); when $n > 1$ we use (3) and the induction hypothesis applies.

7 Projective Morphisms

Exercise 1 (by Yi-Heng)**.**

It suffices to show that $f_P^\#$ is injective for each $P \in X$. Since $f_P^\# : \mathcal{L}_P(= \mathcal{O}_P) \twoheadrightarrow \mathcal{M}_P(= \mathcal{O}_P)$ as a homomorphism of $\mathcal{O}_P$-modules, we have $f_P^\#(a) = af_P^\#(1) = 1$ for some $a \in \mathcal{O}_P$. Thus, $f_P^\#(1)$ is a unit, which implies $f_P^\#$ is injective.

Exercise 3 (by Chi-Kang)**.**

(a) Let $\varphi : \mathbb{P}^n \to \mathbb{P}^m$ be a morphism send x to $[s_0(x), ..., s_m(x)]$, then it is induced by a graded ring homo $\bar{\varphi} : k[x_0, ..., x_m] \to k[x_0, ..., x_n]$ sends x_i to $s_i(x_0, ..., x_n)$ with each s_i homogeneous of degree d for some $d \geq 0$.

Now if φ is non-constant, then at least one of s_i (and hence all s_i) are non-constant, so $d > 0$. And since φ is a morphism, these s_i cannot have common zero other than [0,...,0], so there is at least $n+1$ equations, hence $m \geq n$.

Now we have dim(imφ) = dim(im$\bar{\varphi}$)-1 = dim$k[s_o, ..., s_m] - 1 \leq n$ since φ is a morphism. To show dim(imφ) $\geq n$. By the above inequality, it is equivalent to find $n+1$ algebraically independent element in $k[s_0, ..., s_m]$.

Taking $n+1$ of these s_i's which have no common zero other than zero in the before paragraph. Say $s_0, ..., s_n$. to show they are algebraically independent, we prove by contradiction. Suppose that there is a non zero polynomial F s,t, $F(s_0, ..., s_n) = 0$. Then F defines a hypersurface $H \subset \mathbb{A}^{n+1}$. Note that F must has no constant term, so $0 \in H$, and $F(s_0(a), ..., s_n(a)) = 0$ for all $a \in k^{n+1}$, so we get a morphism $\psi : \mathbb{A}^{n+1} \to H$ by $\psi(x) = (s_0(x), ..., s_n(x))$. Since these s_i have no common zero other than 0, therefore $\psi^{-1}((0, ..., 0)) = 0$ is a single point, so ψ is a finite morphism, but $\dim \mathbb{A}^{n+1} > \dim H$, which is a contradiction. Hence these s_i must be algebraically independent.

(b) by the condition (3), we may assume imφ is not contained in any linear subspace. So by (a), we have each s_i is homogeneous with degree $d > 0$. Now let $M_0, ..., M_N$ be the all degree d monic monomials, we have the d-uple embedding $\rho_d : \mathbb{P}^n_{x_0,...,x_n} \to \mathbb{P}^N_{z_0,...,z_N}$ by sending x to $[M_0(X), ..., M_N(x)]$. Now for each s_i we have $s_i = \sum a_{ij}M_j$ for some $a_{ij} \in k$, consider the subspace $L := V(\{s_i\})$, we get a linear projection $p : \mathbb{P}^N - L \to \mathbb{P}^m$ by $p([z_0, ..., z_n]) = [\sum a_{0j}z_j, ..., \sum a_{mj}z_j]$. Hence $p \circ \rho_d(x) = [\sum a_{0j}M_j, ..., \sum a_{mj}M_j] = [s_0(x), ..., s_m(x)] = \varphi(x)$.

Exercise 4 (by Shuang-Yen)**.**

(a) Let $\mathscr{L}$ be an ample sheaf on X over A, then there exists $m > 0$ such that $\mathscr{L}^{\otimes m}$ is very ample on X over A. So there's an immersion $X \to \mathbb{P}^r_A$ for some r, since $\mathbb{P}^r_A$ is separated over A, hence over $\mathbb{Z}$, X is separated.

(b) Let $X = U_1 \cup U_2$ where $U_i \cong \operatorname{Spec} k[x]$. Let $\mathscr{L} \in \operatorname{Pic} X$, then $\mathscr{L}|_{U_i} \cong f_i^{-1}\mathcal{O}_X(U_i) = f_i^{-1}k[x]$ for some $f_i \in k[x]$ such that

$$f_1/f_2 \in \mathcal{O}(U_1 \cap U_2)^\times = (k[x]_x)^\times = \{x^n \mid n \in \mathbb{Z}\}.$$

So there are $m, n \in \mathbb{Z}$ such that $\mathscr{L}|_{U_1} \cong x^m k[x], \mathscr{L}|_{U_2} \cong x^n k[x]$. Since $\mathscr{L} \cong x^{-m}\mathscr{L}$, we may assume $m = 0$. So $\operatorname{Pic} X \cong \mathbb{Z}$. When $n = 0$, $\mathscr{L}$ is trivial, hence generated by global sections. When $n \neq 0$, WLOG let $n > 0$, then at the point $P \in U_1$ that corresponds to $(x) \in \operatorname{Spec} k[x]$, we have $\mathcal{O}_{X,P} = k[x]_{(x)}$ and $\mathscr{L}_P = k[x]_{(x)}$. Note that $\Gamma(X, \mathscr{L}) = x^n k[x]$, so $\mathscr{L}_p$ is not generated by global sections. It's clear that $\mathcal{O}_X$ is coherent, but $\mathcal{O}_X \otimes (\mathscr{L}^{\otimes d})^{\otimes m} \cong \mathscr{L}^{\otimes md}$ is not generated by global sections, so $\mathscr{L}^{\otimes d}$ is not ample, also, $\mathcal{O}_X^{\otimes m} \otimes \mathscr{L}$ is not generated by global sections, so $\mathcal{O}_X$ is not ample. So X has no ample sheaf on it.

Exercise 5 (by Tzu-Yang Tsai)**.**

(a) Because $\mathscr{L}$ is ample, $\forall \mathscr{F}$ is coherent, $\exists n \in \mathbb{N}$s.t.$\mathscr{F} \otimes \mathscr{L}^n \forall n \geq n_0$ is generated by global sections. Since $\mathscr{F} \otimes \mathscr{L}^n, \mathscr{M}$ are coherent, $\forall \mathscr{F}$ is coherent, and tensor of sheaves that are generated by global sections is still generated by global sections, $\mathscr{F} \otimes (\mathscr{L} \otimes \mathscr{M})^n = (\mathscr{F} \otimes \mathscr{L}^n) \otimes \mathscr{M}^n$ is generated by global sections $\forall n \geq n_0$.

(b) Since $\mathscr{M}$ is coherent, $\mathscr{L}$ is ample, $\exists m_0 \in \mathbb{N}$s.t.$\mathscr{M} \otimes \mathscr{L}^m \forall m \geq m_0$ is generated by global sections. Similarly, $\mathscr{F}$ is coherent, $\exists m_1 \in \mathbb{N}$s.t.$\mathscr{F} \otimes \mathscr{L}^m \forall m \geq m_0$ is generated by global sections.
Then for $m \geq m_0 + m_1, \mathscr{F} \otimes (\mathscr{M} \otimes \mathscr{L}^m)^i = \mathscr{F} \otimes \mathscr{L}^{m_0} \otimes (\mathscr{M} \otimes \mathscr{L}^{m_1})^i \otimes \mathscr{L}^{m_0(i-1)}$ is generated by global sections $\forall i$ large enough, thereby $\mathscr{M} \otimes \mathscr{L}^m$ is ample.

(c) Since $\mathscr{L}, \mathscr{M}$ are ample, $\forall \mathscr{F}$ is coherent, $\exists m_1, m_2 \in \mathbb{N}$ s.t.
$\mathscr{F} \otimes \mathscr{L}^m, (\mathscr{F} \otimes \mathscr{L}^m) \otimes \mathscr{M}^{m'}$ are generated by global sections, then take $m = \max\{m_1, m_2\}, \mathscr{F} \otimes (\mathscr{L} \otimes \mathscr{M})^n$ is generated by global sections $\forall n \geq m$.

(d) Since $\mathscr{L}$ is very ample, $\exists i : X \xrightarrow{closedimm.} P_A^r$ s.t. $i^*(\mathcal{O}(1)) \cong \mathscr{L}$.
On the other hand, due to $\mathscr{M}$ is generated by global sections and finite (by X is of finite type over A), $\exists i : X \to P_A^s$ s.t. $\phi^*(\mathcal{O}(1)) \cong \mathscr{M}$.
Then by Segre embedding, $\exists \psi : X \to P_A^{r+s+rs}$ s.t. $\psi^*(\mathcal{O}(1)) \cong \mathscr{M} \otimes \mathscr{L}$. It's left to show ψ is a closed immersion, then as a consequence, $\mathscr{M} \otimes \mathscr{L}$ is very ample. Observe that ψ can be factor into

$$X \xrightarrow{id \times \phi} X \times P_A^s \xrightarrow{i \times id} P_A^r \times P_A^s \xrightarrow{Segre's} X \times P_A^{r+s+rs}$$

Notice $i \times id$, Segre's embedding are closed embedding, and $id \times \phi$ is $P_A^r \times P_A^s \xrightarrow{Segre's} X \times P_A^{r+s+rs}$ under base change, it's a closed embedding since P_A^r is separated. Thus ψ is closed embedding.

(e) Because $\mathscr{L}$ is ample, $\exists n_0$ s.t. $\mathscr{L}^{n_0}$ is very ample.
Besides, $\exists n_1$ s.t. $\mathscr{L}^n$ is generated by global sections $\forall n \geq n_1$, thus by (d), $\mathscr{L}^i$ is very ample $\forall i \geq n' = n_0 + n_1$.

Exercise 6 (by Pei-Hsuan).

(a) $i : X \hookrightarrow \mathbb{P}^n_k$ is closed immersion, say $X = \operatorname{Proj} S$m where $S = k[x_0, \ldots, x_n]/I(X)$.
$\mathscr{L}$ is very ample, so $\mathscr{L} = i^*\mathscr{O}_{\mathbb{P}^n_k} = S(1)^\sim$.
Also, we have $\Gamma(X, \mathscr{L}^n) = \Gamma(X, S(n)^\sim) \cong S(n)$ Thus, for n large enough, $\dim |nD| = \dim \Gamma(X, \mathscr{L}^n) - 1 = \dim S_n - 1 = P_X(n) - 1$.

(b) If $r|n$, then $|nD| = \{0\}$. Thus, $\dim |nD| = 0$.
If $r \nmid n$. Suppose $E \sim nD$ with E is effective, then $\deg rE > 0$. But $\deg rE =$ degrnD $= 0$ leads a contradiction. Thus, $|nD| = \emptyset$, so $\dim |nD| = -1$.

Exercise 8 (by Jung-Tao).

Given a morphism $f : X \to P(\mathcal{E})$, take $\mathcal{L} = f^*(O(1))$, the map $\mathcal{E} \to \mathcal{L}$ is surjective since there is a natural surjective map from $\pi^*\mathcal{E}$ to $O(1)$.

Conversely, for any local, which is free, to give a surjective morphism $\mathcal{E} = \mathcal{O}_x^n \to \mathcal{L}$ is equivalent to give n sections generate $\mathcal{L}$, and there is an unique such f s.t. $f^*\pi^*\mathcal{E} \to \mathcal{L} = f^*(O(1))$ is consistent with the map $\pi^*\mathcal{E} \to O(1)$.

Remark. It's exactly the case $X = Y,\ g = \mathrm{id}$ in proposition 7.12

Exercise 9 (by Shi-Xin).

1. Denote $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(n)$ simply by $\mathcal{O}(n)$. Consider $\phi : \operatorname{Pic} X \times \mathbb{Z} \to \operatorname{Pic} \mathbb{P}(\mathcal{E})$ by $\phi(\mathcal{L}, n) = \pi^*\mathcal{L} \otimes \mathcal{O}(n)$ where $\pi : \mathbb{P}(\mathcal{E}) \to X$.

   First, we show that this map is injective. If $\phi(\mathcal{L}, n) = \mathcal{O}_{\mathbb{P}(\mathcal{E})}$, we have $\mathcal{O}_X = \pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})} = \pi_*(\pi^*\mathcal{L} \otimes \mathcal{O}(n)) = \mathcal{L} \otimes \pi_*\mathcal{O}(n)$ where the last isomorphism follows from projection formula. Then by proposition 7.11., we know that $\pi_*\mathcal{O}(n) = S^n(\mathcal{E})$ and hence n should be 0 since $\pi_*\mathcal{O}(n)$ need to be an invertible sheaf. Therefore $\mathcal{L} = \mathcal{O}_X$.

   For surjectivity, let $M \in \operatorname{Pic} \mathbb{P}(\mathcal{E})$ and $X = \bigcup U_i = \bigcup SpecA_i$ be covered by finitely many affine open subsets such that $\mathcal{E}|_{U_i} \cong \mathcal{O}_X$. Then $V_i := \mathbb{P}^{r-1}_{A_i} = U_i \times \mathbb{P}^{r-1}$ cover $\operatorname{Pic} \mathbb{P}(\mathcal{E})$ where $r = rank(\mathcal{E})$, and we have $\operatorname{Pic} V_i \cong \operatorname{Pic} U_i \times \mathbb{Z}$. Therefore $M_i := M|_{V_i} \cong \pi_i^*\mathcal{L}_i \otimes \mathcal{O}_{v_i}(n_i)$ for some $\mathcal{L}_i \in \operatorname{Pic} U_i$, $n_i \in \mathbb{Z}$. Moreover, since $\mathcal{M}_i|_{V_i \cap V_j} \cong \mathcal{M}_j|_{V_i \cap V_j}$, it forces that $n_i = n_j = n$ for some n. Thus $\mathcal{O}_{V_i}|_{V_i \cap V_j} \cong \mathcal{O}_{V_j}|_{V_i \cap V_j}$ implies $\mathcal{L}_i|_{V_i \cap V_j} \cong \mathcal{L}_j|_{V_i \cap V_j}$. Finally, let $\mathcal{L}$ be the sheaf glued from $\mathcal{L}_i$, we have $\phi(\mathcal{L}, n) = \mathcal{M}$.

2. ($\Rightarrow$) Let ϕ be the isomorphism from $\mathbb{P}(\mathcal{E})$ to $\mathbb{P}(\mathcal{E}')$. Since $\operatorname{Pic}\mathbb{P}(\mathcal{E}) \cong \operatorname{Pic}\mathbb{P}(\mathcal{E}')$, by using the result in (a), there is an invertible sheaf $\mathcal{L} \in \operatorname{Pic}(X)$ such that

$$\phi^*\mathcal{O}_{\mathbb{P}(\mathcal{E}')} = \mathcal{O}_{\mathbb{P}(\mathcal{E})} \otimes \pi^*\mathcal{L}$$

Thus using projection formula, we have

$$\mathcal{E}' \cong \pi'_*\mathcal{O}_{\mathbb{P}(\mathcal{E}')} \cong \pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})} \otimes \pi^*\mathcal{L}) \cong \mathcal{E} \otimes \mathcal{L}$$

($\Leftarrow$) Remind that $\mathbb{P}(\mathcal{E})$ is defined by $Proj\mathscr{I}$ where $\mathscr{I} = \bigoplus_{d>0}\mathscr{I}^d$ and $\mathscr{I}^d = S^d(\mathcal{E})$. Since $\mathcal{E}' \cong \mathcal{E} \otimes \mathcal{L}$, we must have

$$\mathscr{I}'^d = S^d(\mathcal{E}') \cong S^d(\mathcal{E}) \otimes \mathcal{L}^d = \mathscr{I}^d \otimes \mathcal{L}^d$$

Then by proposition 7.9., $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E}')$

Exercise 12 (by Shuang-Yen)**.**

Since $\widetilde{X} = \mathbf{Proj}(\mathscr{S}(\mathscr{I}_Y + \mathscr{I}_Z))$ is glued by $\operatorname{Proj}(\mathscr{S}(\mathscr{I}_Y(U) + \mathscr{I}_Z(U)))$, where U is an open affine subset of X, so we may assume that $X = \operatorname{Spec} A$ is affine. Let $\mathscr{I}_Y = \widetilde{I_Y}, \mathscr{I}_Z = \widetilde{I_Z}$ where $I_Y, I_Z \lhd A$. Then $\widetilde{X} = \operatorname{Proj}(\mathscr{S}(I_Y + I_Z)), \widetilde{Y} \cong \operatorname{Proj}(\mathscr{S}(I_Y + I_Z/I_Y)), \widetilde{Z} \cong \operatorname{Proj}(\mathscr{S}(I_Y + I_Z/I_Z))$. Suppose that $p \in \widetilde{Y} \cap \widetilde{Z}$, then since

$$\mathscr{S}\left(I_Y + I_Z\,/_{I_Y}\right) = \bigoplus_{d\geq 0}\left(I_Y + I_Z\,/_{I_Y}\right)^d = \bigoplus_{d\geq 0}\left(I_Y + I_Z^d\,/_{I_Y}\right),$$

$p_d \supseteq I_Y^d$ and $p_d \supseteq I_Z^d$. In particular, $p_1 \supseteq I_Y, p_1 \supseteq I_Z$, which implies that $p_1 = I_Y + I_Z$, so $p \supseteq \bigoplus_{d>0}(I_Y + I_Z)^d$, hence $p \notin \operatorname{Proj}(\mathscr{S}(I_Y + I_Z)) = \widetilde{X}$. So $\widetilde{Y} \cap \widetilde{Z} = \varnothing$.

8 Differentials

Exercise 1 (by Yi-Tsung)**.**

(a) (Denote [M] as "Matsumura, Commutative Ring Theory") By [M] theorem 26.1, since $k(B)$ separably generated extension field, it is separable (a k-algebra A is separable if for any extension field k' of k, $A \otimes_k k'$ is reduced.) By [M] theorem 26.9, $k(B)$ is also 0-smooth (a k-algebra A is 0-smooth if for any k-algebra C and any ideal $N \trianglelefteq C$ satisfying $N^2 = 0$, and any k-homomorphism $u : A \to {}^{A}/_{C}$, there exists a lifting k-homomorphism $v : A \to C$ of u to C.) Finally by [M] theorem 25.2, we see that

$$0 \to {}^{\mathfrak{m}}/_{\mathfrak{m}^2} \to \Omega_{B/k} \otimes k(B) \to \Omega_{k(B)/k} \to 0$$

is split, and in particular exact.

Alternative solution (by Shuang-Yen): To show that δ is injective, it suffices to show that $\delta^* : \mathrm{Hom}_{k(B)}(\Omega_{B/k} \otimes k(B), k(B)) \to \mathrm{Hom}_{k(B)}(m/m^2, k(B))$ is surjective. Note that

$$\mathrm{Hom}_{k(B)}(\Omega_{B/k} \otimes k(B), k(B)) \cong \mathrm{Hom}_B(\Omega_{B/k}, k(B)) \cong \mathrm{Der}_k(B, k(B))$$

Given a map $h \in \mathrm{Hom}_{k(B)}(m/m^2, k(B))$, we define $d \in \mathrm{Der}_k(B, k(B))$ as follows: for $b \in B$, $\bar{b} \in B/m^2$. Since B/m^2 is a complete local ring and $k(B/m^2) = (B/m^2)/(m/m^2) \cong B/m = k(B)$ is a separably generated extension of k, there's $k \subseteq K \subseteq B/m^2$ such that $K \cong k(B/m^2) \cong k(B)$. Hence there is a unique way to write $\bar{b} = \lambda + c$ where $\lambda \in K, c \in m/m^2$. Define $d(b) = h(\bar{c})$, then d is a k-derivation and $\delta^*(d) = h$. So δ is injective.

(b) Say $B = A_{\mathfrak{p}}$, where A is finitely generated k-algebra and $\mathfrak{p} \in \operatorname{Spec} A$.
($\Rightarrow$) If B is a regular local ring, since $k(B)$ is separably generated over k, we have $\dim \Omega_{B/k} \otimes k(B) = \dim \Omega_{k(B)/k} + \dim {}^{\mathfrak{m}}/_{\mathfrak{m}^2} = \text{tr.deg } k(B)/k + \dim B$. Let Q be the quotient field of B, then we have $\Omega_{B/k} \otimes_B Q = \Omega_{Q/k}$. Now since k is perfect, Q is separably generated extension field of k, and so $\dim_Q \Omega_{Q/k} = \text{tr.deg } Q/k = \dim A$. Note that $\Omega_{B/k}$ is finitely generated B-module and $\dim \Omega_{B/k} \otimes_B Q = \dim A = \mathrm{ht}\mathfrak{p} + \dim {}^{A}/_{\mathfrak{p}} = \dim B + \text{tr.deg } k(B)/k$. By lemma 8.9, $\Omega_{B/k}$ is free of rank $\dim B + \text{tr.deg } k(B)/k$.
($\Leftarrow$) By (a), $\dim B = \dim \Omega_{B/k} \otimes k(B) - \text{tr.deg } k(B)/k = \dim {}^{\mathfrak{m}}/_{\mathfrak{m}^2} + \dim \Omega_{k(B)/k} - \dim \Omega_{k(B)/k} = \dim {}^{\mathfrak{m}}/_{\mathfrak{m}^2}$. Thus B is regular.

(c) Take an affine neghborhood $U = \operatorname{Spec} A \subseteq X$ of x, then $\mathcal{O}_{X,x} = A_x$. By (b), $\mathcal{O}_{X,x}$ is regular iff $\left(\Omega_{X/k}\right)_x \cong \Omega_{\mathcal{O}_{X,x}/k}$ is free of rank $\dim A_x + \text{tr.deg } k(A)/k = \dim A$. Since now X is irreducible of finite type over k, we have $\dim A = \dim X$. Thus $\mathcal{O}_{X,x}$ is regular iff $(\Omega_{X,x})_x$ is free of rank $n = \dim X$.

(d) By theorem 8.16, there is a dense open subset $V \subseteq X$ which is nonsingular on V, i.e., $V \subseteq U$. Thus U is dense, and now it suffices to show that U is an open subset. For $x \in U$, $\left(\Omega_{X/k}\right)_x$ is free of rank n, then there exists $x \in U' \underset{\text{open}}{\subseteq} X$ such that $\left(\Omega_{X/k}\right)|_{U'}$ is free of rank n. Now for any $x' \in U'$, $\left(\Omega_{X/k}\right)_{x'}$ is free of rank n, hence $x' \in U$, and we see that U is open.

Exercise 2 (by Pei-Hsuan).

Consider $B = \{(x,s) \in X \times V | s_x \in \mathfrak{m}_x \mathscr{E}_x\}$, and $\begin{array}{cccc} \pi_1: & B & \to & X \\ & (x,s) & \mapsto & x \end{array}$ For $x_0 \in X$, $\pi_1^{-1}(x_0) = \{(x_0, s) \in \{x_0\} \times V | s_{x_0} \in \mathfrak{m}_{x_0}\mathscr{E}_{x_0}\} \longleftrightarrow$ global sections that vanish at x_0. Thus, pi_1 is surjective. Also, consider $\begin{array}{cccc} \varphi: & V & \to & \mathscr{E}_{x_0} \\ & s & \mapsto & s_{x_0} \end{array}$. Since $\mathscr{E}$ is generated by V, we get

$$\dim V = \operatorname{rank} \mathscr{E} + \dim \pi_1^{-1}(x_0) = \dim X + \dim V - \operatorname{rank} \mathscr{E} < \dim V.$$

Let $\begin{array}{cccc} \pi_2: & B & \to & V \\ & (x,s) & \mapsto & s \end{array}$, then $\dim \pi_2(B) \le \dim B < \dim V$. Hence, $\exists s \in V \setminus B$ is what we want.

Now, fix this s, consider $\begin{array}{cccc} f: & \mathscr{O}_X & \to & \mathscr{E} \\ & g & \mapsto & g \cdot s \end{array}$. Notice that X is a variety, so X is integral. Thus, for all $p \in X$, $\begin{array}{cccc} f_p: & \mathscr{O}_{X,p} & \to & \mathscr{E}_p \\ & g_p & \mapsto & g_p \cdot s_p \end{array}$ is injective. Also, $\mathscr{E}_p / \mathscr{O}_{X,p} \cong \mathscr{O}_{X,p}^{\oplus(\operatorname{rank}\mathscr{E}-1)}$ is locally free, so coker f is locally free.
Hence, $0 \to \mathscr{O}_X \to \mathscr{E} \to \mathscr{E}' = \text{cokeer } f \to 0$ is exact.

Exercise 3 (by Tzu-Yang Chou).

(a) Use the first exact sequence twice and the fact that $\Omega_{X\times_S Y/X} \simeq p_Y^*\Omega_{Y/S}, \Omega_{X\times_S Y/Y} \simeq p_X^*\Omega_{X/S}$.

(b) $\omega_{X\times Y} \simeq \bigwedge^{n+m} \Omega_{X\times Y/k} \simeq \bigwedge^{n+m}(p_1^*\Omega_{X/k} \oplus p_2^*\Omega_{Y/k}) \simeq \bigwedge^n(p_1^*\Omega_{X/k}) \otimes \bigwedge^m(p_2^*\Omega_{Y/k}) \simeq p_1^*(\bigwedge \Omega_{X/k}) \otimes p_2^*(\bigwedge^m \Omega_{Y/k}) \simeq p^{*1}\omega_X \otimes p_2^*\omega_Y$.

(c) For the arithmetic genus, we first note that $p_a(Y) = 1$ since Y has degree 3. So by Ex(I.7.2) we see that $p_a(X) = p_a(Y \times Y) = -1$.
For the geometric genus, we know that $\omega_Y \simeq \mathscr{O}_Y$ and hence by (b) $\omega_{Y\times Y} \simeq \mathscr{O}_{Y\times Y} \Rightarrow p_g(X) = \dim \Gamma(X, \omega_X) = \dim \Gamma(X, \mathscr{O}_X) = 1$ since X is proper over k. (Here we use Ex(II.4.5)(d).)

Exercise 5 (by Shi-Xin).

1. Since X is nonsingular, we have $\operatorname{Pic}(X) = \operatorname{CaCl}(X) = \operatorname{Cl}(X)$. By proposition 6.5.(c), we have an exact sequence

$$\mathbb{Z} \xrightarrow{\alpha} \operatorname{Pic}(\tilde{X}) \to \operatorname{Pic}(\tilde{X} - Y') \to 0$$

where α is defined by $\alpha(n) = nY'$. In fact, if $\alpha(n) = 0$ for some $n \neq 0$, then nY' must be a principal divisor, says (f) for some $f \in K(\tilde{X})^*$. Note that $K(\tilde{X})^* = K(X)^*$. Hence by pulling back, it follows that Y is given by f, which leads to a contradiction since $codim_X(Y) \geq 2$. Therefore α is injective. Moreover, because $\tilde{X} - Y' \cong X - Y$ and $codim_X(Y) \geq 2$, we must have

$$\operatorname{Pic}(\tilde{X} - Y') \cong \operatorname{Pic}(X - Y) \cong \operatorname{Pic}(X)$$

We conclude that

$$0 \to \mathbb{Z} \to \operatorname{Pic}(\tilde{X}) \to \operatorname{Pic}(X) \to 0$$

On the other hand, the pulling back $\pi^* \operatorname{Pic}(X) \to \operatorname{Pic}(\tilde{X})$ gives the right exactness. Thus $\operatorname{Pic}(\tilde{X}) \cong \operatorname{Pic}(X) \oplus \mathbb{Z}$

2. By (a), we can write $\omega_{\tilde{X}} \cong \pi^* M \otimes \mathcal{L}(qY')$ for some invertible sheaf M on X and $q \in \mathbb{Z}$. Note that we have $\operatorname{Pic}(\tilde{X} - Y') \cong \operatorname{Pic}(X - Y) \cong \operatorname{Pic}(X)$. It follows that

$$\omega_X \cong \omega_X|_{X-Y} \cong \omega_{\tilde{X}}|_{\tilde{X}-Y'} \cong (\pi^* M \otimes \mathcal{L}(qY'))|_{X-Y} \cong M|_{X-Y} \cong M$$

Now, we may write $\omega_{\tilde{X}} \cong \pi^* \omega_X \otimes \mathcal{L}(qY')$. We are going to show that $\omega_{\tilde{X}} \cong \pi^* \omega_X \otimes \mathcal{O}_{Y'}(-q-1)$. In fact, by adjunction formula, we deduce that

$$\begin{aligned}\omega_{Y'} &\cong \omega_{\tilde{X}} \otimes \mathcal{L}(Y') \otimes \mathcal{O}_{Y'} \\ &\cong \pi^* \omega_X \otimes \mathcal{L}((q+1)Y') \otimes \mathcal{O}_{Y'} \\ &\cong \pi^* \omega_X \otimes \mathcal{O}_{Y'}(-q-1)\end{aligned}$$

Then take a closed point $y \in Y$ and let $Z = \pi^{-1}(y) = \{y\} \times_Y Y'$ be the fiber of Y' over y. Hence by Exercise 2.8.3(b), we have $\omega_Z \cong \pi_1^* \omega_y \otimes \pi_2^* \omega_{Y'} \cong \mathcal{O}_Z(-q-1)$ Since $Z \cong \mathbb{P}^{r-1}$, it is clear that $\omega_Z \cong \mathcal{O}_Z(-r)$. Thus $q = r-1$, and hence we show that $\omega_{\tilde{X}} \cong \pi^* \omega_X \otimes \mathcal{L}((r-1)Y')$.

Exercise 6 (by Shuang-Yen).

(a) ($\Rightarrow$) We have

$$\begin{aligned}
&\theta(aa') - a\theta(a') - a'\theta(a) \\
&= g(a)g(a') - g'(a)g'(a') - f(a)(g(a') - g'(a')) - f(a')(g(a) - g'(a)) \\
&= [(g(a) - f(a))\theta(a') + g(a)g'(a')] + [(g'(a) - f(a'))\theta(a) - g(a)g'(a')] \\
&= (g(a) - f(a))\theta(a') + (g'(a') - f(a'))\theta(a) \in I^2 = 0.
\end{aligned}$$

So $\theta \in \mathrm{Der}_k(A, I)$.

($\Leftarrow$) For any $\theta \in \mathrm{Hom}_A(\Omega_{A/k}, I) \cong \mathrm{Der}_k(A, I)$,

$$\begin{aligned}
g'(aa') &= g(aa') + \theta(aa') \\
&= g(a)g(a') + f(a)\theta(a') + f(a')\theta(a) \\
&= (g+\theta)(a)(g+\theta)(a') + (f(a) - g(a))\theta(a') \\
&\qquad\qquad + (f(a') - g(a'))\theta(a) - \theta(a)\theta(a') \\
&= g'(a)g'(a').
\end{aligned}$$

So g' is a homomorphism.

(b) For any i, pick $b \in B'$ such that $\overline{b} = f(\overline{x_i})$, define $h(x_i) = b$, then we may extend it to $h : k[x_1, \ldots, x_n] \to B'$. It commutes by the construction. Note that $\overline{h(\alpha)} = f(\overline{\alpha}) = f(0) = 0$ and $\overline{h}(\alpha\beta) = \overline{h}(\alpha)\overline{h}(\beta) \in I^2 = 0$, so $\overline{h}$ is well-defined. For any $\beta \in A, \overline{\alpha} \in J/J^2$, $\overline{h}(\beta\overline{\alpha}) = h(\beta\alpha) = f(\beta)h(\alpha) = f(\beta)\overline{h}(\overline{\alpha})$, which is A-linear.

(c) Since Spec$A$ is nonsingular in Spec$P = \mathbb{A}_k^n$, which is also nonsingular, so we have the exact sequence

$$0 \longrightarrow \mathscr{I}/\mathscr{I}^2 \longrightarrow \Omega_{\mathbb{A}_k^n/k} \otimes \mathcal{O}_{\mathrm{Spec}A} \longrightarrow \Omega_{\mathrm{Spec}A/k} \longrightarrow 0,$$

where $\mathscr{I} = \widetilde{J}$. Taking global sections, we have

$$0 \longrightarrow J/J^2 \longrightarrow \Omega_{P/k} \otimes A \longrightarrow \Omega_{A/k} \longrightarrow 0$$

is exact since they're quasi-coherent. Then

$$0 \to \mathrm{Hom}_A(\Omega_{A/k}, I) \to \mathrm{Hom}_P(\Omega_{P/k}, I) \to \mathrm{Hom}_A(J/J^2, I) \to \mathrm{Ext}_A^1(\Omega_{A/k}, I).$$

Note that $\Omega_{A/k}$ is locally free, which means $\Omega_{A/k}$ is projective, so $\mathrm{Ext}_A^1(\Omega_{A/k}, I) = 0$. Let $\theta \in \mathrm{Hom}_P(\Omega_{P/k}, I) \cong \mathrm{Der}_k(P, I)$ that maps to $\overline{h} : J/J^2 \to I$. Let $h' = h - \theta$, then h' is a homomorphism by letting $A = P$ and $f = f \circ \pi$ in (a) where $\pi : P \to A$ is the projection. Note that $\forall \alpha \in J$,

$$h'(\alpha) = h(\alpha) - \theta(\alpha) = h(\alpha) - \theta(d\alpha) = \overline{h}(\overline{\alpha}) - \overline{h}(\overline{\alpha}) = 0.$$

So $h'(J) = 0$, hence $g = \overline{h'} : A \to B'$ lifts f.

Exercise 7 (by Jung-Tao)**.**

We can translate the statement, $\mathcal{F}$ is coherent, and X is affine, so we ma assume $\mathcal{F} = \widetilde{M}$, and $\mathcal{P} = I$ the sheaf of ideal is isomorphic to $\mathcal{F}$ with $I^2 = 0$.

According to exercise 2.8.6, we can lift the exact sequence

$$0 \to I \to O_{X'} \to O_X \to 0$$

so the exact sequence split and the extension is the trivial one.

Exercise 8 (by Chun-Yi)**.**

Consider the rational map $X \to X'$. Let $V \subset X$ be the largest open set such that $f : V \to X'$ represents the rational map. By first fundamental sequence, we have a map $f^*\Omega_{X'/k} \to \Omega_{V/k}$. Taking q^{th} exteior power, we get $f* :$

9 Formal Schemes