Algebraic Geometry I Homework
Chapter I Varieties

A course by prof. Chin-Lung Wang

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Exercise 0 (by Kuan-Wen)**.**

This is an example of proof.

Remark. This is an example for how to write in this format.

1 Affine Varieties

Exercise 1 (by Jung-Tao)**.**

(a) $A(Y) = k[x,y]/(y - x^2) \cong k[x]$

(b) $A(Z) = k[x,y]/(xy - 1)$, which is generated by invertible elements x, y, is not isomorphic to $k[x]$

(c) If char$(k) \neq 2$, irreducible(non-degenerate) quadratic polynomial $ax^2 + bxy + cy^2 + dx + ey + f$ in $k[x,y]$ can be represented as either $x^2 - y = 0$ (if $b^2 = 4ac$) or $x^2 - y^2 = t$ (else) under linear parameterization, and notice that $xy - 1 = 0$ is equivalent to $(x+y)^2 - (x-y)^2 = 4$, so we are done.

Else if char$(k) = 2$, consider the quadratic polynomial $ax^2 + bxy + cy^2 + dx + ey + f$,

if $b \neq 0$, we can assume $d = e = 0$ under translation, $ax^2 + bxy + cy^2 = g(x,y)h(x,y)$, where g, h are linear. If $g = zh$, for some $z \in k$, the quadratic polynomial is not irreducible, so $g \neq zh$, and the coordinate ring $k[x,y]/(gh - f)$ is isomorphic to $k[x,y]/(xy - 1)$, because $k[x,y] = k[g,h]$.

Else $b = 0$, we can assume $e = 0$, else $e \neq 0$, and let $y' = y + \frac{d}{e}x$, and d becomes zero. And then we can assume $a = 0$, so the quadratic polynomial becomes $cy^2 + dx + f$, and the coordinate ring is obviously $k[x]$

Exercise 2 (by Yu-Ting)**.**

Let $f = y - x^2$ and $g = z - x^3$. Note that $Y = V(f, g)$, then Y is closed. $A := k[x, y, z]/(f, g) \xrightarrow{\sim} k[x]$, which is a domain, then (f, g) is a prime ideal, hence $Y = V(f, g)$ is an affine variety. Also, we have $I(Y) = I(V(f, g)) = \sqrt{(f, g)} = (f, g)$, and $\dim Y = \dim A(Y) = \dim k[x] = 1)$.

Exercise 3 (by Te-Lun)**.**

$$\begin{aligned} V(x^2 - yz, xz - x) &= V(x^2 - yz) \cap V(xz - x) \\ &= V(x^2 - yz) \cap [V(x) \cup V(z - 1)] \\ &= [V(x^2 - yz) \cap V(x)] \cup [V(x^2 - yz) \cap V(z - 1)] \\ &= [V(x, y) \cup V(x, z)] \cup V(x^2 - y, z - 1) \end{aligned}$$

where $(x, y), (x, z), (x^2 - y, z - 1)$ are obviously prime by taking their quotient domain.

Exercise 4 (by Shi-Xin)**.**

Clearly, the Zariski closed set in $\mathbb{A}^1$ are the sets of finitely many points. So the Zariski open sets in $\mathbb{A}^1$ are the complement of finitely many points. Since open sets of $\mathbb{A}^1 \times \mathbb{A}^1$ are the union of $U_i \times V_i$ where U_i, V_i are open in $\mathbb{A}^1$, in the sense of the product topology of their Zariski topology, the closed sets in $\mathbb{A}^1 \times \mathbb{A}^1$ consist of some point points, some horizontal lines and some vertical lines. However, there are more varied closed sets in $\mathbb{A}^2$ such like the curve defined by $y - x^2$.

Exercise 5 (by Pei-Hsuan)**.**

($\Rightarrow$) Say $B \cong k[x_1, \ldots, x_n]/I(Y)$ for some algebraic set Y in $\mathbb{A}^n$. Clearly, B is finitely generated, and since $I(Y)$ is a radical ideal, B has no nilpotent element.

($\Leftarrow$) Since B is finitely generated, we can write B as $k[x_1, \ldots, x_n]/\mathfrak{a}$ for some $n \in \mathbb{N}$ and for some ideal $\mathfrak{a}$ in $k[x_a, \ldots, x_n]$. Since B is reduced, $\mathfrak{a}$ is radical. Let $Y = V(\mathfrak{a})$, then $B \cong A(Y)$.

Exercise 7 (by Tzu-Yang Tsai)**.**

(a) To show these are equivalent, we shall show the following:

(i)$\Rightarrow$(ii) For any closed set chain in the family of closed subsets, there's a minimal element. Then the intersection of minimal elements of closed set chains, which satisfies d.c.c, has a minimal element, which is the minimal element in the family of closed subsets.

(ii)⇒(i) For any closed set chain $\{F_i\}_{i\in I}$, expand it into a family of closed set, which has a minimal element. The minimal element can be denoted as $\cap_{i=1}^n F_i$ for some $n \in \mathbb{N}$, in other word, the chain has a minimal element.

(i)⇔(iii) (And also (ii)⇔(iv)) Easily shown by taking complement.

(b) If $\{F_i\}_{i\in I}$ is a set of open cover of A, we assume the result is wrong, then $\exists\{F_i\}_{i\in\mathbb{Z}}$ s.t. $\{\cup F_i\}_{i\in\mathbb{Z}}$ is an a.c.c. in X that don't have maximal element, which contradicts to Noetherian by (a)(iii).

(c) If $Y \subset X$ is not Noetherian, i.e. $\exists Y_1 \subsetneq Y_2 \subsetneq \dots$ that is not stationary, then take closures of Y_i in X, which leads to a chain with d.c.c., which is not stationary thus a contradiction.

(d) Suppose it's not finite, for an infinite set of points $\{x_i\}_{i\in I}$, take open sets $\{F_i\}_{i\in I}$ s.t. $x_i \in F_i$ and $x_j \notin F_i \forall i \neq j$. Then $\cup_{i=1}^n F_i \subset X$, which is a chain with a.c.c, by (a)(iii), it's a contradiction. Therefore X is finite, thus has a discrete topology.

Exercise 8 (by Zi-Li)**.**

Let $Y = V(\mathfrak{P})$ be an affine variety of dimension r in $\mathbb{A}^n$, $H = V(f)$, f is an irreducible polynomial. $V(\mathfrak{P}) \cap V(f) = V(\mathfrak{P} + (f))$, say $\sqrt{\mathfrak{P} + (f)} = \cap_{i=1}^k P_i$, P_i is minimal prime divisors of $\sqrt{\mathfrak{P} + (f)}$. Work in $A/\mathfrak{P}$, $\sqrt{\mathfrak{P} + (f)}/\mathfrak{P} = \sqrt{(\bar{f})} = \cap_{i=1}^k P_i/\mathfrak{P}$, where $\bar{f}$ denotes the image of f in $A/\mathfrak{P}$. We can assume $Y \cap H \neq \emptyset$ and $Y \nsubseteq H$, i.e. $\bar{f}$ is not a unit or zero divisor in $A/\mathfrak{P}$. Note that $(\bar{f})$ and $\sqrt{(\bar{f})}$ have same minimal prime divisors, hence $htP_i/\mathfrak{P} = 1$. Hence, $htP_i = ht\mathfrak{P} + 1$, every irreducible component of $Y \cap H$ has dimension of $r - 1$.

Exercise 9 (by Shuang-Yen)**.**

Induction on r, the case $r = 0$ is trivial. Let $\mathfrak{a}$ be generated by $f_1, f_2, \dots, f_r$ and let $\mathfrak{a}' = \langle f_1, \dots, f_{r-1}\rangle$, then every irreducible component of $V(\mathfrak{a}')$ has dimension $\geq n - r + 1$. For an irreducible component V of $V(\mathfrak{a})$, suppose $\mathfrak{p} = I(V)$. Let $V(\mathfrak{a}') = W_1 \cup W_2 \cup \dots \cup W_m$ be all the irreducible component and suppose $\mathfrak{q}_k = I(W_k)$, then $\cap\mathfrak{q}_k \subset \mathfrak{p}$ implies that $\mathfrak{q}_k \subset \mathfrak{p}$ for some k. Hence $\mathfrak{p}/\mathfrak{q}_k$ is one of the minimal prime over $\langle \overline{f_r}\rangle \lhd k[x_1, \dots, x_n]/\mathfrak{q}_k$. If $\overline{f_r}$ is neither a zero divisor or a unit, then $\mathrm{ht}\mathfrak{p}/\mathfrak{q}_k = 1$. If $\overline{f_r}$ is a zero divisor, then $\overline{f_r} = 0$, which means $\mathrm{ht}\mathfrak{p}/\mathfrak{q}_k = 0$. If $\overline{f_r}$ is a unit, then $\mathfrak{p} = k[x_1, \dots, x_n]$, a contradiction. So $\mathrm{ht}\mathfrak{p}/\mathfrak{q}_k \leq 1$, which implies

$$\mathrm{ht}\mathfrak{p} \leq \mathrm{ht}\mathfrak{q}_k + 1 \leq r \implies \mathrm{dim}V \geq n - r.$$

Exercise 10 (by Yi-Tsung)**.**

(a) If $\dim X = \infty$, then there is nothing to prove. Suppose $\dim X < \infty$, for any chain $C_0 \subsetneq C_1 \subsetneq \dots$ of irreducible closed subsets in Y, we have $\overline{C_0} \subseteq \overline{C_1} \subseteq \dots$ a chain of irreducible closed subsets in X. If there exists $n \in \mathbb{N} \cup \{0\}$ such that $\overline{C_n} = \overline{C_{n+1}}$, note that $\overline{C_n \cap Y} = \overline{C_n}^Y = C_n$, where $\overline{C_n}^Y$ denote the closure of C_n in Y, then $C_n = C_{n+1}$, contradiction. Thus $\dim Y \leq \dim X$.

(b) It suffices to show that $\dim X \leq \sup_i \dim U_i$. For any chain $C_0 \subsetneq C_1 \subsetneq \dots$ of irreducible closed subsets in X, take $x \in C_0$ and i such that $x \in U_i$. Then $C_0 \cap U_i \subseteq C_1 \cap U_i \subseteq \dots$ is a chain of irreducible closed subsets in U_i. If there exists $n \in \mathbb{N} \cap \{0\}$ such that $C_n \cap U_i = C_{n+1} \cap U_i$, note that $C_j \cap U_i$ is dense in C_j, then

$$C_n = \overline{C_n \cap U_i}^{C_n} = \overline{C_n \cap U_i}^X = \overline{C_{n+1} \cap U_i}^X = C_{n+1}$$

which is a contradiction. Hence $C_0 \cap U_i \subsetneq C_1 \cap U_i \subsetneq \dots$ is a chain of irreducible closed subsets in U_i, then $\dim X \leq \dim U_i \leq \sup_i \dim U_i$. Thus $\dim X = \sup_i \dim U_i$.

(c) Consider $X = \{1, 2\}$ with open subsets $\{1, 2\}, \{2\}, \emptyset$ and $U = \{1\}$. Then $\dim U = 0$ but $\dim X = 1$.

(d) If $Y \neq X$, let $C_0 \subsetneq C_1 \subsetneq \dots \subsetneq C_{\dim Y}$ be a chain of irreducible closed subsets in Y, then $C_0 \subsetneq C_1 \subsetneq \dots \subsetneq C_{\dim Y} \subsetneq X$ is a chain of irreducible closed subsets in X, yielding that $\dim X > \dim Y$, contradiction. Hence $Y = X$.

(e) Let $X = \mathbb{N}$ with closed subsets $\{\{1, 2, \dots, n\} | \, n \in \mathbb{N}\}$. Clearly X is noetherian but $\{1\} \subsetneq \{1, 2\} \subsetneq \dots$ is a chain of irreducible closed subsets in X, therefore $\dim X = \infty$.

Exercise 11 (by Chi-Kang)**.**

First we show that Y is irreducible i,e, $I(Y)$ is prime. Now suppose $Y = V_1 \cup V_2$ with V_i proper closed subset of Y, then each V_i is of the form $V \cap V(I_i)$ for some ideal I_i of $k[x, y, z]$. Now since Y is not a subset of $V(I_i)$, we have $I(Y) \not\supseteq I_i$ for each i. Now let $f_i \in I_i \backslash I(Y)$, we have $V(f_i) \supset V(I_i)$. So we have

$$V_i \subset Y \cap V(f_i) = \{(t^3, t^4, t^5) | t \in k, f_i(t^3, t^4, t^5) = 0\}.$$

But $f_i(t^3, t^4, t^5) \neq 0$ (otherwise $f_i \in I(Y)$), there are only finitely many root of $f_i(t^3, t^4, t^5)$, thus $Y \cap V(f_i)$ must be a finite set, so does V_i, but Y is an infinite set since algebraically closed field always infinite, so it cannot be a union of two finite subsets, hence we get a contradiction and so Y is irreducible. And in the above proof we also show that every proper closed subset of Y is finite, thus any non-empty proper closed irreducible subset of Y must be a single point, so $\dim(Y) = 1$, hence $\mathrm{ht}(I(Y)) = 3 - 1 = 2$.

To show $I(Y)$ cannot be generated by 2 elements, now we give the new "degree" of variables by $\deg x = 3$, $\deg y = 4$, $\deg z = 5$. Then since 8 is the smallest integer which can be written in more than one distinct non-negative integer combination of 3,4,5, every monomial term of element in $I(Y)$ has degree at least 8. Now it is obviously that $xz - y^2, x^3 - yz, x^2y - z^2 \in I(Y)$.
Claim: for any $f, g \in I(Y)$, at least one of $xz - y^2, x^3 - yz, x^2y - z^2$ not lies in the ideal (f, g).
To prove this claim, first we write $f = \sum f_d, g = \sum g_d$ for f_d, g_d be the homogeneous part of degree d in f, g. Then since $f_d(t^3, t^4, t^5) = g_d(t^3, t^4, t^5) = 0$ for each homogeneous part of f, g, and xz, y^2 are the only two degree 8 monomial in $k[x, y, z]$, x^3, yz are the only two degree 9 monomial in $k[x, y, z]$, x^2y, z^2 are the only two degree 10 monomial in $k[x, y, z]$, we must have

$$f_8 = a_1(xz - y^2), f_9 = a_2(x^3 - yz), f_{10} = a_3(x^2y - z^2)$$
$$g_8 = b_1(xz - y^2), g_9 = b_2(x^3 - yz), g_{10} = b_3(x^2y - z^2)$$

for some $a_i, b_i \in k$. Now if $xz - y^2, x^3 - yz, x^2y - z^2$ all lies in (f, g), then for $xz - y^2$, there is some $h_1, h_2 \in k[x, y, z]$ s,t, $h_1 f + h_2 g = xz - y^2$, since in $k[x, y, z]$ any polynomial has no degree 1,2 terms, let c_i be the constant term of h_i, we must have

$$\begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Similarly if $x^3 - yz, x^2y - z^2 \in (f, g)$, there is $d_i, e_i \in k$ s,t,

$$\begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \\ e_1 & e_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

But by counting the rank it is obviously impossible. Hence at least one of $xz - y^2, x^3 - yz, x^2y - z^2$ not lies in the ideal (f, g), so any 2 elements of $I(Y)$ cannot generate the complete $I(Y)$.

Exercise 12 (by Wei-Ping)**.**

Take $xy + 1 = 0$ to see that there are two components. Assume $xy + 1 = p(s, y)q(x, y)$, then counting degree of x and y we must have p and q are both linear and is in form $xy + 1 = (ax + b)(cx + d)$. Then $ac = bd = 1, bc + ad = 0 \Rightarrow \frac{b}{a} + \frac{a}{b} = 0$, which is impossible in $\mathbb{R}$. Hence $xy + 1$ is irreducible but its zero set isn't irreducible.

2 Projective Varieties

Exercise 1 (by Chun-Yi).

If $f \in S$ is a homogeneous polynomial with positive degree such that $f(p) \in Z(\mathfrak{a})$ in $\mathbb{P}^n$, then $f(p) \in \mathbb{A}^{n+1}$. By usual Hilbert Nullstellensatz, $f^q \in \mathfrak{a}$ for some $q > 0$.

Exercise 2 (by Chun-Yi).

Exercise 4 (by Jung-Tao).

(a) If $Z(a) \neq \phi$, we have $I(Z(a)) = \sqrt{a} = a, Z(I(Y)) = \bar{Y} = Y$. And

$$Z(a) = \phi \Leftrightarrow a = S \text{ or } S_+$$

$I(\phi) = S$, so we got a one-to-one inclusion-reversing correspondence between Y an algebraic set in P^n and a a homogeneous radical ideal of S not equal to S_+ by [Ex 1.2.3].

(b)

$$Y \text{ is not irreducible } \Leftrightarrow Y = Y_1 \cup Y_2 \Leftrightarrow I(Y) = I(Y_1) \cap I(Y_2)$$

Where Y_1, Y_2 are closed and $\neq Y$.
if $I(Y) = I(Y_1) \cap I(Y_2),\ \exists x \in I(Y_1) \setminus I(Y), y \in I(Y_2) \setminus I(Y) \Rightarrow xy \in I(Y)$
So $I(Y)$ is not a prime ideal.
On the other side, if $I(Y)$ is not a prime ideal, consider its primary decomposition. The fact $I(Y)$ is reduced $\implies$ it is intersection of prime ideals more than one, so $I(Y) = I(Y_1) \cap I(Y_2)$, for some closed Y_1, Y_2, and Y is not irreducible.

(c) $I(P^n) = 0$ is a prime ideal $\implies$ P^n is irreducible

Exercise 5 (by Jung-Tao).

(a) A descending chain of irreducible closed sets in P^n corresponds to an ascending chain of prime ideals in S. So the S is a noetherian ring implies P^n is noetherian.

(b) algebraic set Y in P^n corresponds to a radical ideal in S. Consider the primary decomposition of $I(Y) = P_1 \cap ... \cap P_t$, P_i is prime since $I(Y)$ is a radical ideal. so $P_i = Z(Y_i)$, and $I(Y) = I(Y_i) \cap ... \cap I(Y_t)$, and $Y = Y_1 \cup ... \cup Y_t$, where Y_i is irreducible, and no one contains another one.
If there is a "redundantless" way to represent Y as finite unions of irreducible closed sets, it will corresponds to a primary decomposition of $I(Y)$, by the uniqueness of primary decomposition, we concluded that every algebraic set in P^n can be uniquely written as a finite union of irreducible closed sets, no one containing another.

Exercise 6 (by Tzu-Yang Tsai)**.**

Let $Y_i = \phi(U_i \cup Y), A(Y_i)$ is the coordinate ring of Y_i in U_i, then $A(Y_i) \cong ((S(Y)_{x_i})_{(0)}$, which implies $A(Y_i)[x_i, x_i^{-1}] \cong S(Y_i)_{x_i}$. Take function field of both side, we get$K(A(Y_i)[x_i, x_i^{-1}]) \cong K(A(Y_i)[x_i]) \cong K(S(Y_i)_{x_i})$.
Consequently, $\dim S(Y)$ =$\operatorname{trdeg}_k K(S(Y))$ =$\operatorname{trdeg}_k K(A(Y_i)[x_i]) = \dim(A(Y_i) + 1 = \dim Y_i + 1$. It remains to show that $\exists i \in \mathbb{N}$s.t. $\dim Y_i = \dim Y$. But $Y = \cup_{i=1}^n U_i \cup Y$ is an open cover, by Exercise 1.10, we have $\dim Y = \sup \dim Y_i$, therefore $\exists i \in \mathbb{N}$s.t. $\dim Y_i = \dim Y$.

Exercise 7 (by Yi-Heng)**.**

(a) By Ex2.6, $dim\, \mathbb{P}^n = dim\, S(\mathbb{P}^n) - 1 = dim\, k[x_0, ..., x_n] - 1 = n.$

(b) Consider $Y_i = \varphi(Y \bigcap U_i) \neq \phi$. By Ex2.6, $\dim Y = \dim Y_i = \dim \bar{Y_i} = \dim \bar{Y}$.

Exercise 8 (by Yi-Heng)**.**

Note that $\dim Y = n - 1 \Leftrightarrow \operatorname{ht}(I(Y)) = 1$.

($\Rightarrow$) Since S is an UFD, $I(Y) = (f)$ for some $f \in S$ irreducible by Prop1.12A. Thus, $Y = Z(f)$.

($\Leftarrow$) Let $Y = Z(f)$ for some $f \in S$ irreducible. Then, by Thm1.11A, $\operatorname{ht} I(Y) = \operatorname{ht}(f) = 1$ since $\deg(f) > 0$ and S is a domain.

Exercise 9 (by Yu-Chi)**.**

(a) Notice that if we given a polynomial $f(y_1, \ldots, y_n) \in k[y_1, \ldots, y_n]$, then

$$\alpha(\beta(f)) = \alpha(x_0^{\deg(f)} f(x_1/x_0, \ldots, x_n/x_0)) = f.$$

Therefore, given $f \in I(Y)$, $\beta(f)$ vanishes on $\overline{Y} \cap U_0 = Y$. Since Y is dense in $\overline{Y}$, f vanishes on whole $\overline{Y}$. This gives $\beta(I(Y)) \subset I(\overline{Y})$; hence, $\langle \beta(I(Y)) \rangle \subset I(\overline{Y})$.

For the reverse inclusion, consider $g \in I(\overline{Y})$, then $\alpha(g) = g(1, y_1, \ldots, y_n)$ vanishes on Y. As a result, $\alpha(g) \in I(Y)$. Say $\alpha(g)$ has degree r (note that $r \leq d$), then

$$\beta(\alpha(g)) = \beta(g(1, y_1, \ldots, y_n)) = x_0^r g(1, x_1/x_0, \ldots, x_n/x_0).$$

Thus, $g(x_0, \ldots, x_n) = x_0^{d-r} \beta(g) \in \langle \beta(I(Y)) \rangle$.

(b) From previous exercise I.1.2, we know that the defining ideal $I(Y)$ of affine twisted cubic Y is given by $I(Y) = \langle f_1 := x^2 - y, f_2 := x^3 - z\rangle$. Then their homogenizations are $\beta(f_1) = x^2 - yw$; $\beta(f_2) = x^3 - zw^2$.

Notice that $y^2 - xz = (y + x^2)f_2 - xf_1 \in I(Y)$ and $\beta(xz - y^2) = xz - y^2$. However, $xz - y^2 \notin \langle \beta(f_1), \beta(f_2)\rangle$. To see this, suppose there exists $f, g \in k[x, y, z, w]$ such that

$$xz - y^2 = f(x, y, z, w)(wy - x^2) + g(x, y, z, w)(w^2y - x^3).$$

By comparing degree, one see that g must be 0 and f must have degree 0. This gives a contradiction.

To obtain a generator of $\langle \beta(I(Y)\rangle$, we first see note that $\{g_1 := xz-y^2, g_2 := xy-z, g_3 := x^2-y\}$ is a set generators of $I(Y)$ since we have $x^3-z = xg_3+g_2$ and $g_3 = f_1$. In fact, $\{g_1, g_2, g_3\}$ is reduced Gröbner basis for $I(Y)$ with respect to grevlex as one can verify by hand or using Macaulay 2.

We now claim that their homogenizations $\{g_1 = xz - y^2, \beta(g_2) = xy - zw, \beta(g_3) = x^2 - yw\}$ is a set of generators for $I(\overline{Y}) = \langle \beta(I(Y))\rangle$. Let $J := \langle xz - y^2, xy - zw, x^2 - yw\rangle$, then on affine part $U(w) := \mathbb{P}^3 \setminus Z(w)$, $Z(J) \cap U(w) \cong Z(xz - y^2, xy - z, x^2 - y) \subset \mathbb{A}^3$, which is just affine twisted cubic. For points $P = [x : y : z : 0]$ on $Z(J) \cap Z(w)$, we see that the homogeneous coordinates of P satisfies

$$xz - y^2; xy = 0; x^2 = 0.$$

This gives $x = y = 0$. Hence, $P = [0 : 0 : 1 : 0]$. Therefore,

$$V(J) = \{[t : t^2 : t^3 : 1] | t \in k\} \cup \{[0 : 0 : 1 : 0]\}.$$

Since $V(J)$ is a closed subset containing Y, we have $Z(J) \supseteq \overline{Y}$, and thus $I(Z(J)) \subset I(\overline{Y})$. Conversely, for any homogeneous polynomial $f \in k[x, y, z, w]^d$ such that f vanishes on Y, i.e., $g(t) := f(t, t^2, t^3, 1) = 0$ for all $t \in k$. Since $k = \bar{k}$, k is an infinite field, $g(t)$ must be zero polynomial. Suppose that $f(0, 0, 1, 0) \neq 0$, then f must contains a term of the form cz^d for some $c \neq 0$. Then $g(t) = ct^{3d} + h(t)$, where $\deg h(t) < 3d$. Therefore, it cannot be zero polynomial. Hence, $I(\overline{Y}) \subset I(Z(J))$ and $\overline{Y} = Z(J)$. Now, $Z(J) = \overline{Y}$ is irreducible since Y is. This shows that J is a prime ideal, and hence $I(\overline{Y}) = I(Z(J)) = J$.

Although this example shows that if $\{f_1, f_2\}$ generates $I(Y)$, then $\{\beta(f_1), \beta(f_2)\}$ does not necessarily generates $I(\overline{Y})$. However, one can prove that if $\{g_1, \ldots, g_r\}$ is a Gröbner basis of I with respect to some graded monomial ordering (eg., grlex, grevlex), then their homogenization generates the ideal $\langle \beta(I)\rangle$, as illustrated by above example.

Exercise 10 (by Pei-Hsuan)**.**

(a) Clearly, $f \in I(Y)$ is generated by homogeneous polynomial, so $f(0, \ldots 0) = 0$. Also, $f(\theta^{-1}(Y) = 0$ by definition of $I(Y)$ and θ. So $I(Y) \subseteq I(C(Y))$.

On the other hand, for $g \in I(CY))$, write

$$g = \sum_{d=0}^{r} g_d,$$

where g_d is homoeneous with degree d.

Since for any $(a_0, \ldots a_n) \in C(Y)$, then $(\lambda a_0, \ldots, \lambda a_n) \in C(Y)$, $\forall \lambda \in k$. So we can see that

$$0 = \sum_{d=0}^{r} g_d(\lambda a_0, \ldots, \lambda a_n) = \sum_{d=0}^{r} g_d(a_0, \ldots, a_n)\lambda^d.$$

RHS is a polynomial in $k[\lambda]$, but it vanish on all k, so it must be a zero polynomial. Thus, $g_d(a_0, \ldots a_n) = 0$, $\forall d$. Then $g_d \in I(Y), \forall d \Rightarrow g \in I(Y)$.

(b) $C(Y)$ is irreducible. $\Leftrightarrow$ $I(C(Y))$ is prime. $\Leftrightarrow$ $I(Y)$ is prime. $\Leftrightarrow$ Y is irreducible.

(c)
$$\begin{aligned} \dim C(Y) &= \dim k[x_0, \ldots, x_n]/I(C(Y)) = \dim k[x_0, \ldots, x_n]/I(Y) \\ &= \dim S(Y) = \dim Y + 1 \quad \text{(The last equality is due to exercise 1.2.6.)} \end{aligned}$$

Exercise 12 (by Wei-Ping)**.**

(a) $k[y_0, \ldots, y_N]/\ker\theta \to k[x_0, \ldots, x_n]$ is injective, since $k[x_0, \ldots, x_n]$ is domain, image of θ is also a domain, hence kernel is a prime ideal(so a radical ideal). For any $\theta(f) = 0$, their homogeneous part must also map to 0, hence kernel is a homogeneous ideal.

(b) $\mathrm{Im}\ \rho_d \subseteq Z(\ker\theta)$ since for any $f \in \ker\theta$, choose any $(M_0(a) \ldots M_n(a)) \in \mathrm{Im}\ \rho_d$, $f(M_0(a) \ldots M_n(a)) = \theta(f)(a) = 0$. To prove the converse, we construct preimage. First we pair number from 0 to N with n-tuple $(e_0, \ldots, e_n)$ such that $\sum_{i=0}^{n} e_i = d$, and let $M_k(b) = \prod_{i=0}^{n} b_i^{e_i}$, where $b = (b_0, \ldots, b_n)$. Consider point $a = (a_0, \ldots, a_N) \in Z(\ker\theta)$, construct point $\bar{a} = (a_{(d,0,\ldots 0)}, a_{(d-1,1,0,\ldots 0)}, a_{(d-1,0,1,\ldots,0)}, \ldots, a_{(d-1,0,\ldots,0,1)})$.

Now we claim $\rho_d(\bar{a}) = a$ by saying $\rho_d(\bar{a})_{(e_0,\ldots,e_n)} = a^{d-1}_{(d,0,\ldots,0)} a_{(e_0,\ldots,e_n)}$ for any $(e_0, \ldots, e_n)$. Then it suffices to show that $f = x^{e_0}_{(d,0,\ldots,0)} \prod_{i=1}^{n} x^{e_i}_{(d-1,0,\ldots,1,\ldots,0)} - x^{d-1}_{(d,0,\ldots,0)} x_{(e_0,\ldots,e_n)} \in \ker\theta$ for any $(e_0, \ldots, e_n)$, which holds since both two terms are sent to $y_0^{e_0+d(d-1)}\prod_{i=1}^n y_i^{e_i}$. Note that this method works with $a_{(d,0,\dots 0)}\neq 0$, since not all $a_{(0,\dots,d,\dots,0)}=0$, by suitable reordering we can always assume it is nonzero.

(c) By (b) and injectivity(fixing some $b_k\neq 0$, and compare $b_k^{d-1}b_j$ for $0\leq j\leq n$ to see that ratio of $(b_0,\dots,b_n)$ is fixed), the map is bijective. Now the map in (b) is inverse map, and both are continuous since both map is in polynomial form, so any preimage of closed set define by equations is still defined by polynomial equations, which is closed. Hence the map is homeomorphism, in fact, it is an isomorphism.

(d) The embedding is $(x_0,x_1)\to(x_0^3,x_0^2x_1,x_0x_1^2,x_1^3)$, assume $x_0\neq 0$, then it is twist cubic curve $(1,t,t^2,t^3)$ where $t=\frac{x_1}{x_0}$.

Exercise 13 (by Shuang-Yen)**.**

Let $Y=V(\mathfrak{a})$ as in the previous exercise and let $Z=Y\cap V(I)=V(\sqrt{\mathfrak{a}+I})$ for some radical homogeneous ideal I. Since $\dim Z=1$ and $\dim Y=\dim\mathbb{P}^2=2$, $\mathrm{ht}\sqrt{\overline{I}}=1$ in $k[y_0,\dots,y_5]/\mathfrak{a}\cong\bigoplus_{m\geq 0}S_{2m}\cong k[z_0,z_1,z_2]$ which is a UFD. So $\sqrt{\overline{I}}=\langle\overline{f}\rangle$ for some $f\in k[y_0,\dots,y_5]$ implies that $\sqrt{\mathfrak{a}+I}=\sqrt{\mathfrak{a}+\langle f\rangle}$, so

$$Z=V(\sqrt{\mathfrak{a}+\langle f\rangle})=V(\mathfrak{a}+\langle f\rangle)=V(\mathfrak{a})\cap V(f)=Y\cap V,$$

where $V=V(f)$ is a hypersurface.

Exercise 14 (by Tzu-Yang Chou)**.**

We need to show that the ideal defined by the kernel of the map $\phi:k[z_{ij}|i=0,\cdots,r,j=0,\cdots,s]\to k[x_0,\cdots,x_r,y_0,\cdots,y_s]$ will define the image of Segre embedding.

First, we note that the image of ψ is equal to $Z(I)$, where I is the ideal generated by binomials of the form $z_{ij}z_{kl}-z_{il}z_{kj}$. One side of the inclusion is clear. For the converse, given any $p=[z_{00}:\cdots:z_{rs}]\in Z(I)$, we may choose some $z_{ij}\neq 0$ and then define $a_k:=\frac{z_{kj}}{z_{ij}}$ for $k\neq i$ and $a_i:=1$. Similarly, we define $b_l:=\frac{z_{il}}{z_{ij}}$ for $l\neq j$ and $b_j:=1$. Then we found that $([a_0:\cdots:a_r],[b_0:\cdots:b_s])$ will map to the point p under the map ψ.

Now, it remains to show that kernel of ϕ is exactly the same as I since then I will be a prime ideal and hence our assertion is proved. Again, one inclusion is obvious. Conversely, for a polynomial f with $\phi(f)=0$, we may assume f is homogeneous by looking at each of its homogeneous part. By the algebraic independence of x_i and y_j, we see that the sum of the coefficients of terms, whose second indices differ by a permutation, must be zero, that is, the sum of coefficients of $z_{i_1\sigma(i_1)}z_{i_2\sigma(i_2)}z_{i_3\sigma(i_3)}\cdots z_{i_k\sigma(i_k)}$ for all $\sigma \in S_k$ is 0. So we reduce to the problem that whether a binomial which is difference of two such terms can be generated by $z_{ij}z_{kl} - z_{il}z_{kj}$, namely, those of degree two. However, since the symmetric groups are generated by 2-cycles, we obtain that $z_{i_1\sigma(i_1)}z_{i_2\sigma(i_2)}z_{i_3\sigma(i_3)}\cdots z_{i_k\sigma(i_k)}$ can be generated by those binomials of degree $k-1$. This completes the proof.

Exercise 15 (by Shi-Xin)**.**

(a) Let $\phi : \mathbb{P}^1_{X,Y} \times \mathbb{P}^1_{Z,W} \to \mathbb{P}^3_{w,x,y,z}$ be the map sending $((X,Y),(Z,W))$ to (XZ, XW, YZ, YW). Then by exercise 2.14, one can show $\operatorname{Im}\phi = Z(xy - zw) = Q$ immediately.

(b) For $t, u \in \mathbb{P}^1$, let $\{L_t\}$ and $\{M_u\}$ be the image of $\{t\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{u\}$ via ϕ, respectively. Then

$$L_t = \{(w,x,y,z) \in \mathbb{P}^3 \mid t_2 x = t_1 z, t_1 y = t_2 w\}$$
$$M_u = \{(w,x,y,z) \in \mathbb{P}^3 \mid u_2 w = u_1 x, u_2 y = u_1 z\}$$

It is clear that $L_t = L_{t'}$ if and only if $t = t'$ in $\mathbb{P}^1$, and $L_t \cap L_{t'} = \emptyset$ whenever $t \neq t'$. Similarly, the same result holds for M_u. Moreover, $L_t \cap M_u$ only meet at the point $(t_1u_1, t_1u_2, t_2u_1, t_2u_2)$

(c) Consider the set $T := Z(x-y) \cap Q$ which is a closed set in Q. It is easy to show that $\operatorname{Im}(S) = T$ where $S = \{(\lambda,\lambda) \mid \lambda \in \mathbb{P}^1\}$. If S is closed, it can be write as $U \times V$ for some closed subsets $U, V \subset \mathbb{P}^1$. Choose two distinct point $(\lambda_1,\lambda_1), (\lambda_2,\lambda_2)$ in S. Then (λ_1,λ_2) should be in $U \times V$, which leads to a contradiction since (λ_1,λ_2) should not be in S. So S is not a closed set in the product topology.

Exercise 16 (by Yi-Tsung)**.**

(a) Let $Q_1 = V\left(x^2 - yw\right), Q_2 = (xy - zw)$, then $Q_1 \cap Q_2 = V\left(x^2 - yw, xy - zw\right)$. For $(x,y,z,w) \in Q_1 \cap Q_2$, we have

$$\begin{aligned} & xzw = x^2 y = y^2 w \\ \Rightarrow\ & w = 0 \text{ or } y^2 = xz \\ \Rightarrow\ & x = w = 0 \text{ or } y^2 = xz \\ \Rightarrow\ & Q_1 \cap Q_2 \subseteq V(x,w) \cap V\left(y^2 - xz\right) \end{aligned}$$

Conversely, it is clear that $Q_1 \cap Q_2 \supseteq V(x,w) \cap V\left(y^2 - xz\right)$, hence we see that $Q_1 \cap Q_2 = V(x,w) \cap V\left(y^2 - xz\right)$ is the union of a twisted cubic curve and a line.

(b) Let $C = V\left(x^2 - yz\right), L = V(y) \subseteq \mathbb{P}^2 \Rightarrow C \cap L$ has exactly one point $P = (0,0,1) \Rightarrow I(P) = (x,y)$ and $I(C) + I(L) = (x^2 - yz) + (y)$. However $x \in I(P)$ but $x \notin I(C) + I(L)$, hence $I(C) + I(L) \neq I(P)$.

Exercise 17 (by Chi-Kang)**.**

(a) Let $C(Y)$ be the affine cone of Y in $\mathbb{A}^{n+1}$, then we have

$$\dim Y = \dim C(Y) - 1 = \dim A(C(Y)) - 1 \geq n + 1 - q - 1 = n - q$$

(b) Let Y be a variety of dimension r in $\mathbb{P}^n$ which is a strict complete intersection. Then $I(Y) = (f_1, ..., f_{n-r})$. Thus $Y = V(I(Y)) = V(f_1) \cap ... \cap V(f_{n-r})$, which is a complete intersection as set.

(c) To show Y is not a complete intersection, similar to 1.11, we can see that when we give the weight v of variables by $v(w) = 0, v(x) = 1, v(y) = 2, v(z) = 3$, then x^2, yw are the only 2 degree 2 monomial of weight 2. xy, zw are the only 2 degree 2 monomial of weight 3. And xz, y^2 are the only 2 degree 2 monomial of weight 4. Since every variables have distinct weight, in $I(Y)$ there ia no monomial with degree or weight less that 1.
Now let $f, g \in I(Y)$, we claim that at least one of $x^2 - yw, xy - zw, xz - y^2$ is not in (f, g).
To prove this, first we write $f = \sum f_{d,v}, g = \sum g_{d,v}$ for $f_{d,v}, g_{d,v}$ be the part of degree d and weight v in f, g. Then since $f_{d,v}(s, st, st^2, st^3) = g_{d,v}(s, st, st^2, st^3) = 0$ for all $t \in k, s \in k - 0$ (s parametrized the degree, t parametrized the weight.)Similar to 1.11 we must have

$$\begin{aligned} &f_{2,2} = a_1(x^2 - yw), f_{2,3} = a_2(xy - zw), f_{2,4} = a_3(xz - y^2) \\ &g_{2,2} = b_1(x^2 - yw), g_{2,3} = b_2(xy - zw), g_{2,4} = b_3(xz - y^2) \end{aligned}$$

for some $a_i, b_i \in k$. Now if $xz - y^2, x^3 - yz, x^2y - z^2$ all lies in (f, g), then for $xz - y^2$, there is some $h_1, h_2 \in k[x, y, z]$ s,t, $h_1 f + h_2 g = xz - y^2$, since $x^2 - yw, xy - zw, xz - y^2$ are the lowest degree term in I, let c_i be the constant term of h_i we must have

$$\begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Similarly if $xy - zw, xz - y^2 \in (f, g)$, there is $d_i, e_i \in k$ s,t,

$$\begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \\ e_1 & e_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

But by counting the rank it is obviously impossible. Hence at least one of $xz - y^2, x^3 - yz, x^2y - z^2$ not lies in the ideal (f, g), so any 2 elements of $I(Y)$ cannot generate the complete $I(Y)$.
Thus Y is not a strict complete intersection. To show Y is a set theoretic complete intersection. Let H_2 be defined by $x^2 - yw$, H_3 be defined by $y(xz - y^2) + z(xy - zw) = 2xyz - y^3 - z^2w$, the we have

$$\begin{aligned} H_2 \cap H_3 =& \{[1, x, y, z] | y = x^2, z^2 + y^3 = 2xyz\} \\ & \cup \{[0, x, y, z] | x^2 = y(2xz - y) = 0\} \\ =& \{[1, t, t^2, z] | (z - t^3)^2 = 0\} \cup \{[0, 0, y, z] | y^3 = 0\} \\ =& \{[1, t, t^2, t^3] | t \in K\} \cup \{[0, 0, 0, 1]\} = Y. \end{aligned}$$

Hence Y is a set-theotetic complete intersection.

3 Morphisms

Exercise 1 (by Zi-Li)**.**

(a) By exercise 1.1, any conic Y is isomorphic to $\mathbb{A}^1$ or $xy-1=0$. Say $Y = Z(xy-1)$, let $\varphi : Y \longrightarrow \mathbb{A}^1 - \{0\}, (x,y) \longmapsto x$, $\psi : \mathbb{A}^1 - \{0\} \longrightarrow Y, x \longmapsto (x, 1/x)$.The compositions of φ and ψ are identities, hence, $Y \simeq \mathbb{A}^1 - \{0\}$

(b) Let Y be an open set of $\mathbb{A}^1$, we may assume $0 \notin Y$, then $f : Y \longrightarrow k, y \longmapsto 1/y$ is a regular function on Y. Suppose $\varphi : \mathbb{A}^1 \xrightarrow{\sim} Y$, then $1/\varphi(x) = g(x)$ for some $g(x) \in k[x]$. However, the zeros of $g(x)$ lead to a contradiction.

(c) Let $F(x,y,z) = ax^2 + by^2 + cz^2 + dxy + eyz + fzx$ be an irreducible polynomial. Consider $F(1,s,t)$, by technique in classification of conic of $\mathbb{A}^2$, there exists transformation $\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} P \\ Q \end{bmatrix}$ such that $F = u - v^2$ or $F = uv - 1$, where $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is invertible matrix. Let $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ P & A & B \\ Q & C & D \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$, then $F = uv - w^2$ or $F = vw - u^2$, hence, we can assume conic $Y = Z(xy - z^2)$. Then, $2-Uple$ embedding of $\mathbb{P}^1$ gives us the isomorphism between $\mathbb{P}^1$ and Y.

(d) Any two curves in $\mathbb{P}^2$ intersects, however, there are two curves which do not intersect in $\mathbb{A}^2$.

(e) If affine variety Y is isomorphic to projective variety, then $A(Y) \simeq k$, hence $I(Y)$ is maximal, and hence Y is one point.

Exercise 2 (by Wei)**.**

(a) Recall that the map φ is defined by

$$\varphi : \mathbb{A}^1 \to V(y^2 - x^3) \subseteq \mathbb{A}^2,\ t \mapsto (t^2, t^3)$$

(a.1) **Bijectivity** : the inverse set map is given by

$$\varphi^{-1}(x,y) = \begin{cases} yx^{-1}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

(a.2) **Bicontinuity** : φ is clearly continuous. If we can show continuity of φ^{-1}, then we are done, that is, the image of a closed subset of $\mathbb{A}^1$ (i.e. a finite set of points) is closed in $V(y^2 - x^2)$, which is obvious.

(a.3) **Not an isomorphism** : if we can show that the induced map on rings isn't an isomorphism, then we are done. Notice that the induced map is

$$\varphi^* : k[x,y]/(y^2 - x^3) \to k[t],\ x \mapsto t^2,\ y \mapsto t^3$$

but this maps misses t.

(b) Recall that the map F (Frobenius morphism) is defined by

$$F : \mathbb{A}^1 \to \mathbb{A}^1,\ t \mapsto t^p$$

where k is algebraically closed and has characteristic $p > 0$.

(b.1) **Bijectivity** : Notice that F is surjective under the assumption that k is algebraically closed, and that F is injection under the assumption that $\mathrm{char}(F) = p$.

(b.2) **Bicontinuity** : By the fact that F is bijective and that the closed subsets of $\mathbb{A}^1$ are finite points, this is true.

(b.3) **Not an isomorphism** : if we can show that the induced map on rings isn't an isomorphism, then we are done. Notice that the induced map is

$$F^* : k[Y] \to k[X],\ Y \mapsto X^p$$

which is clearly not surjective (since F^* misses X).

Exercise 3 (by Yi-Tsung)**.**

(a) For $\langle U, f\rangle \in \mathcal{O}_{\varphi(P),Y}$, let $\varphi_P^*(\langle U, f\rangle) := \langle \varphi^{-1}(U), f \circ \varphi\rangle$. f is regular on $U \Rightarrow f \circ \varphi$ is regular on $\varphi^{-1}(U)$. Hence φ_P^* is a map from $\mathcal{O}_{\varphi(P),Y} \to \mathcal{O}_{P,X}$, and it is clear that φ_P^* is a homomorphism.

(b) ($\Rightarrow$) Since φ is bicontinuous and bijective, φ is a homeomorphism.
($\Leftarrow$) For $X \xrightarrow{f} k$ regular, since $\varphi_P^*(f)$ is regular and ${\varphi_P^*}^{-1}(f)$ and ${\varphi_P^*}^{-1}(f) = f \circ \varphi^{-1}$, hence φ^{-1} is a morphism, and hence φ is an isomorphism.

(c) For $\varphi_P^*(f) = 0$,

$$\begin{aligned}
&f \circ \varphi = 0 \text{ on an open subset of } X\\
\Rightarrow\ & f \circ \varphi = 0 \text{ on } X\\
\Rightarrow\ & f = 0 \text{ on } \varphi(X)\\
\Rightarrow\ & f = 0 \text{ on } Y \text{ since } \varphi(X) \text{ is dense in } Y\\
\Rightarrow\ & \varphi_P^* \text{ is injective.}
\end{aligned}$$

Exercise 4 (by Wei).

Previously, we've seen that the d-uple embedding

$$\rho_d : \mathbb{P}^n \to \mathbb{P}^N,\ a \mapsto (M_i(a))_{i=0}^N$$

is a homeomorphism of $\mathbb{P}^n$ onto its image $P := \rho_d(\mathbb{P}^n)$. It is clear from definition that ρ_d is a morphism, so we have to check if its inverse is also a morphism. Being a morphism is a local property; that is, if for a chosen open cover $\{U_i\}_{i=0}^n$ of $\mathbb{P}^n$ that ρ_d^{-1} is a morphism from $\rho_d(U_i)$ to U_i, then ρ_d^{-1} is a morphism.
Choose for $\mathbb{P}^n$ the typical affine open cover $\{U_i\}_{i=0}^n$ with isomorphisms to affine spaces $\varphi_i : U_i \to \mathbb{A}^n$, define P_i as $\rho_d(U_i)$. Assume that

$$i = 0,\ (M_0, M_1, M_2, ..., M_n) = (x_0 x_0^{d-1}, x_1 x_0^{d-1}, x_2 x_0^{d-1}, ..., x_n x_0^{d-1})$$

Then among restricting ρ_d^{-1} to P_0 and composing with φ_0, we obtain

$$\varphi_0 \rho_d^{-1} : P_0 \to \mathbb{A}^n,\ [y_0 : y_1 :, ..., : y_N] \mapsto [1 : y_1/y_0, : ..., : y_n/y_0] \mapsto (y_1/y_0, ..., y_n/y_0)$$

This is a map into an affine variety with regular component functions. Therefore $\rho_d^{-1}|_{P_0}$ is a morphism.

Exercise 7 (by Wei).

(b) Suppose $Y \cap H = \emptyset$, then $Y \subset \mathbb{P}^n \setminus H$. Y being closed, irreducible in $\mathbb{P}^n$, it is also closed, irreducible in $\mathbb{P}^n \setminus H$, so along with Exercise 3.5, Y is then an affine variety. Being both an affine variety and a projective variety, we have by Exercise 3.1(e) that $\dim(Y) = 0$.

(a) Special case of (b).

Exercise 7 (by Yu-Chi).

I would like to give a direct proof on part (a) without using part (b). Given two plane curves $X, Y \subset \mathbb{P}^2$. By Exercise I.2.8, we know that there exists irreducible homogeneous polynomials $f, g \in k[x, y, z]$ with positive degree such that $X = Z(f), Y = Z(g)$. Then $X \cap Y = Z(f, g)$. The homogeneous coordinate ring of $X \cap Y$ is given by $k[x, y, z]/(f, g) \cong k[\bar{x}, \bar{y}, \bar{z}]/(\bar{g})$, where the bar means the image in the quotient ring $k[x, y, z]/(f)$. Note that the quotient ring is a domain since f is irreducible.

If $g \in (f)$, then $X \subset Y$, and hence $X \cap Y = X \neq \emptyset$. Therefore, we assume $g \notin (f)$, then $\bar{g} \neq 0$ in $k[x, y, z]$. On the other hand, if g is a unit in the quotient ring, then there exists some $h \in k[x, y, z]$ such that $hg + rf = 1$ for some $r \in k[x, y, z]$. However, by comparing degree, f, g are homogeneous of positive degree, and thus above situation is impossible.

Thus, $\bar{g}$ is neither a zero divisor nor a unit in $k[x,y,z]/(f)$. By Krull's principal ideal theorem, any minimal prime in the quotient ring containing $\bar{g}$ has height 1. Therefore, $\mathrm{ht}(f,g) \leq 2$. Suppose that $X \cap Y = \emptyset$, then $(f,g) \supset (x,y,z)$, and the latter ideal has height 3. This gives the contradiction.

Exercise 8 (by Tzu-Yang Tsai)**.**

We extend this function to global and write it as $f = \frac{h}{g}$ in $P \setminus (H_i \cap H_j)$ for some f, g are homogeneous polynomials that have same degree. Then g has no solution on $P \setminus H_i \Rightarrow g = x_i^n$, similarly, g has no solution on $P \setminus H_j \Rightarrow g = x_j^n$, then these two conditions are mutually contradicts. Thus g is a constant, which leads to f is a constant.

Exercise 9 (by Shi-Xin)**.**

The idea is to show that $S(Y) = k[x,y,z]/(xz - y^2)$ is not a UFD by checking that $\bar{x}$ is an irreducible element but not a prime element in $S(Y)$. Clearly, $\bar{x}$ is not prime since $\bar{y} \cdot \bar{y} = \bar{x} \cdot \bar{z} \in (\bar{x})$, but $\bar{y}$ doesn't lie in $(\bar{x})$. Now, suppose $\bar{x} = \bar{f} \cdot \bar{g}$ for some $\bar{f}, \bar{g} \in S(Y)$. Then $x - fg = (xz - y^2)h$ where $f, g, h \in k[x,y,z]$. Replacing y^2 by xz, we can assume that f, g have at most degree 1 w.r.t y, and hence $h \in k[x,z]$ since $x - fg$ has at most degree 2 w.r.t y. We may write

$$x - (a + by)(c + dy) = (xz - y^2) \cdot h \quad \text{for } a, b, c, d, h \in k[x,z]$$

In the view of polynomials of y, we have the following equation from the coefficients:

$$\begin{cases} x - ac = xz \cdot h \\ ad + bc = 0 \\ bd = h \end{cases}$$

If $h \neq 0$, let n, m be the degree of h, ac w.r.t x. From the first equation, it follows that $m = n + 1$. Moreover, one can derive $m \equiv n (mod\ 2)$ from the other two equations. However, it is impossible that $n + 1 \equiv n (mod\ 2)$, so h should be zero. Since x is irreducible in $k[x,y,z]$, either f or g is a unit, and hence either $\bar{f}$ or $\bar{g}$ is a unit. Thus $\bar{x}$ is irreducible.

Exercise 10 (by Yi-Heng)**.**

For every open set $V \subset Y'$, and for every regular function $f : V \to k$, we want to show that $f \circ \varphi|_{X'} : (\varphi|_{X'})^{-1}(V) \to k$ is regular.

Let $P \in (\varphi|_{X'})^{-1}(V)$ and $Q = \varphi(P) \in V$. Since f is regular on V, we have $f = g/h$ on some open neighborhood U of Q in V. Note that g/h is regular on $W := Y - V(h)$. Thus, $(g/h) \circ \varphi$ is regular on $\varphi^{-1}(W)$, and $(g/h) \circ \varphi = f \circ \varphi|_{X'}$ on $\varphi^{-1}(W) \cap (\varphi|_{X'})^{-1}(V) \supset (\varphi|_{X'})^{-1}(U) \ni P$. Hence, $f \circ \varphi|_{X'}$ is regular.

Exercise 11 (by Jung-Tao)**.**

Consider an open affine cover Y of P, $\mathcal{O}_p \cong A(Y)_{m_p}$, so prime ideals of $\mathcal{O}_p$ corresponds to prime ideas of $A(Y)_{m_p}$ corresponds to prime ideals of $A(Y)$ inside m_p corresponds to the closed subvarieties of Y containing P. So we have to prove that the closed subvarieties of Y containing P corresponds to closed subvarieties of X containing P.

We have to prove that there are no closed subvarieties U and V, $U \cap Y = V \cap Y$. If there are such U and V, Y^C and U covers V, V is irreducible and $P \in U \cap Y$ so $V \nsubseteq Y^C$ and $V \subset U$, similarly $U \subset V \implies U = V$, and we are done.

Exercise 12 (by Pei-Hsuan)**.**

If X is affine, then by Theorem 3.2(c), it is done!

Now, suppose $X \subseteq \mathbb{P}^2$. For point $p \in X$, we use affine cover. There exists an affine piece such that $P \in X_i = X \cap U_i$. By Theorem 3.2(c), $\dim \mathcal{O}_{p,X_i} = \dim X_i = \dim X$. So it is sufficient to show that $\mathcal{O}_{p,X} = \mathcal{O}_{p,X_i}$. $\mathcal{O}_{p,X} \subseteq \mathcal{O}_{p,X_i}$ is clear. On the other hand, for $< U, f > \in \mathcal{O}_{p,X_i}$ (i.e. f is regular on $U \subseteq X_i$.) But U is also an open subset of X. Thus, $< U, f > \in \mathcal{O}_{p,X}$. The proof is complete.

Exercise 13 (by Yi-Tsung)**.**

For any nonunit $\langle U, f\rangle \in \mathcal{O}_{Y,X}$, since f is vanishing on $Y \Rightarrow 1 - f \neq 0$ on $Y \Rightarrow \dfrac{1}{1-f}$ is defined on Y. Since f is regular on U, so is $\dfrac{1}{1-f}$. Hence $\left\langle U, \dfrac{1}{1-f}\right\rangle \in \mathcal{O}_{Y,X}$. That is, $\langle U, 1-f\rangle$ is a unit in $\mathcal{O}_{Y,X}$. Therefore $\mathcal{O}_{Y,X}$ is a local ring. Moreover,

$$\begin{aligned}\dim X = \dim A(X) &= \dim A(x) \big/ \overline{I(Y)} + \mathrm{ht} I(Y) \\ &= \dim A(Y) + \dim \mathcal{O}_{Y,X} \\ &= \dim Y + \dim \mathcal{O}_{Y,X}\end{aligned}$$

Hence $\dim \mathcal{O}_{Y,X} = \dim X - \dim Y$.

Exercise 14 (by Yi-Tsung)**.**

(a) By changing the coordinates, we may suppose that $P = (1, 0, \ldots, 0)$ and $\mathbb{P}^n = V(x_0)$. Then $\varphi(a_0, a_1, \ldots, a_n) = (0, a_1, \ldots, a_n)$. Hence φ is clear a morphism.

(b) $Y : (x, y, z, w) = (t^3, t^2u, tu^2, u^3), P = (1, 0, 0, 0)$. Then $\varphi(Y) = \{(t^3, t^2u, 0, u^3)\} = V(y^3 - x^2w)$ is a cusp cubic curve with equation $y^3 - x^2w$ in $\mathbb{P}^2 = \{z = 0\}$.

Exercise 15 (by Tzu-Yang Chou).

(a) Assume that $X \times Y = Z_1 \cup Z_2$ with Z_i: closed. Let $X_i := \{x \in X | x \times Y \subseteq Z_i\}$. Then since X_i are closed in X, we have $X = X_1$ or X_2 and hence $X \times Y = Z_1$ or $Z_2 \Rightarrow X \times Y$ is irreducible.

(b) Define $\alpha : A(X) \otimes_k A(Y) \longrightarrow A(X \times Y)$ by $\alpha(f \otimes g)(x, y) := f(x)g(y)$. Then we see that α is epic. To show that α is a monomorphism, we express an nonzero element in kernel as the sum of $f_i \otimes g_i, i = 1, \cdots, n$, with shortest expression. Now, by plugging in a point of Y we can write g_n as a linear combination of $g_1, \cdots, g_{n-1}$, and hence this leads to a contradiction.

(c) The projections are defined by polynomials and hence are morphisms. For the universal property, given Z and $f : Z \longrightarrow X$,$g : Z \longrightarrow Y$, we may define a map $Z \longrightarrow X \times Y$ by $z \mapsto (f(z), g(z))$. Moreover, this map is a morphis, since both f, g are defined by polynomials.

(d) Set $a := \dim X$, $b := \dim Y$. Noether Normalization says that $A(X)$(resp. $A(Y)$) is integral over a polynomial ring of a(resp. b) variables. This implies $A(X) \otimes_k A(Y)$ is integral over a polynomial ring of $a + b$ variables. Hence $\dim X \times Y = \dim A(X \times Y) = \dim A(X) \otimes_k A(Y) = \dim A(X) + \dim A(Y) = \dim X + \dim Y$.

Exercise 17 (by AY).

(a) Since any conic in P^2 is isomorphic to P^1, we only need to show that P^1 is normal. But since $S(P^1) = k[x]$ is integrally closed, $\mathcal{O}_P \cong S(P^1)_{(\mathfrak{m}_P)}$ is also integrally closed.

(b) (1) The coordinate rings of the affine covers of Q_1 are all $k[x, y, z]/(xy - z)$, which is isomorphic to $k[x, y, xy] = k[x, y]$ which is a UFD and thus integrally closed. Hence Q_1 is normal.

(2) The coordinate rings of the affine cover of Q_2 are $k[x, y, z]/(x - z^2)$, $k[x, y, z]/(xy - 1)$, and $k[x, y, z]/(xy - z^2)$. The first two are integrally closed since they are isomorphic to UFDs $k[x, y]$ and $(k[x, 1/x])[z]$, respectively. Let $\alpha \in Q(R)$ be integral over $R = k[x, y, z]/(xy - z^2)$, then $\alpha = az + b$ where $a, b \in Q(k[x, y])$. Consider the primitive polynomial $(\alpha - b)^2 - a^2xy$. It is the minimal polynomial of α in $Q(k[x, y])[\alpha]$ and hence in $k[x, y][\alpha]$. Thus the coefficients $2b$ and $b^2 - a^2xy$ are in $k[x, y]$, and since xy is square-free, $a, b \in k[x, y]$. As a result, $\alpha \in R$, and R is integrally closed. Hence Q_2 is normal.

(c) Let $Y = Z(x^2 - y^3)$. Then $A(Y) = k[x,y]/(x^2 - y^3)$ is isomorphic to $A = k[t^2, t^3]$, which is a proper subset of $B = k[t]$ since $t \notin A$. But $Q(B) = Q(A)$. Hence B is the integral closure of A and A is not integrally closed.

(d) The statement follows from the fact that $A = A' \Leftrightarrow \forall$ maximal $\mathfrak{m} \in A[A_{\mathfrak{m}} = A'_{\mathfrak{m}}]$. $\Rightarrow$ is a direct consequence of the preservation of integral closure under localization. There remains the proof of $\Leftarrow$. Suppose by contradiction that there is a $t \in Q(A) \setminus A$ that is integral over A , and consider the ideal $I = \{a \in A | at \in A\}$. This is a proper ideal since $1 \notin I$, and hence I is in some maximal ideal $\mathfrak{m}$. Then $t \notin A_{\mathfrak{m}}$, but t is integral over $A_{\mathfrak{m}}$ since A is a subring of $A_{\mathfrak{m}}$, which gives a contradiction.

(e) Let $\tilde{A} = (A(Y))'$, then according to Theorem 3.9A, the domain A is finitely generated k-algebra, and hence is a coordinate ring of some affine variety. This affine variety is $\tilde{Y}$, and π is the morphism associated with the natural homomorphism $h : A(Y) \to \tilde{A}$. The remaining is to prove that for all integrally closed finitely generated domain A and an injective homomorphism $f : A(Y) \to A$, there exists a unique homomorphism $g : \tilde{A} \to A$ such that $g \circ h = f$. This is clear since $A(Y) \cong f(A(Y))$, $\tilde{A} = (A(Y))' \cong (f(A(Y)))'$, and g is the natural homomorphism between them.

Exercise 18 (by AY)**.**

(a) $S(Y) = (S(Y))' \Rightarrow S(Y)_{\mathfrak{m}_P} = (S(Y)_{\mathfrak{m}_P})' \Rightarrow S(Y)_{(\mathfrak{m}_P)} = (S(Y)_{(\mathfrak{m}_P)})'$

(c) They are isomorphic with the map $(t,u) \leftrightarrow (x,y,z,w) = (t^4, t^3u, tu^3, u^4)$. A regular function $f(x,y,z,w)/g(x,y,z,w)$ on Y can be pulled back to a regular function $f(t^4, t^3u, tu^3, u^4)/g(t^4, t^3u, tu^3, u^4)$ on P^1, and a regular function $h(t,u) = f(t,u)/g(t,u)$ on P^1 can be pulled back to a regular function, the equivalence class of $(h(x/y, 1), Y \setminus Z(y = 0)), (h(1, y/x), Y \setminus Z(x = 0)), (h(z/w, 1), Y \setminus Z(w = 0)), (h(1, w/z), Y \setminus Z(z = 0))$.

(b) It's normal since it is isomorphic to P^1, which is normal. $S(Y) = k[x,y,z,w]/(xw - yz, xz^2 - wy^2, y^3 - x^2z, z^3 - w^2y)$, and $X = xz/y \in Q(S(Y)) \setminus S(Y)$ is integral over $S(Y)$ with the polynomial $X^2 - xw = 0$. Hence $S(Y)$ is not integrally closed. As a result, Y is not projectively normal.

Exercise 20 (by Chi-Kang)**.**

(TO BE CONTINUED) (a) Since every variety are assumed be quasi-projective, so P has a quasi-affine neighborhood, thus we may assume Y is quasi affine. Now let $Y \subset \mathbb{A}^n$ be quasi affine, then $f = \frac{g}{h}$ for some $g, h \in k[x_1, ..., x_n]$ and $h(x) \neq 0$

for all $x \in Y - P$.
What we need to show is $h(P) \neq 0$ i,e, $h \notin m_P$. Since Y is birational to $\bar{Y}$, we have $\mathcal{O}_P = A(\bar{Y})_{m_P} = \cap A(\bar{Y}_Q)$, where the last equality is taking the intersection over all height 1 prime ideal Q s,t, $Q \subset m_P$.
So we can change to show that $h \notin Q$ for all such prime ideal. If $h \in Q$ for some Q, then $h|_{V(Q)=0}$, since $h(x) \neq 0$ for any $x \in Y - P$, we have $V(Q) \cap Y = \{P\}$. But $V(Q) \cap \bar{Y}$ has each component has codimension 1, so since $\{P\}$ is an component of $V(Q) \cap \bar{Y}$, we conclude that $\dim Y$=1, which is a contradiction.

(b) Let $Y = \mathbb{A}^1, f = \frac{1}{x}, P = 0$, then this is obviously f is a regular function on $Y - P$ which connot extend to P.

Exercise 21 (by Shuang-Yen)**.**

(a) Since $x + y$ and $-y$ are polynomials, $\mu : \mathbb{A}^2 \to \mathbb{A}^1$ and $\cdot^{-1} : \mathbb{A}^1 \to \mathbb{A}^1$ are morphisms, so G_a is a group variety.

(b) Since $1/x$ is a rational function that defines on all of $\mathbb{A}^1 - \{(0)\}$ and xy is a polynomial, $\mu : (\mathbb{A}^1 - \{(0)\})^2 \to \mathbb{A}^1 - \{(0)\}$ and $\cdot^{-1} : \mathbb{A}^1 - \{(0)\} \to \mathbb{A}^1 - \{(0)\}$ are morphisms, so G_m is a group variety.

(c) Let $\varphi, \psi \in \operatorname{Hom}(X, G)$, define $(\varphi \cdot \psi)(p) = \varphi(p) \cdot \psi(p)$. Since $[p \mapsto (\varphi(p), \psi(p))] \in \operatorname{Hom}(X, G \times G)$, by the universal property of product variety, $\varphi \cdot \psi = \mu \circ [p \mapsto (\varphi(p), \psi(p))] \in \operatorname{Hom}(X, G)$. Clearly, $(\varphi \cdot \psi) \cdot \eta = \varphi \cdot (\psi \cdot \eta)$. Let $e \in G$ be the identity, then the map $\tilde{e} = [p \mapsto e]$ is a morphism and $\varphi \cdot \tilde{e} = \tilde{e} \cdot \varphi = \varphi$ for any $\varphi \in \operatorname{Hom}(X, G)$. The inverse of $\varphi \in \operatorname{Hom}(X, G)$ is defined by $(\varphi^{-1})(p) = (\varphi(p))^{-1}$, it's a morphism since inverse on G is a morphism, also $\varphi \cdot \varphi^{-1} = \varphi^{-1} \cdot \varphi = \tilde{e}$, so φ^{-1} is the inverse of φ in $\operatorname{Hom}(X, G)$. Hence, $\operatorname{Hom}(X, G)$ is a group.

(d) Define $\theta : \operatorname{Hom}(X, G_a) \to \mathcal{O}(X)$ by $\theta(\varphi) = \varphi$, then it's well-defined since regular functions are morphisms to $\mathbb{A}^1$. It's clerly a bijective homomorphism, so $\operatorname{Hom}(X, G_a) \cong \mathcal{O}(X)$.

(e) Define $\theta : \operatorname{Hom}(X, G_m) \to \mathcal{O}(X)^\times$ by $\theta(\varphi) = \varphi$, then it's well-defined since regular functions are morphisms to $\mathbb{A}^1$ and $0 \notin \operatorname{Im}\varphi$. It's clerly a bijective homomorphism, so $\operatorname{Hom}(X, G_m) \cong \mathcal{O}(X)^\times$.

4 Rational maps

Exercise 1 (by Yu-Ting)**.**

Let F be a function defined on $U \cup V$ such that $F|_U = f$ and $F|_V = g$. Then F is regular on U and V, hence F is regular on $U \cup V$.

Now, let f be a rational map with eqivalence class $\langle U_\alpha, f_\alpha \rangle$, $\alpha \in \Lambda$. According to the above argument, f is regular on $\bigcup_{\alpha \in \Lambda}$, which is open in X. If $\exists V \supset \bigcup_{\alpha \in \Lambda} U_\alpha$ such that f is regular on V, then $\langle U, f_V \rangle = \langle U_\alpha, f \rangle$, contradiction. Hence, $\bigcup_{\alpha \in \Lambda}$ is the largest.

Exercise 2 (by Yu-Ting)**.**

For $\langle U, \phi_U \rangle = \langle V, \phi_V \rangle$, define $\phi_{U \cup V}$ such that $\phi_{U \cup V}|_U = \phi_U$ and $\phi_{U \cup V}|_V = \phi_V$. If f is regular on open subset $W \subset Y$, either $\phi^{-1}_{U \cup V}(W) \subset U$ or $\phi^{-1}_{U \cup V}(W) \subset V$, then $f \circ \phi$ is regular on $\phi^{-1}_{U \cup V}(W)$, hence $\phi_{U \cup V}$ is a morphism on $U \cup V$. Let ϕ be $\langle U_\alpha, \phi_\alpha \rangle$, $\alpha \in \Lambda$. Similar to Exercise 1, $\bigcup_{\alpha \in \Lambda} U_\alpha$ is the largest open set on which ϕ is represented by a morphism.

Exercise 3 (by Te-Lun)**.**

(a) Note that f is itself a regular function on the open set $\mathbb{P} \setminus Z(x_0)$, and definitly not regular on any point in $Z(x_0)$, so the set of points where f is defined is then $\mathbb{P} \setminus Z(x_0)$ and the regular function be $f(x_0 : x_1 : x_2) = \frac{x_1}{x_0}$

(b) Compose the following:

$$\begin{array}{cccccc} \varphi : & \mathbb{P}^2 & \overset{f}{\dashrightarrow} & \mathbb{A}^1 & \hookrightarrow & \mathbb{P}^1 \\ & (x_0 : x_1 : x_2) & \mapsto & \frac{x_1}{x_0} & & \\ & & & a & \mapsto & (a : 1) \end{array}$$

The resulting rational map is $\varphi(x_0 : x_1 : x_2) = \left(\frac{x_1}{x_0} : 1 \right) = (x_1 : x_0)$, which is regular on $\mathbb{P}^2 \setminus \{(0 : 0 : 1)\}$.

Exercise 4 (by Pei-Hsuan)**.**

(a) By exercise 1.3.1(b), any conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$. Since a morphsim is a rational map by definition. Thus, it is rational.

(b) Let $Y = Z(y^2 - x^3) \subseteq \mathbb{A}^2$.

Consider $\begin{array}{cccc} \varphi : & \mathbb{P}^2 & \dashrightarrow & Y \\ & (x_0, x_1) & \mapsto & ((\frac{x_1}{x_0})^2, (\frac{x_1}{x_0})^3)) \end{array}$ and $\begin{array}{cccc} \psi : & Y & \dashrightarrow & \mathbb{P}^2 \\ & (x, y) & \mapsto & (x, y) \end{array}$

Clearly, φ and ψ are rational maps, and it is easy to check $\varphi \circ \psi = \mathrm{id}_{\mathbb{P}^1}$ and $\psi \circ \varphi = \mathrm{id}_Y$ as rational maps.

(c) Following the hint, it is easy to see $\varphi(x,y,z) = (x,y)$.

Define $\begin{array}{cccc} \psi : & \mathbb{P}^1 & \dashrightarrow & Y \\ & (x_0,x_1) & \mapsto & ((x_1^2-x_0^2)x_0,(x_1^2-x_0^2)x_1,x_0^3) \end{array}$, then $\varphi\circ\psi = \mathrm{id}_Y$ and $\psi\circ\varphi = \mathrm{id}_{\mathbb{P}^1}$ as rational maps. Thus, Y is rational.

Exercise 5 (by Pei-Hsuan)**.**

Since we know that Q is isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$ through the Serge embedding. $\mathbb{P}^1\times\mathbb{P}^1$ is birational to $\mathbb{P}^2$ since their function fields are isomorphic to each other. (i.e. $K(\mathbb{P}^1\times\mathbb{P}^1)\cong K(\mathbb{P}^1)\otimes_k K(\mathbb{P}^1)\cong k(x,y)\cong K(\mathbb{P}^2)$.)

However, $\mathbb{P}^1\times\mathbb{P}^1$ is not isomorphism to $\mathbb{P}^2$, since any two curve in $\mathbb{P}^2$ must intersect, but $\{s\}\times\mathbb{P}^1$ and $\{t\}\times\mathbb{P}^1$ has no interestion in $\mathbb{P}^1\times\mathbb{P}^1$ whenever $s\neq t$.

Remark. You can also prove that Q birational to $\mathbb{P}^2$ by considering the following maps:

$\begin{array}{cccc} \varphi : & Q & \dashrightarrow & \mathbb{P}^2 \\ & (x,y,z,w) & \mapsto & (x,y,z) \end{array}$ and $\begin{array}{cccc} \psi : & \mathbb{P}^2 & \dashrightarrow & Q \\ & (x,y,z) & \mapsto & (zx,zy,z^2,xy) \end{array}$

Then check that $\varphi\circ\psi = \mathrm{id}_{\mathbb{P}^2}$, $\psi\circ\varphi = \mathrm{id}_Q$.

Exercise 6 (by Tai-Ning)**.**

(a) Observe that $\varphi : [a_0,a_1,a_2]\mapsto[a_1a_2,a_0a_2,a_0a_1]$ is a well-defined map from $\mathbb{P}^2-\{[0,0,1],[0,1,0],[1,0,0]\}$ to $\mathbb{P}^2$, and it is defined by polynomials, hence φ is a morphism. For $[a_0,a_1,a_2]\in\mathbb{P}-V(xyz)$,

$$\varphi^2([a_0,a_1,a_2]) = [a_0^2a_1a_2, a_0a_1^2a_2, a_0a_1a_2^2] = [a_0,a_1,a_2].$$

So φ is it's own inverse.

(b) Let $U = V = \mathbb{P}-V(xyz)$, then φ is a morphism from U to V, and φ is it's own inverse.

(c) φ can be defined on $\mathbb{P}^2-\{[0,0,1],[0,1,0],[1,0,0]\}$. To see that φ can not extend further, suppose otherwise, we can extend φ to $P=[1,0,0]$, then, let $Q=\varphi(P)$. Since Q has at least one non-zero coordinate, choose a regular function f to be either $\frac{y}{x}$, $\frac{z}{y}$ or $\frac{y}{z}$, so that f can be defined on Q. By definition, $g = f\circ\varphi$ is also a regular function. Since f can be defined on Q, g can be defined on P, so g can be written as $\frac{G}{H}$ on V_1 near P. But on $\mathbb{P}^2-\{[0,0,1],[0,1,0],[1,0,0]\}$, g is either $\frac{a_0}{a_1}$, $\frac{a_1}{a_2}$ or $\frac{a_2}{a_1}$ on some V_2. Let's say $g=\frac{a_0}{a_1}$ (the other two cases are similar), so $g=\frac{G}{H}=\frac{a_0}{a_1}$ on their intersection $V_1\cap V_2$, so $a_1G-a_0H=0$ on $V_1\cap V_2$, which is dense in $\mathbb{P}^2$, therefore, $a_1G-a_0H=0$. Thus, $a_1\mid H$, but H cannot be zero at P. Contradiction. So, φ cannot extend.

Exercise 7 (by Tai-Ning)**.**

Take affine open neighborhood $U \subseteq$ for P, since $\mathcal{O}_{P,X} = \mathcal{O}_{P,X\cap U}$, we may assume assume $X \subseteq \mathbb{A}^n, Y \subseteq \mathbb{A}^m$ are affine.

For any k-algebra homomorphism $h : \mathcal{O}_{P,X} \to \mathcal{O}_{Q,Y}$, let $\overline{x_i}$ be coordinate functions on X, which can always defined on the entire X, let $h(X, \overline{x_i}) = (V_i, \xi_i)$, where ξ_i is a regular function defined on V_i near Q, take $V = \cap_i V_i$. Now we show that $\theta(h) : V \to X$ defined by $\theta(h)(y) = (\xi_1(y), \ldots, \xi_n(y))$ is a morphism sends Q to P. As the proof of prop 3.5 , we can see that $\theta(h)$ is a morphism, we only need to check that $\theta(h)$ sends Q to P. Suppose not, say $\theta(h)(Q) = P'$, which means $P' = (\xi_1(Q), \ldots, \xi_n(Q))$. Then suppose P and P' is different at index i. consider

$$\begin{aligned} h(1) &= h(\frac{1}{x_i - \xi_i(Q)}(x_i - \xi_i(Q))) \\ &= h(\frac{1}{x_i - \xi_i(Q)})h(x_i - \xi_i(Q)) \\ &= h(\frac{1}{x_i - \xi_i(Q)})(\xi_i - \xi_i(Q)) \end{aligned}$$

Notice that $\frac{1}{x_i - \xi_i(Q)}$ is a well-defined regular function in $\mathcal{O}_{P,X}$. But the right-hand-side is zero at Q, and left-hand-side is a constant 1, contradiction.

Finally, we can see that $\theta(hg) = \theta(g)\theta(h)$, so it is a contravariant functor, hence isomorphism of k-algebras will induced an isomorphism of varieties.

Exercise 10 (by Wei-Ping)**.**

Let $(t, u) \in \mathbb{P}^1$, and solve system of equation $y^2 = x^3, xu = yt$. Consider $t \neq 0$, then set $t = 1$ and put the image in $\mathbb{A}^3$, which is (u^2, u^3, u) or $x = y = 0$, u arbitrary. Clearly the latter is exceptional curve. Similarly do these with assumption $u \neq 0$, and get $\bar{Y} = (u^2, u^3, 1, u)$ and $E \cap \bar{Y} = (0, 0, 1, 0)$. Now let $u \to (u^2, u^3, 1, u)$, a isomorphism from $\bar{Y}$ to $\mathbb{A}^1$.(Since all coordinate can be express in polynomials.). Now map from $\mathbb{A}^1$ to $\bar{Y}$ is $u \to (u^2, u^3)$, which is bijective, bicontinuous, but not isomophism by 3.2(a).

5 Nonsingular Varieties

Exercise 1 (by Jung-Tao).

(a) $f = x^4 + y^4 - x^2, f_x = 4x^2 - 2x, f_y = 4y^3$
$f_y = 0 \implies y = 0 \implies x^4 - x^2 = 4x^3 - 2x = 0 \implies x = 0$, and the only singular point is $(0, 0)$, its graph is a Tacnode.

(b) $f = x^6 + y^6 - xy, f_x = 6x^5 - y, f_y = 6y^5 - x$
$f_x = f_y = 0 \implies 6x^5 = y, 6y^5 = x \implies 6x^6 = xy = 6y^6$, $x = 0$ iff $y = 0$, so we may assume $x^6 = y^6$, and $f = -4x^6 = 0 \implies x = 0$, and the only singular point is $(0, 0)$, its graph is a Node.

(c) $f = x^4 - x^3 + y^4 + y^2, f_x = 4x^3 - 3x^2, f_y = 4y^3 + 2y$,$x = 0$ iff $y = 0$,
if $y \neq 0 \implies 2y^2 = -1 \implies x^4 - x^3 = \frac{1}{4}$, which is impossible. So the only singular point is $(0, 0)$, and its graph is a Cusp.

(d) $f = x^4+y^4-x^2y-xy^2, f_x = 4x^3-2xy-y^2, f_y = 4y^3-2xy-x^2$, $x = 0$ iff $y = 0$, else $f_x = f_y = 0 \implies 4x^3 = y(2x + y), 4y^3 = x(2y + x) \implies 4xy(x + y) = 4x^4 + 4y^4 = xy(3x + 3y) \implies x + y = 0$, and $xy(x + y) = x^4 + y^4 = 2x^4 = 0 \implies x = 0$ and the only singular point is $(0, 0)$, its graph is a Triple point.

Exercise 2 (by Jung-Tao).

(a) $f = xy^2 - z^2, f_x = y^2, f_y = 2xy, f_z = -2z$
$f_x = f_y = f_z = 0 \implies y = z = 0$, and its graph is a pinch point.

(b) $f = x^2 + y^2 - z^2, f_x = f_y = f_z = 0 \implies x = y = z = 0$, and its graph is a conical double point.

(c) $f = xy + x^3 + y^3, f_x = y + 3x^2, f_y = x + 3y^2$, $x = 0$ iff $y = 0$,
else $x = -3y^2, y = -3x^2 \implies -xy = 3x^3 = 3y^3$, $f = -x^3 = 0 \implies x = 0$.
So the singular points are of the form $(0, 0, z)$, and its graph is a double line.

Exercise 3 (by Yi-Heng).

(a) $\mu_P(Y) = 1 \Leftrightarrow f_1 = ax + by$ with a, b not all zero $\Leftrightarrow Df|_{(0,0)} = (a,\, b)$ is of rank 1.

(b) 2,2,2,3 (the multiplicity is the smallest degree in each equation)

Exercise 4 (by Yi-Heng).

(a) To prove that $(Y \cdot Z)$ is finite, it suffices to show that $\mathcal{O}_P/(f,g)$ is noetherian and has dimention $= 0$. Note that $\mathcal{O}_P \simeq A_{\mathfrak{m}_P}$ is noetherian and $dim(\mathcal{O}_P/(f,g)) = dim(A(Y)_{\mathfrak{m}_P}/g) = dim(Y) - ht(g) = 0$.

Next, let $I = (x,y)$, $m = \mu_P(Y)$ and $n = \mu_P(Z)$. Then $(Y \cdot Z)_P \geq l(\mathcal{O}_P/(I^{m+n}, f, g)) \geq l(k[x,y]/(I^{m+n}, f, g)) = dim_k(k[x,y]/(I^{m+n}, f, g))$. Consider the exact sequence

$$k[x,y]/I^n \times k[x,y]/I^m \longrightarrow k[x,y]/I^{n+m} \longrightarrow k[x,y]/(I^{n+m}, f, g) \longrightarrow 0$$

$$(\overline{A}, \overline{B}) \longmapsto \overline{Af + Bg}$$

Therefore, $dim_k(k[x,y]/(I^{m+n}, f, g)) = dim_k(k[x,y]/I^{m+n}) - dim_k(k[x,y]/I^n) - dim_k(k[x,y]/I^m) = mn$.

(b) Note that ${}^{\mathcal{O}_P}/_{(f,\, ax+b)} = ({}^{k[x]}/_{f(x,\, \frac{-a}{b}x)})_{\mathfrak{m}_P}$ if $ab \neq 0$. Thus $(L \cdot Y)_P = \mu_P(Y)$ for $ab \neq 0$ and $f_n(x, \frac{-a}{b}x)) \neq 0$ $(n = \mu_P(Y))$.

(c) We may assume $L = V(x)$ by changing coordinate. Then $P = (0, P_1, P_2) \in Y \cap L$ has $P_1 \neq 0$ or $P_2 \neq 0$.

If $P_1 \neq 0$, then $(L \cdot Y)_P = (V(x) \cdot V(f(x,1,z)))_{(0,P_2)} = \mu_{(x=0,z=P_2)}(f(0,1,z))$ by the discussion in (b). Similarly, if $P_1 = 0$, then $(L{\cdot}Y)_P = \mu_{(x=0,y=P_1)}(f(0,y,1))$

In conclusion, $(L \cdot Y)_P = \sum_{P_1 \neq 0}(L \cdot Y)_P + \sum_{P_1=0}(L \cdot Y)_P =$ (the largest deg of z in $f(0,y,z)$) + (the smallest deg of y in $f(0,y,z)$) $= d$.

Exercise 5 (by Pei-Hsuan)**.**

case 1 If $p \nmid d$ or $p = 0$, then choose $f(x_0,x_1,x_2) = x_0^d + x_1^d + x_2^d$. Notice that x_0, x_1, x_2 are not all zeros, so $Df = \left(dx_0^{d-1}, dx_1^{d-1}, dx_2^{d-1}\right)$ has rank 1.

case 2 If $p \mid d$, then choose $f(x_0,x_1,x_2) = x_0^{d-1}x_1 + x_1^{d-1}x_2 + x_2^d$. Notice that $\frac{\partial f}{\partial x_2} = dx_2^{d-1} + x_1^{d-1} = x^{d-1}$. Thus, $\frac{\partial f}{\partial x_2} = 0 \Leftrightarrow x_1 = 0$. So

$$Df = (0,0,0) \Leftrightarrow x_1 = 0 \text{ and } x_0 = 0.$$

But $f(0,0,x_2) = 0 \Leftrightarrow x_2 = 0$. So, $Df|_p = 0 \Leftrightarrow p = (0,0,0)$ which is not in $\mathbb{P}^2$.

Exercise 6 (by Shuang-Yen)**.**

(a) $Y = V(x^3 - (y^2 + x^4 + y^4))$ is a cusp. Let $t = y/x$, then the equation $x^3 = y^2 + x^4 + y^4$ can be written as $(x - (t^2 + x^2 + x^2t^4))x^2 = 0$, so one of the affine chart of the blowing-up image is isomorphic to $V(x - (t^2 + x^2 + x^2t^4))$. It suffices to show the nonsingularity at $(x, t) = (0, 0)$, which is clear since $f_x(0, 0) = 1 \neq 0$, where $f(x, t) = x - (t^2 + x^2 + x^2t^4)$. Another affine chart of the blowing-up image is defined by $ys^3 = 1 + y^2s^4 + y^2$, which doesn't contain the preimage of O. Hence the blowing-up of Y is nonsingular.

(b) WLOG let $P = (0, 0)$. Write $f = f_2 + \cdots$ and $f_d = c_{d,0}x^d + c_{d-1,1}x^{d-1}y + \cdots + c_{0,d}y^d$, then P is a node implies that $f_1 = 0$ has distinct solutions in $\mathbb{P}^1$, say λ_1, λ_2. WLOG let $\lambda_i \in D(x)$. Let $t = y/x$, then the equation $f = 0$ can be written as

$$x^2 \left(\sum_{e_1,e_2} c_{e_1,e_2} x^{e_1+e_2-2} t^{e_2} \right) = 0,$$

so one of the affine chart of the blowing-up image is isomorphic to $V(\tilde{f})$, where

$$\tilde{f}(x, t) = \sum_{e_1,e_2} c_{e_1,e_2} x^{e_1+e_2-2} t^{e_2}.$$

The preimage of P in the affine chart is $V(\tilde{f}) \cap V(x)$, which is $(0, \lambda_i)$, so the preimage of P has two points. Since

$$\tilde{f}_t(0, \lambda_i) = 2c_{2,0}\lambda_i + c_{1,1} \neq 0,$$

$(0, \lambda_i)$ is nonsingular. So blowing-up resolves the singularity,

(c) The variety Y defined the equation $x^2 = x^4 + y^4$ has a tacnode at $P = (0, 0)$. Let $t = y/x, s = x/y$, then the equation can be written as $x^2(1 - x^2 - x^2t^4) = 0, y^2(s^2 - y^2s^4 - y^2) = 0$, so the preimage of P appears in the affine chart which is defined by the equation $g := s^2 - y^2s^4 - y^2 = 0$, then it's a node since $s^2 - y^2 = 0$ has two solutions $[1 : \pm 1]$.

(d) The multiplicity of $(0, 0)$ on Y is 3. Let $t = y/x, s = x/y$, then the equation $y^3 = x^5$ can be written as $x^3(t^3 - x^2) = 0, y^3(1 - y^2s^5) = 0$, so the preimage of O appears in the affine chart which is defined by the equation $g := t^3 - x^2$, and hence a cusp. One further blowing-up resolves the singularity by the way similar to (a).

Exercise 7 (by Wei-Ping)**.**

(a) Clearly f is non-singular at point not equal to P. Since $\deg f > 1$, $\frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = \frac{\partial f}{\partial z}(P) = 0$. Thus P is the only non-singular point.

(b) Consider system of equations $xv = yw, xw = zu, yw = zv, f(x, y, z) = 0$. Consider the open affine set that $u \neq 0$, then set $u = 1$ and get $f(x, xv, xw) = x^{\deg f} f(1, v, w) = 0$. If $x = 0 \Rightarrow y = 0, z = 0$, which is the exceptional curve. Assume $x \neq 0$, let $(x, y, z, v, w) \to (x, v, w)$ be isomorphism from $\mathbb{A}^5$ to $\mathbb{A}^3$. The blow up of X, say $\bar{X}$, maps into closed set defined by equation $f(1, v, w) = 0$.
To say that this set is non-singular, calculate $(\frac{\partial f(1,v,w)}{\partial x}, \frac{\partial f(1,v,w)}{\partial v}, \frac{\partial f(1,v,w)}{\partial w}) = (0, f_y(1, v, w), f_z(1, v, w))$. Note that $(f_x(1, v, w), f_y(1, v, w), f_z(1, v, w)) \neq (0, 0, 0)$ since f is non-singular, and $f_x(1, v, w) + v f_y(1, v, w) + w f_z(1, v, w) = \deg f \cdot f(1, v, w) = 0$. This implies that $(0, f_y(1, v, w), f_z(1, v, w)) \neq (0, 0, 0)$, now cover $\bar{X}$ with open affine sets to conclude it is non-singular.

(c) $\phi^{-1}(p)$ is union of three sets: $\{(0, 0, 0, 1, v, w) \mid f(1, v, w) = 0\}$, $\{(0, 0, 0, u, 1, w) \mid f(u, 1, w) = 0\}$, $\{(0, 0, 0, u, v, 1) \mid f(u, v, 1) = 0\}$. This is just $(0, 0, 0) \times Y$.

Exercise 8 (by Yu-Ting)**.**

Let $P = (a_0, \dots, a_n)$. For every $i, j, \lambda \neq 0$, $\frac{\partial f_i}{\partial x_j}(\lambda a_0, \dots, \lambda a_n) = \lambda^{d-1} \frac{\partial f_i}{\partial x_j}(a_0, \dots, a_n)$, hence, the rank of $\left\| \frac{\partial f_i}{\partial x_j}(a_0, \dots, a_n) \right\|$ is independent of the choice of coordinates.

$U_k := \mathbb{P}^n - Z(x_k) \underset{\phi_k}{\xrightarrow{\sim}} \mathbb{A}^n$ and $Y_k := \phi_k(Y \cap U_k)$. For every $k, 1 \neq i \neq t$, define $g_{i,k}(x_0, \dots, \hat{x_k}, \dots, x_n) = f_i(x_0, \dots, x_{k-1}, 1, x_{k+1}, \dots, x_n)$. P is nonsingular on Y if and only if P is nonsingular on Y_k for all k satisfying $a_k \neq 0$, i.e. $rk \left\| \frac{\partial g_{i,k}}{\partial x_j}(\frac{a_0}{a_k}, \dots, \frac{a_n}{a_k}) \right\| = n-r$. For $j \neq k$, $\frac{\partial g_{i,k}}{\partial x_j}(\frac{a_0}{a_k}, \dots, \frac{a_n}{a_k}) = \frac{\partial f_i}{\partial x_j}(\frac{a_0}{a_k}, \dots, \frac{a_{k-1}}{a_k}, 1, \frac{a_{k+1}}{a_k} \dots \frac{a_n}{a_k}) = \frac{\partial f_i}{\partial x_j}(P)$. By Euler's lemma, $\sum_{j=0}^n a_j \left(\frac{\partial f_i}{\partial x_j} \right) = df(P) = 0$, then the column $\left(\frac{\partial f_i}{\partial x_k} \right)$ is redundant. We have $rk \left\| \frac{\partial f_i}{\partial x_j}(a_0, \dots, a_n) \right\| = n - r$.

Exercise 9 (by Yu-Ting)**.**

Suppose f is reducible and $f = gh$, where $g, h \neq f$. By Exercise 3.7, two curves have intersection, then there exists $P \in (Z(g) \cap Z(h)) \subset Z(f)$. $f_x(P) = g_x(P)h(P) + g(P)h_x(P) = 0$. Similarly, $f_y(P) = f_z(P) = 0$, contradiction. Hence, f is irreducible.

Exercise 10 (by Wei)**.**

Given variety X and point $P \in X$, recall that the Zariski tangent space on P denoted by $T_P(X)$ is the k-vector space $(\mathfrak{m}_P/\mathfrak{m}_P^2)^\wedge := \mathrm{Hom}_k(\mathfrak{m}_P/\mathfrak{m}_P^2, k)$.

(a) We know that since $\mathfrak{m}_P/\mathfrak{m}_P^2$ is finite dimensional (by Noetherianess), we have

$$\dim_k T_p(X) = \dim_k(\mathfrak{m}_P/\mathfrak{m}_P^2)^\wedge = \dim_k \mathfrak{m}_P/\mathfrak{m}_P^2$$

On the other hand, we have the inequality

$$\dim_k \mathfrak{m}_P/\mathfrak{m}_P^2 \geq \dim_k \mathcal{O}_{P,X}$$

where equality holds iff P is a non-singular point.

(b) Suppose given

$$X \xrightarrow{\phi} Y$$

We get a diagram

$$\begin{tikzcd} \mathcal{O}_{P,X} & \mathcal{O}_{\phi(P),Y} \arrow[l, "\phi_P^*"'] \\ \mathfrak{m}_P \arrow[u] & \mathfrak{m}_{\phi(P)} \arrow[l] \arrow[u] \\ \mathfrak{m}_P^2 \arrow[u] & \mathfrak{m}_{\phi(P)}^2 \arrow[l] \arrow[u] \end{tikzcd}$$

In order to construct a map $T_P(X) \to T_{\phi(P)}(Y)$, it suffices (by duality) to construct a map $\mathfrak{m}_P/\mathfrak{m}_P^2 \leftarrow \mathfrak{m}_{\phi(P)}/\mathfrak{m}_{\phi(P)}^2$. The composition of maps

$$\mathfrak{m}_P/\mathfrak{m}_P^2 \longleftarrow \mathfrak{m}_P \xleftarrow{\phi_P^*} \mathfrak{m}_{\phi(P)} \longleftarrow \mathfrak{m}_{\phi(P)}^2$$

is 0 by above, so we may define $\mathfrak{m}_P/\mathfrak{m}_P^2 \leftarrow \mathfrak{m}_{\phi(P)}/\mathfrak{m}_{\phi(P)}^2$ to be the map making the diagram commutative

$$\begin{tikzcd} \mathfrak{m}_P \arrow[d] & \mathfrak{m}_{\phi(P)} \arrow[l] \arrow[d, "\phi_P^*"] \\ \mathfrak{m}_P/\mathfrak{m}_P^2 & \mathfrak{m}_{\phi(P)}/\mathfrak{m}_{\phi(P)}^2 \arrow[l, dashed, "(T_P(\phi))^\wedge"'] \end{tikzcd}$$

(c) Write X, Y as the following varieties

$$X = V(x - y^2) \subseteq \mathbb{A}^2,\ Y = \mathbb{A}^1$$

with morphism

$$X \xrightarrow{\phi} Y\ ,\ (x, y) \mapsto x$$

Denote P as the point $(0,0) \in X$, then ϕ induces a map

$$(k[x,y]/(y^2 - x))_{(x,y)} = \mathcal{O}_{P,X} \xleftarrow{\phi_P^*} \mathcal{O}_{\phi(P),Y} = (k[t])_t$$

given by $x \leftarrow\!\shortmid t$. To show $T_P(\phi)$ is 0, it suffices to show $(T_P(\phi))^\wedge$ is 0, or

$$\begin{array}{ccccc} & \mathfrak{m}_P/\mathfrak{m}_P^2 \xleftarrow{(T_P(\phi)^\wedge)} \mathfrak{m}_{\phi(P)}/\mathfrak{m}_{\phi(P)}^2 \longleftarrow \mathfrak{m}_{\phi(P)} \\ = & \mathfrak{m}_P/\mathfrak{m}_P^2 \longleftarrow \mathfrak{m}_P \xleftarrow{\phi_P^*} \mathfrak{m}_{\phi(P)} \end{array}$$

is 0 (the equality above is by our definition in (b)). Choose $f \in \mathfrak{m}_{\phi(P)}$, that is, a function vanishing on $\phi(P) = 0$, then $f = tg$ for some g, so

$$\phi_P^*(f) = \phi_P^*(tg) = \phi_P^*(t)\phi_P^*(g) = x\phi_P^*(g) = y^2\phi_P^*(g) \in \mathfrak{m}_P^2$$

Therefore, we have $(T_P(\phi))^\wedge = 0$.

Exercise 11 (by Tai-Ning)**.**

The statement is true only when char $k \neq 2$. Let's assume char $k \neq 2$,

$$Y = \{[x, y, z, w] \in \mathbb{P}^3 : x^2 - xz - yw = 0, yz - xw - zw = 0\}.$$

$$Z = \{[x, y, z] \in \mathbb{P}^2 : y^2z - x^3 + xz^2 = 0\}$$

Now we show that $\varphi : Y - [0, 0, 0, 1] \to Z - [1, 0, -1]$ defined by $\varphi : [x, y, z, w] \mapsto [x, y, z]$, and we could find φ^{-1} by

$$\varphi^{-1}([x, y, z]) = \begin{cases} [x, y, z, \frac{x^2-xz}{y}] & \text{,if } y \neq 0. \\ [x, y, z, \frac{yz}{x+z}] & \text{,if } x + z \neq 0. \end{cases}$$

Now we check well-defined.

- For any $[x, y, z, w] \in Y - [0, 0, 0, 1]$, it's impossible that $y = 0$ and $x + z = 0$ at the same time, since otherwise, $x^2 = xz + yw = -x^2$, so $2x^2 = 0$, so x, y, z are all zero, contradiction. Therefore, if $y \neq 0$, we have $w = \frac{x^2-xz}{y}$, substitute and get $yz = (x + z)\frac{x^2-xz}{y}$, so $y^2z - x^3 + xz^2 = 0$. If otherwise $x + z \neq 0$, we have $w = \frac{yz}{x+z}$ and get $x^2 - xz - y\frac{yz}{x+z}$, also have $y^2z - x^3 + xz^2 = 0$. And $[x, y, z] \neq [1, 0, -1]$. Thus, φ is a morphism.
- For any $[x, y, z] \in Z - [1, 0, -1]$, it's impossible that $y = 0$ and $x + z = 0$ at the same time, for the first case, we check $x^2 - xz - y\frac{x^2-xz}{y} = 0$ and $yz - (x + z)\frac{x^2-xz}{y} = 0$, which is true because $[x, y, z] \in Z$. Another case is similar. Therefore φ^{-1} is a well-defined morphism.

Since Z is defined by an irreducible polynomial, so Z is irreducible. And the derivative matrix is

$$[-3x^2 + z^2, \quad 2yz, \quad y^2 + 2xz]$$

The only possible to be all zero is $x = y = z = 0$. So Z is nonsingular. By the isomorphism of φ, $Y - [0, 0, 0, 1]$ is also irreducible and nonsingular. So it's sufficient to check Y is nonsingular at $[0, 0, 0, 1]$. The derivative matrix of Y is

$$\begin{bmatrix} 2x & -w & -x & -y \\ -w & z & y-w & -x-z \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \end{bmatrix}.$$

which has rank 2, so is nonsingular.

Exercise 12 (by Shuang-Yen)**.**

(a) A quadratic form over a field k with characteristic $\neq 2$ has an orthogonal basis, after linear transformation, may assume $f = x_0^2 + \cdots + x_r^2$.

(b) For $r = 0, 1$, f is clearly reducible. For $r \geq 2$, induction on r. Suppose not, say $f = gh$ and $g, h \notin k^\times$, denote $\deg_i$ the degree of x_i, then $\deg_i g + \deg_i h = 2$ for each i. If $\deg_i g \neq 1$ for all i, then $\deg_i g = 2, \deg_j h = 2$ for some $i \neq j$, then there will be a term $x_i^2 x_j^2$ in f with coefficient in $k[x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_r]$, which is a contratiction. WLOG let $\deg_0 g = \deg_0 h = 1$, write $g = ax_0 + b, h = cx_0 + d$, then $ac = 1$ implies $a, c \in k^\times$, after scaling, may let $a = c = 1$, then $b + d = 0$ and $bd = x_1^2 + \cdots + x_r^2$. If $r > 2$, by induction hypothesis, one of b, d is a unit, then $b + d = 0$ implies the another one is a unit, but bd is not a unit. If $r = 2$, then $-b^2 = bd = x_1^2 + x_2^2 = (x_1 + ix_2)(x_1 - ix_2)$, but $k[x_1, x_2]$ is a UFD. So f is irreducible if and only if $r \geq 2$.

(c) $Z = \operatorname{Sing} Q$ is defined by the equations, f and $\partial f / \partial x_i$, which is

$$Z = V(x_0^2 + \cdots + x_r^2, 2x_0, \ldots, 2x_r, 0, \ldots, 0) = V(x_0, \ldots, x_r),$$

a linear variety of dimension $n - r - 1$.

(d) Let $Q' = V(x_0^2 + \cdots + x_r^2, x_{r+1}, \ldots, x_n) \subset V(x_{r+1}, \ldots, x_n) \cong \mathbb{P}^r$. For any $A = (a_0, \ldots, a_r, 0, \ldots, 0) \in Q'$ and $B = (0, \ldots, 0, b_{r+1}, \ldots, b_n) \in Z$, every point C on the line AB can be written as $(sa_0, \ldots, sa_r, tb_{r+1}, \ldots, tb_n)$, then

$$(sa_0)^2 + \cdots + (sa_r)^2 = s^2(a_0^2 + \cdots + a_r^2) = 0 \implies C \in Q,$$

so the cone of Q' and Z is contianed in Q. For any point $C = (c_0, \ldots, c_n) \in Q$, if at least one of $c_0, \ldots, c_r$ is not 0, then C lies on the line jointing $(c_0, \ldots, c_r, 0, \ldots, 0) \in Q'$ and $(0, \ldots, 0, b'_{r+1}, \ldots, b'_n) \in Z$, where $(tb'_{r+1}, \ldots, tb'_n) =$

$(b_{r+1},\ldots,b_n)$ for some t such that $(b'_{r+1},\ldots,b'_n) \neq (0,\ldots,0)$. If all of $c_0,\ldots,c_r$ is 0, then C lies on the line jointing $(1,i,0,\ldots,0) \in Q'$ and $(0,\ldots,0,c_{r+1},\ldots,c_n) \in Z$.

Exercise 13 (by Zi-Li)**.**

We can assume the variety X is affine, let R be the integral closure of $A(X)$ in its quotient field. By 3.9A, $R = \sum_{i=1}^{k} A(X) f_i$, we may assume $f_i = g_i/F$.Then, $X - Z(F)$ is nonempty and every points of $X - Z(F)$ is normal because $1/F \in \mathcal{O}_P$ for every $P \in X - Z(F)$. Besides, if P is a normal point, write $f_i = a_i/B, B(P) \neq 0$, then $P \in X - Z(B)$ and every points of $X - Z(B)$ are normal points. Hence, non-normal points of a variety form a proper closed set.

Exercise 14 (by Shi-Xin)**.**

(a) Suppose $P \in Y = Z(f)$ and $Q \in Z = Z(g)$ are analytically isomorphic. By suitable change of variables, we can assume $P = Q = (0,0)$ and write

$$f = f_s + h.o.t., g = g_t + h.o.t.$$

where $f_s, g_t \in k[x,y]$ are the homogeneous polynomials of minimal degree s,t in f,g respectively. WLOG, assume y is not their common tangent direction. Denote $L = Z(y)$. Since $k[[x,y]]/(f) \cong k[[x,y]]/(g)$, we have

$$k[[x]]/(x^s) \cong k[[x]]/(f(x,0)) \cong k[[x,y]]/(f,L) \cong k[[x,y]]/(g,L) \cong k[[x]]/(x^t)$$

Then it forces $s = t$, and hence $\mu_P(Y) = s = t = \mu_Q(Z)$.

(b) Consider the same process in the textbook. The idea is that for any $k > r$, find g_{k-t}, h_{k-s} such that

$$f_k = g_s h_{k-s} + g_{k-t} h_t + \sum_{i=s+1}^{k-t-1} g_i h_{k-i}$$

when all the other terms are known. Then it suffices to show that for any homogeneous f of degree $k > r$, there are homogeneous polynomials g_{k-t}, h_{k-s} of degree $k-t, k-s$ respectively such that

$$f = g_s h_{k-s} + g_{k-t} h_t.$$

Write $g_s = y^s \tilde{g}(z), h_t = y^t \tilde{h}(z), f = y^k \tilde{f}(z)$ where $z = x/y$ and $\tilde{g}, \tilde{h}, \tilde{f} \in k[z]$. Let $m := \deg \tilde{g}, n := \deg \tilde{h}$ and consider the linear map

$$\phi : P_n \times P_m \to P_{n+m} \text{ defined by } \phi(A,B) = \tilde{g}A + \tilde{h}B$$

where P_n denote the polynomial in z of degree less than n.

Since $0 = deg(\gcd(\tilde{g},\tilde{h})) = m + n - \operatorname{rank}(\phi)$, ϕ is surjective, and hence there are A, B such that $\tilde{g}A + \tilde{h}B = \tilde{f}$. Just let $g_{k-t} = y^{k-t}B, h_{k-s} = y^{k-s}A$.

(c) (i) (2-fold) According to b, we may write $f = f_1 f_2$ with two distinct linear factors f_1, f_2. Then there is an automorphism of $k[[x,y]]$ sending f_1, f_2 to x, y respectively. It induces an isomorphism

$$k[[x,y]]/f \cong k[[x,y]]/(xy).$$

Thus every ordinary 2-fold point is analytically isomorphic to $(0,0)$ of $Z(xy)$.

(ii) (3-fold) Write $f = f_1 f_2 f_3$ with three distinct linear factors f_1, f_2, f_3. Then every ordinary 3-fold point is analytically isomorphic to $(0,0)$ of $Z(xy(x+y))$ by considering an automorphism of $k[[x,y]]$ sending f_1, f_2, f_3 to $ax, by, x+y$ for some $a, b \in k$.

(iii) (4-fold) Write $f = f_1 f_2 f_3 f_4$ with three distinct linear factors f_1, f_2, f_3, f_4. For the same reason, every ordinary 4-fold point is analytically isomorphic to $(0,0)$ of $Z(xy(x+y)(x+\alpha y))$ for some $\alpha \in k$ by considering an automorphism of $k[[x,y]]$ sending f_1, f_2, f_3, f_4 to $ax, by, x+y, c(x+\alpha y)$ for some $a, b, c \in k$. Denote $f_\alpha = xy(x+y)(x+\alpha y)$, and consider $(0,0)$ in $Z(f_\alpha)$ and $(0,0)$ in $Z(f_\beta)$. If they are analytically isomorphic, then there is a map $\phi : k[[x,y]]/(f_\alpha) \to k[[x,y]]/(f_\alpha)$ such that $\phi(x,y) = (x,y)$ and $\phi((f_\alpha)) = (f_\beta)$. It forces that $\alpha = \beta$. Thus there is a one-parameter family of mutually nonisomorphic ordinary 4-fold points.

(d) Given any $f = f_2 + h.o.t.$ with $f_2 \neq 0$. If f_2 has two distinct linear factors, according to $(c)(i)$, it is isomorphic to an ordinary 2-fold point, which is isomorphic to the singularity $(0,0)$ of $Z(y^2 = x^2)$. Now suppose f_2 has only one linear factor, by taking a suitable automorphism, we may assume $f = y^2 + yg_1(x,y) + h_1(x)$ where $\deg g_1 > 2$ w.r.t y. By sending $y + g_1/2$ to y, we have $f = y^2 + yg_2(x,y) + h_2(x)$ where $\deg g_2 > 3$ w.r.t y. Continuing the process, we can assume $f = y^2 + yg(x,y) + h(x)$ where $\deg g$ is sufficiently large. Now we refer to some results in a useful textbook [1]. Since the Milnor number of f, denoted by μ, is finite, f is right $(\mu+1)$-determined, which means we only need to care the part of f of degree no more than $(\mu+1)$. Hence by above process, f is right equivalent to $y^2 + h(x)$ for some $h \in k[x]$. Thus f defines a singularity which is isomorphic to $(0,0)$ of $Z(y^2 = x^r)$.

[1] Greuel, G.M., Lossen, C., Shustin, E., 2007. Introduction to Singularities and Deformations, Springer, Berlin, ch2

6 Nonsingular Curves

Exercise 2 (by Tzu-Yang Chou).

(a) Let $f := x^3 - x - y^2$. Then $f_x = 3x^2 - 1 . f_y = 2y$. However, f, f_x, f_y cannot all be zero, that is, any $P \in Y = Z(f)$ is nonsingular. Also, smoothness implies normality.

(b) x is clearly transcendental over k since if some $g(x) \in (f)$, then $g = 0$. Thus $k[x]$ is a polynomial ring. Recall that UFD is integrally closed, so it suffices to show that A is contained in the integral closure of $k[x]$ in K. Now, $y \in A$ satisfies the equation $-t^2 + x^3 - x$ and hence it is integral over $k \Rightarrow A = \overline{k(x)}$.

(c) The given map $\sigma : A \longrightarrow A$ is its own inverse and hence is an automorphsim. If $a \in k[x]$, them $\sigma(a) = a \Rightarrow N(a) = a^2 \in k[x]$; if $a \in A$, then we may write $a = py + q$ with $p, q \in k[x] \Rightarrow N(a) \in k[x]$. $N(1) = 1$ and $N(ab) = N(a)N(b)$ follows from $\sigma(1) = 1$ and $\sigma(ab) = \sigma(a)\sigma(b)$.

(d) Given $a \in A$ with $ab = 1$ for some $b \in A$, we have $N(a)N(b) = 1$, that is, $N(a)$ is a unit in $k[x]$. Again write $a = py + q$, we obtain that $p(x) = 0, q(x) \in k \Rightarrow 0 \neq a \in k$.
The irreducibility of x, y follows from a degree argument on $N(x), N(y)$. Then $A(Y)$ is not a UFD since $y^2 = x^3 - x = x(x^2 - 1)$.

(e) If Y is rational, then Y is either $\mathbb{P}^1$ or an open subset of $\mathbb{A}^1$ and then $A(Y)$ is a UFD, which leads to a contradiction.

Exercise 3 (by Tai-Ning).

(a) Let $X = \mathbb{P}^2$, which is nonsingular and

$$\begin{array}{rccc} \varphi : & X - [0,0,1] & \longrightarrow & \mathbb{P}^1 \\ & [x,y,z] & \longmapsto & [x,y] \end{array}$$

cannot extend to $[0,0,1]$.

(b) Let $X = \mathbb{P}^1$, $Y = \mathbb{A}^1$. Then,

$$\begin{array}{rccc} \varphi : & X - [0,1] & \longrightarrow & \mathbb{A}^1 \\ & [x,y] & \longmapsto & \frac{y}{x} \end{array}$$

cannot extend to $[0,1]$. Since a morphism to $\mathbb{A}^1$ is a regular function, but the only regular function that can be defined on the entire $\mathbb{P}^1$ is constant.

Exercise 4 (by Chi-Kang).

$\varphi : Y \to \mathbb{P}^1$ is a morphism is just by the Riemann mapping theorem. To show is ia surjective with finite fibres. Note that when $Y \subset \mathbb{P}^n$, a rational function is induced by a rational function of $\mathbb{P}^n$, so it is in the form $\frac{f}{g}$, $f, g \in k[x_0, ..., x_n]$, and f, g are homogeneous with same degree. Now realize $\mathbb{P}^1 = \mathbb{A}^1 \cup \infty = \{[a,1] | a \in \mathbb{A}^1\} \cup \{[1,0]\}$. Then $\varphi^{-1}(a)$ in $\mathbb{P}^n$ is the set $\frac{f(x)}{g(x)} = a$, which is equal to the hypersurface defined by $f - ag$. (and $\varphi^{-1}(\infty) = V(g)$).

Exercise 6 (by Yi-Tsung)**.**

(a) Let $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$, $x \mapsto \dfrac{ax+b}{cx+d}$, then consider $\psi : \mathbb{P}^1 \to \mathbb{P}^1$, $x \mapsto \dfrac{1}{ad-bc}\dfrac{dx-b}{-cx+a}$. Then clearly $\varphi \circ \psi = \psi \circ \varphi = \mathrm{id}_{\mathbb{P}^1}$. Hence ϕ is an isomorphism of $\mathbb{P}^1$.

(b) By cor 6.12, giving $\varphi \in \mathrm{Aut}\,(\mathbb{P}^1)$ is equivalent to giving $\varphi' \in \mathrm{Aut}\,(k(x))$. Thus $\mathrm{Aut}\,(\mathbb{P}^1) \cong \mathrm{Aut}\,(k(x))$.

(c) For $\varphi \in \mathrm{Aut}\,(k(x))$, write $\varphi(x) = \dfrac{f(x)}{g(x)}$ with $\gcd\,(f(x), g(x)) = 1$. For $y = \dfrac{f(x)}{g(x)}$, we have $f(x) - yg(x) = 0$. View y as a variable, then since $f - yg \in k\,[y]\,[x]$, we may write $f - yg = \sum_{i=0}^{m} h_i(y)x^i$ for some $h_i(y) \in k\,[y]$ with $\deg h_i \leq 1$. Since $(h_m\,(y)\,x)^m + \sum_{i=0}^{m-1} (h_m\,(y)\,x)^i\, h_m^{m-i-1}\,(y)\,h_i\,(y) = 0$ and $k\,[y]$ is integrally closed, we see that $h_m\,(y)\,x \in k\,[y]$. Say $h_m\,(y)\,x = a\,(y) \in k\,[y]$. Now $a\,(y)^m + \sum_{i=0}^{m-1} a\,(y)^i\, h_m^{m-i-1}\,(y)\,h_i\,(y) = 0$. If $\deg\,(a) \geq 2$, since $\deg\left(\sum_{i=0}^{m-1} a\,(y)^i\, h_m^{m-i-1}\,(y)\,h_i\,(y)\right)$ is at most $\max\limits_{1 \leq i \leq m-1} \{m - i + i\deg\,(a)\} = 1 + (m-1)\deg\,(a)$, then we must have $m\deg\,(a) \leq 1 + (m-1)\deg\,(a)$, which is a contradiction. Hence $\deg\,(a) \leq 1$. Now $x = \dfrac{a\,(y)}{h_m\,(y)}$ with $\deg\,(a)\,, \deg\,(h_m) \leq 1$, by part (a), $y = \phi\,(x)$ is a fractional linear transformation. Thus $\mathrm{PGL}(1) \cong \mathrm{Aut}\,(k\,(x))$, and thus $\mathrm{PGL}(1) \xrightarrow{\sim} \mathrm{Aut}\mathbb{P}^1$.

Exercise 7 (by Jung-Tao)**.**

Denote $P_{r+1} = Q_{s+1} = \infty$, the map from $P^1 - \{P_1, ..., P_{r+1}\}$ to $P^1 - \{Q_1, ..., Q_{r+1}\}$ be ϕ, and its inverse map ϕ^{-1}.

All but finite point in P^1 is a nonsingular quasi-projective curve, and is an abstract nonsingular curve, so we can extend ϕ $(r+1)$-times to get a morphism

$\bar{\phi}$ from P^1 to itself, similarly we can extend ϕ^{-1} to $\overline{\phi^{-1}}$, then $\bar{\phi}\overline{\phi^{-1}}$ is identity at all but finite point, which is an open set. So $\bar{\phi}$ is an isomorphism and $\overline{\phi^{-1}}$ is its inverse, that means $\bar{\phi}$ sends $\{P_1, ..., P_{r+1}\}$ points to $\{Q_1, ..., Q_{s+1}\}$ bijectively, and $r + 1 = s + 1,\ r = s$.

Note that from 6.6, every automorphism of P^1 is of the form $x \mapsto \dfrac{ax+b}{cx+d}$, in most of the cases, three points will determine the map, and if we choose the point arbitrarily, there won't be a map intuitively. To prove it directly, we want to determine the condition when three points identify an automorphism. Suppose $f(x_1) = -y_1, f(x_2) = -y_2, f(x_3) = -y_3$, where x_i's and y_i's are different respectively, we will get $b + x_i a + x_i y_i c + y_i d = 0$, consider the row $[x_i, y_i, x_i y_i]$,
if $x_i = y_i = \infty$, it becomes $[0, 0, 1]$
if $x_i = \infty \neq y_i$, it becomes $[1, 0, y_i]$
if $x_i \neq \infty = y_i$, it becomes $[0, 1, x_i]$
if $x_i, y_i \neq \infty$, it is simply $[x_i, y_i, x_i y_i]$

Notice that even if we force one of the x_i's and y_i's to be ∞, the determinant of the matrix determined by three rows $[x_i, y_i, x_i y_i]$ is not a zero polynomial.

Denote $x_1 = y_1 = \infty$, pick x_2, x_3, x_4, y_2, y_3 as indeterminate, enumerate the bijections from three of the $\{x_1, ..., x_4\}$ to $\{y_1, ..., y_3\}$, every such bijection determines a matrix, note that all of those determinants are not zero polynomial, so the product of them is not a zero polynomial, and we can find x_2, x_3, x_4, y_2, y_3 s.t. the product of the determinants is not zero, that means all of the bijections from three of the $\{x_1, ..., x_4\}$ to $\{y_1, ..., y_3\}$ determines an unique automorphism, and in any situation, f(the left one) is determined, and we just choose y_4 to avoid those finite possibilities.

7 Intersections in Projective Spaces

Exercise 1 (by Yu-Ting)**.**

(a) Denote imρ_d by Y.

$$P_Y(x) = P_{\mathbb{P}^n}(dx) = \binom{dx+n}{n} = \frac{d^n}{n!}x^n + \text{lower degree terms.}$$

Then $\deg Y = d^n$.

(b) Denote the image of segre embbeding by Y. $\dim Y = r + s$.

$$P_Y(x) = P_{\mathbb{P}^r}(x) \cdot P_{\mathbb{P}^s}(x) = \binom{x+r}{x} \cdot \binom{x+s}{x} = \frac{x^{r+s}}{r!s!} + \text{lower degree terms.}$$

Then $\deg Y = \frac{(r+s)!}{r!s!} = \binom{r+s}{r}$.

Exercise 2 (by Yu-Ting)**.**

(a) $p_a(\mathbb{P}^n) = (-1)^n(P_{\mathbb{P}^n}(0) - 1) = (-1)^n \left(\binom{n+0}{n} - 1\right) = 0$.

(b) Let $Y = Z(f)$, where $n = 2$. The exact sequence $0 \to S(-d) \xrightarrow{f} S \to S/(f) \to 0$ implies $\varphi_{S/(f)}(l) = \varphi_S(l) - \varphi_S(l-d) = \binom{l+2}{2} - \binom{l-d+2}{2}$. Then $p_a(Y) = (-1)^1(P_Y(0) - 1) = -\left(\binom{2}{2} - \binom{2-d}{2} - 1\right) = \frac{1}{2}(d-1)(d-2)$.

(c) Let $H = Z(f)$. Similar to (b), we have $\varphi_{S/(f)}(l) = \binom{l+n}{n} - \binom{l-d+n}{n}$. Then $p_a(H) = (-1)^{n-1}\left(\binom{n}{n} - \binom{n-d}{n} - 1\right) = (-1)^n\binom{n-d}{n} = \binom{d-1}{n}$.

(d) Suppose $Y = H_1 \cap H_2$, where H_1 and H_2 are of degree a and b respectively. Let $H_1 = A(f)$ and $H_1 = Z(g)$. Consider the following short exact sequence:

$$0 \to S/(fg) \to S/(f) \oplus S/(g) \to S/(f,g)$$

Then $\varphi_{S/(f,g)}(l) = \varphi_{S/(f)}(l) + \varphi_{S/(g)}(l) - \varphi_{S/(fg)}(l) = \left(\binom{l+3}{3} - \binom{l-a+3}{3}\right) + \left(\binom{l+3}{3} - \binom{l-b+3}{3}\right) - \left(\binom{l+3}{3} - \binom{l-a-b+3}{3}\right)$.
Hence, $p_a(Y) = (-1)\left(-\binom{3-a}{3} - \binom{3-b}{3} + \binom{3-a-b}{3}\right) = \frac{1}{2}ab(a+b-4) + 1$.

(e) $\varphi_{Y\times Z}(l) = \varphi_Y(l) \cdot \varphi_Z(l)$. $p_a(Y \times Z) = (-1)^{r+s}(P_Y(0) \cdot P_Z(0) - 1) = p_a(Y)p_a(Z) + (-1)^s p_a(Y) + (-1)^r p_a(Z)$.

Exercise 3 (by Shuang-Yen)**.**

Let $Y = V(f)$ and let $P = (b_0, b_1, b_2) \in Y$. Let $g = a_0x_0 + a_1x_1 + a_2x_2$ and $L = V(g)$ be a line pass through P, then

$$S(Y \cap L) = k[x_0, x_1, x_2] \Big/ \langle f, g\rangle \cong k[x_0, x_1, x_2]/\langle g\rangle \Big/ \langle \overline{f}\rangle.$$

Since one of a_0, a_1, a_2 is not 0, say $a_0 \neq 0$, then $k[x_0, x_1, x_2]/\langle g\rangle \cong k[x_1, x_2]$ and $\overline{f}$ factors into linear functions $\{h_i\}_{i=1}^d$, where $d = \deg f$. By Chinese remainder theorem,

$$S(Y \cap L) \cong \prod_{j=1}^{d} k[x_1, x_2] \Big/ \langle h_j\rangle$$
$$\Longrightarrow i(Y, L; P) = \#\{i \mid h_i(b_1, b_2) = 0\}.$$

Note that $\overline{f}$ has a multiple root at $(x_1, x_2) = (b_1, b_2)$ if and only if

$$(b_1, b_2) = \frac{\partial \overline{f}}{\partial x_2}(b_1, b_2) = 0.$$

Since

$$\overline{f}(x_1, x_2) = f\left(-\frac{1}{a_0}(a_1x_1 + a_2x_2), x_1, x_2\right),\ \overline{f}(b_1, b_2) = f(P).$$

Also,

$$\frac{\partial \overline{f}}{\partial x_1}(b_1, b_2) = -\frac{a_1}{a_0}\frac{\partial f}{\partial x_0}(P) + \frac{\partial f}{\partial x_1}(P),\ \frac{\partial \overline{f}}{\partial x_2}(b_1, b_2) = -\frac{a_2}{a_0}\frac{\partial f}{\partial x_0}(P) + \frac{\partial f}{\partial x_2}(P).$$

So both of the derivatives equal to zero is equivalent to

$$[a_0 : a_1 : a_2] = \left[\frac{\partial f}{\partial x_0}(P), \frac{\partial f}{\partial x_1}(P), \frac{\partial f}{\partial x_2}(P)\right]$$

Hence the unique line $T_P(Y)$ is

$$V\left(\frac{\partial f}{\partial x_0}(P)x_0 + \frac{\partial f}{\partial x_1}(P)x_1 + \frac{\partial f}{\partial x_2}(P)x_2\right).$$

So the map

$$P \mapsto \left[\frac{\partial f}{\partial x_0}(P), \frac{\partial f}{\partial x_1}(P), \frac{\partial f}{\partial x_2}(P)\right]$$

is a morphism from Reg Y to $(\mathbb{P}^2)^*$ since $\partial f/\partial x_i$'s are polynomials.

Exercise 4 (by Pei-Hsuan).

If a line is not tangent to Y and doesn't pass through any singular point of Y, then by Bezout's Theorem, it must have exactly d intersection with Y. So our goal is to show that the tangent lines and the lines pass through the singular points are contained in a proper closed subset of $(\mathbb{P}^2)^*$.

By exercise 1.7.3, the tangent lines of Y is contained in Y^* which is proper. Also, we have the following fact:

$$1 = \dim Y > \dim \operatorname{Sing} Y.$$

Thus, $\dim \operatorname{Sing} Y = 0$ which means $\operatorname{Sing} Y$ has only finitely many point. For each singular point p, $\{\text{Lines pass through } p\} \cong \mathbb{P}^1$.

Thus, $U = (\mathbb{P}^2)^* \setminus (Y^* \cup \{\text{Lines pass through } \operatorname{Sing} Y\}$ is what we require.

Exercise 6 (by Shi-Xin)**.**

($\Rightarrow$) By Prop.7.6(b) in textbook, Y is irreducible. Assume $\dim Y = 1$ first, and let P, Q be two distinct points in Y. Consider H is a hyperplane containing P, Q. Then if $Y \not\subseteq H$, every irreducible component of $Y \cap H$ has dimension 0. Also, by Thm7.7, $\deg(Y \cap H) = 1$. It follows that $Y \cap H$ is a point, which leads to a contradiction since $P, Q \in Y \cap H$. Therefore, Y is contained in any such hyperplane, and hence it must be a linear variety.

Now, assume $\dim Y = r$. By induction hypothesis, for any hyperplane H doesn't contain Y, we have $Y \cap H$ is a linear variety of dimension $r - 1$. Note that for any two distinct points P, Q in Y, we can choose several hyperplanes which doesn't contain Y such that their intersection is the line $\overline{PQ}$. It follows that $Y \cap \overline{PQ}$ is linear for any $P \neq Q \in Y$. Thus f must be a linear variety.

($\Leftarrow$) Since Y is a linear variety, Y is the intersection of some hyperplanes. Note that every hyperplane has degree 1. Thus by Thm7.7, it forces $\deg Y = 1$.

Exercise 7 (by Chi-Kang)**.**

(a) X is an algebraic set by the definition, to show X is irreducible, suppose $X = X_1 \cup X_2$ be union of closed subsets. Then for any $Q \in Y$, PQ is an irreducible variety, so $PQ \subset X_i$ for some i.
Let $Y_i := \{Q \in Y | PQ \subset X_i\}$, then $Y = Y_1 \cup Y_2$, since Y is irreducible, some Y_i is equal to Y, say Y_1, then we have $X_1 \supset \cup PQ$, since X_1 is closed, we have $X_1 = X$, so X is irreducible.

To show $\dim X = r + 1$, since Y is a proper closed subset of X, we have $\dim X >$r. And for each PQ, there is an isomorphism $\phi_Q : \mathbb{P}^1 \to PQ$ s,t, $\phi_Q(0) = P, \phi_Q(1) = Q$. Now we consider the map

$$\begin{aligned} \varphi : Y \times \mathbb{P}^1 &\to X \\ [Q, z] &\to \phi_Q(z) \\ [P, z] &\to P. \end{aligned}$$

Then we have $r+1=$dim$Y\times\mathbb{P}^1\geq$ dim(imφ)=dim$\cup PQ$=dim$X$, so we conclude $r<$dim$X\leq r+1$, hence dim$X$=r+1.

(b) For convinience to apply induction on zero dimensional case we remove the condition Y is irreducible. When $r=0$ we have deg$Y$ is the number of the points of Y, snd X is a union of deg$Y$-1 lines, so deg$X<$deg$Y$.
Suppose the concequence holds for dim$Y<r$, then for dim$Y$=r, let H b a hyperplane s,t, $P\in H$, $Y\not\supseteq H$, and the intersection multiplicity of H and X,Y of all irreducible component is 1. Then by thm7.7 we have deg$Y$=deg$H\cap Y$, deg$X$=deg$H\cap X$. Since $P\in H$ we have $H\cap X=\cup_{Q\in H\cap Y}PQ$. So by the induction hypothesis we have deg$Y\cap H>$ deg $X\cap H$, so deg$Y>$deg$X$.

Exercise 8 (by Chi-Kang)**.**

Applying the construction of exercise7.7, we have X is a variety of dimension r+1 with deg$X<2$. So by exercise7.6 we have X is a linear variety i,e, $X\cong\mathbb{P}^{r+1}$, and it is obviously $Y\subset X$.