Algebraic Geometry II Homework
Beauville

A course by prof. Chin-Lung Wang

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Exercise 0 (by Kuan-Wen)**.**

This is an example of proof.

Remark. This is an example for how to write in this format.

V Castelnuovo's theorem and applications

Exercise 1 (by Yi-Tsung Wang)**.**

Since $-K$ is ample, we have $K^2 > 0$. If $h^0(2K) = p_2 \neq 0$, then $2K \sim$ effecitve. Moreover, $-mK$ is very ample for sufficiently large m, hence $(-mK).(2K) \geq 0$, which gives $k^2 \leq 0$, contradiction. Therefore $p_2 = 0$. In particular, $p_g = 0$, and by lemma IV.1, we have $q = 0$. By Castelnuovo's rationality criterion, S is rational. Then S is obtained form $\mathbb{P}^2$ or $\mathbb{F}_n (n \neq 1)$ by blowing up some points. If S is obtained from $\mathbb{F}_n$ for some $n \geq 2$ by blowing up some points, since $K_{\mathbb{F}_n} \equiv -2C_0 - (2+n) f$, we have $(-K_{\mathbb{F}_n}).C_0 = -2n + 2 + n = 2 - n \leq 0$, yielding that $-K_{\mathbb{F}_n}$ is not ample, nor is $-K_S$, which is a contradiction. Therefore S is obtained from $\mathbb{P}^2$ or $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ by blowing up some points. Note that the blow up of $\mathbb{P}^2$ at two points is the same as the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at a point, hence S is $\mathbb{P}^1 \times \mathbb{P}^1$ or obtained from $\mathbb{P}^2$ by blowing up r points. In the latter case, $(-K_S)^2 = 9 - r > 0$, so $r \leq 8$. Conversely, for $S = \mathbb{P}^1 \times \mathbb{P}^1$, we have shown that $-K_S$ is ample. For S being obtained from $\mathbb{P}^2$ by blowing up r points ($r \leq 8$), it is a Del Pezzo surface, thus $-K_S$ is ample.

Exercise 3 (by Yu-Chi Hou)**.**

For any surface S, we first decompose $\mathrm{Aut}(S)$ into the identity component $\mathrm{Aut}(S)^\circ$ of the automorphism group $\mathrm{Aut}(S)$ of S and the component group $\mathrm{Aut}(S)/\mathrm{Aut}(S)^\circ$. Then the quotient $\mathrm{Aut}(S)/\mathrm{Aut}(S)^\circ$ is a discrete subgroup. By Chevalley's structure theorem on algebraic groups[1], which states that any connected algebraic groups G has a unique normal affine algebraic subgroup G_{Aff} such that $A := G/G_{\mathrm{Aff}}$ is an abelian variety, we can furthermore decompose $G := \mathrm{Aut}(S)^\circ$ into the exact sequence of groups

$$1 \to G_{\mathrm{Aff}} \to \mathrm{Aut}(S)^\circ \to A \to 1,$$

[1]For the proof of Chevalley's theorem in modern language, one can consult Milne's article `https://www.jmilne.org/math/articles/2013c.pdf` or Brian Conrad's article `http://math.stanford.edu/~conrad/papers/chev.pdf`

where G_{Aff} is a normal affine subgroup of $\text{Aut}(S)^\circ$ and A is an abelian variety. Now, we claim that H must have zero dimension if S is a non–ruled surface. In this case, since G_{Aff} is affine variety of dimension 0, G_{Aff} must be a finite set.

Suppose G_{Aff} has positive dimension, then it must contains an one dimensional subgroup H. By classification of one dimensional affine algebraic group H (cf. for instance, Springer, *Linear Algebriac Groups*, Proposition 3.1.3), $H = \mathbb{G}_a$ or $\mathbb{G}_m$. In general, S/G does not exists as geometric quotient. However, a fact due to Rosenlicht which states the following:

Fact 1 (Rosenlicht, 1956[2])**.** *For any affine algebraic group G acting on an irreducible variety X, there exists an non–empty G-stable open subset $U \subset X_{\text{reg}}$ such that U/G exists as a geometric quotient.*

Thus, applying this to S acting by H, there exists a non-empty $G-$stabl open subset U of S such that U/H exists and has one–dimensional. Thus, U is birational to $H \times U/H$. Also, since U/H is one-dimensional, there exists a smooth projective model C such that $K(C) = K(U/H)$. It is clear that $H = \mathbb{G}_a$ or $\mathbb{G}_m$ is also birational to $\mathbb{P}^1$. This shows that S is birational to $C \times \mathbb{P}^1$, contradiction to the assumption that S is non–ruled.

Now, for the abelian variety A, this induces a map from $A \to \text{Aut}(\text{Alb}(S))$, where $\text{Alb}(S)$ is the Albense variety of S. Since A is connected, A maps into the identity component $\text{Aut}(\text{Alb}(S))^0 = \text{Alb}(S)$. This shows that A must be a abelian subvariety of $\text{Alb}(S)$ and thus shows that A is an abelian variety of dimension$\leq q$.

Exercise 4 (by Po-Sheng Wu)**.**

An automorphisms on $\mathbb{F}_n$ must fixes the unique curve C_0 with negative self-intersection, and permutes the curves of zero self-intersection, that is, the fibers of $\mathbb{F}_n \to \mathbb{P}^1$, hence induces an automorphism on $\mathbb{P}^1$.

The map $\text{Aut}(\mathbb{F}_n) \to \text{Aut}(\mathbb{P}_1) \cong \text{PSL}(2, \mathbb{C})$ is surjective since $\pi^*(\mathcal{O} \oplus \mathcal{O}(n)) \cong \mathcal{O} \oplus \mathcal{O}(n)$ for any $\pi \in \text{Aut}(\mathbb{P}_1)$. The kernel T of this map is determined by an automorphism of $\mathcal{O} \oplus \mathcal{O}(n)$, mod global sections of $\mathcal{O}^*$, that is, $\mathbb{C}^*$, so T has elements of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ in $\text{End}(\mathcal{O} \oplus \mathcal{O}(n))$, where $a \in \mathbb{C}^*, b \in \text{Hom}(\mathcal{O}, \mathcal{O}(n)) = H^0(\mathcal{O}(n))$. Verify the composition rule and we find that T is a semidirect product of $\mathbb{C}^*$ with $H^0(\mathcal{O}(n))$.

Exercise 5 (by Shuang-Yen Lee)**.**

Let X be a surface containing infinitely many exceptional rational curves, say $C_1, C_2, \ldots$. To show that X is biration to $\mathbb{P}^2$, we need to show that $p_2 = q = 0$. If $p_2 \neq 0$, then $2K$ is equivalent to a effective divisor D. Since there are infinitely C_i such that C_i is not a component of D,

$$-2 = (C_i + K).C_i = C_i^2 + K.C_i \geq C_i^2 = -1,$$

a contradiction, so $p_2 = 0$. If $q \neq 0$, then we have the Albanese map $\alpha : X \to \text{Alb}(X)$. Since $p_g = 0$, $\alpha(X)$ is a smooth curve of genus q. Note that $q > 0$, $\alpha|_{C_i}$ is constant for all i, so C_i is contained in a fiber. $C_i^2 = -1$ implies that C_i is a component of a singular fiber, which is finite, a contradiction. So $q = 0$.

[2]The proof of Rosenlicht's result can be found in theorem 4.4 in the survey *Invariant Theory* by Popov and Vinberg appearing in *Algebraic Geometry IV* `https://link.springer.com/book/10.1007/978-3-662-03073-8`

Construction: Let $P_1, \ldots, P_8$ be in general position in $\mathbb{P}^2$ such that P_9 is the basepoint of $|3\ell - \sum_{i=1}^{8} P_i|$, let $S = \mathrm{Bl}_{P_1 \cdots P_9} \mathbb{P}^2$. If $D \in \mathrm{Div}\, S$ such that $D^2 = -1$ and $K_S.D = -1$, then $(K_s + D).D = -2$, which means $p_a(D) = 0$. To show that D is linear equivalent to a rational curve, it suffices to show that D is linear equivalent to an irreducible curve. Write $D \sim aL - \sum_{i=1}^{9} b_i E_i$, then

$$a^2 - \sum_{i=1}^{9} b_i^2 = -1, \quad -3a + \sum_{i=1}^{9} b_i = -1$$

and so for all j,

$$a^2 + 1 = \sum_{i=1}^{9} b_i^2 = \sum_{i \neq j} b_i^2 + (b_j + 1)^2 - 2b_j - 1 \geq \left(\frac{\sum_{i=1}^{9} b_i + 1}{3} \right) - 2b_j - 1 = a^2 - 2b_j - 1,$$

or $b_j \geq -1$. By R-R, $\ell(D) + \ell(K - D) = 1 + s(D) \geq 1$. If $\ell(K - D) \neq 0$, then $K - D \sim C$ for some effective $C \sim (-a-3)L + \sum_{i=1}^{9} (1 + b_i) E_i$. Now,

$$0 \leq -a - 3 = -\frac{\sum_{i=1}^{9} b_i + 1}{3} - 3 \leq -\frac{-8}{3} - 3 < 0,$$

a contradiction. So $\ell(D) > 0$, so $D \sim \sum n_i C_i$ for some C_i irreducible. Note that $-K$ is effective and $\dim |-K| = 1$, $(-K)^2 = 0$, so $C_i(-K) \geq 0$ and "=" iff $C_i \sim -K$, which implies $D \sim C_1 + n(-K)$, for some $n \geq 0$. Then

$$D^2 = -1 \implies -1 = C_1^2 + 2nC_1(-K) = C_1^2 + 2n.$$

By adjunction formula,

$$-2 \leq 2p_a(C_1) - 2 = (C_1 + K).C_1 = -1 - 2n - 1 = -2n - 2,$$

so $n = 0$ and hence $D \sim C_1$ is irreducible and rational. Now, consider the map $D \mapsto D + (\delta_{ijk}.D)\delta_{ijk}$ where $\delta_{ijk} = L - E_i - E_j - E_k$. Then, if $D^2 = D.K = -1$,

$$(D + (\delta_{ijk}.D)\delta_{ijk})^2 = (D + (\delta_{ijk}.D)\delta_{ijk}).K = -1.$$

Write $D \sim aL - \sum_{i=1}^{9} b_\ell E_\ell$, then $D + (\delta_{ijk}.D)\delta_{ijk} \sim (2a - b_i - b_j - b_k)L - \sum_\ell b'_\ell E_\ell$. Since $a = (\sum b_i + 1)/3$, we can always find i, j, k such that $b_i + b_j + b_k < a$, or $2a - b_i - b_j - b_k > a$. So we can always find a D' such that $D'.L > D.L$, hence S has infinitely many -1 rational curve.

VI Surfaces with $p_g = 0$ and $q \geq 1$

Exercise 0 (by Yi-Heng Tsai)**.**

[Serre's lemma: Let $M \in GL(N.\mathbb{Z})$ and r be the order of M. Assume $M \equiv id_N \,(mod\, n)$ for some $n \geq 3$, then $M = id_N$.]
Write $M = id_N + A$ with $A = n(a_{ij})$. Let p be a prime dividing n, then we can write $n = p^\alpha b$ and $a_{ij} = p^{\alpha_{ij}} b_{ij}$ for some $\alpha \in \mathbb{N}, \alpha_{ij} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $p \nmid b, b_{ij}$.($\alpha_{ij} := \infty$ if $a_{ij} = 0$) Assume $A \neq O_N$, then let $\alpha_{lk} \in \mathbb{Z}_{\geq 0}$ be a minimal one among all a_{ij}'s. Hence,

$$O_N = rA + C_2^r A^2 + \cdots + C_r^r (A^r) \tag{1}$$

Then we get $p^{\alpha+\alpha_{ij}} | rbb_{lk}$, so $p^{\alpha+\alpha_{ij}} | r$. Now, by induction on r, we may assume r is a prime. Therefore, $\alpha = 1, \alpha_{ij} = 0$ and $r = n = p$. By (1), $p | a_{lk}$, which is a contradiction.

Exercise 1 (by Yu-Ting Huang)**.**

Let H be a hyperplane section, then $H^2 + H.K_S = 2g(H) - 2 = 0$. This implies $H.K_S < 0$. By Noether-Enrique, S is a ruled surface and $P_n = 0$ for all n. In particular, $p_g = 0$.
Assume $q \geq 2$. Consider the Albanese map, $\alpha : S \to Alb(S)$, where $\alpha(S)$ is a smooth curve of genus q, and the fibers of the map are connected. Note that H is contained in some fiber. Write the fiber $F = \sum_i n_i C_i + mH$, where C_i are irreducible (There might be no C_i.) Then $m^2 H^2 = mH(F - \sum_i n_i C_i) \leq 0$. This contrdicts that $H^2 > 0$.
Now, for $q = 0$, consider the exact sequence

$$0 \to H^0(S, \mathscr{O}_S(K_S)) \to H^0(S, \mathscr{O}_S(K_S + H)) \to H^0(H, \mathscr{O}_H(K_H)) \to H^1(S, \mathscr{O}_S(K_S)),$$

where $h^0(\mathscr{O}_S(K_S)) = p_g = 0$, $h^0(\mathscr{O}_H(K_H)) = g(H) = 1$ and $h^1(\mathscr{O}_S(K_S)) = h^0(\mathscr{O}_S) = p_g - q = 0$. Hence $h^0(\mathscr{O}_S)(K_S + H) = 1$. i.e. $\dim |K_S + H| = 0$. But $H^2 + H.K_S = 0$, so $K_S + H \; 0$ i.e. $K_S \; -H$. By Ex. V.21(2), S is S_d or S'_8.
For $q = 1$, we have already known that S is a ruled surface. Suppose S is birational to $C \times \mathbb{P}^1$. Then by

$$H^0(S, \Omega_S) = H^0(C, \omega_C) \oplus H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}),$$

we have $g(C) = q = 1$. Then S is an elliptic ruled surface.

Remark (by Pei-Hsuan Chang). Another example for the case $q(S) = 1$: Let E be a elliptic curve, $S = E \times \mathbb{P}^1$. Let $P \in E$, $Q \in \mathbb{P}^1$, $D = p_1^*(3P) + p_2^*(Q)$, where p_1, p_2 are the projection of the first and the second position, Since $\deg 3P \geq 2g(E) + 1$, $3P$ is very ample on E. Notice that the map induced by D is a Segre embedding induced by $3P$ and Q on E and $\mathbb{P}^1$ respectively. Thus, D is very ample. Now, using D to embed S into $\mathbb{P}^N$, then $H \in |D|$ is hyperplane section. Notice that $K_S = p_1^*(K_E) + p_2^*(K_{\mathbb{P}^1})$, then we have

$$2g(H) - 2 = H.(K_s + H) = (p_1^*(3P) + p_2^*(Q)).(p_1^*(K_E) + p_2^*(K_{\mathbb{P}^1}) + p_1^*(3P) + p_2^*(Q)) = -6 + 3 + 3 = 0.$$

Hence, $g(H) = 1$.

Remark (by Yu–Chi Hou). Generalizing the previous example by Pei-Hsuan, we consider any elliptic ruled surface with self–intersection number of the distinguished section $C_0^2 = -e$. Hartshorne Exercise V.2.12 shows that any linear system $|C_0 + bF|$ is very ample if and only if $b \geq e + 3$. Observe that for general element $D \in |C_0 + bF|$, D is non–singular by Bertini's theorem. Also, one can compute the genus of D by adjunctin formula:

$$2g(D) - 2 = (C_0 + bF)(C_0 + bF + K_2) = (C_0 + bF)(-C_0 + (b - e)F) = e + b - e - b = 0.$$

Hence, general element of very ample linear system $|C_0 + bF|$ are non–singular elliptic curve.

Exercise 2 (by Shuang-Yen Lee).

(a) Let $C = H \cap S$ be nonsingular, $D = H \cap C \in \mathrm{Pic}(C)$, then $d = \deg D = C^2$. Since S is not ruled, $p_{12} \neq 0$, which implies $H.K \geq 0$. By adjunction formula,

$$C.(C + K) = 2g(C) - 2 \implies \deg D = 2g(C) - 2 - H.K \leq 2g(C - 2).$$

If D is special, then by Clifford, $\dim |D| \leq d/2$. Note that $|D| : C \to \mathbb{P}^{n-1} = H$ is an embedding, $\dim |D| = n - 1$, or $d \geq 2n - 2$. If D is nonspecial, then

$$\dim |D| = d - g(C) \leq d - \left(1 + \frac{d}{2}\right) = \frac{d}{2} - 1$$

by R-R. So $n \leq d/2$, or $d \geq 2n > 2n - 2$. When "=", since D is very ample, we have $D = K_C$, so

$$\deg D = 2g(C) - 2 \implies H.K = 0 \implies K \sim 0$$

since $12K$ is equivalent to an effective divisor.

(b) :)

Exercise 3 (by Yi-Heng Tsai).

If S is bielliptic, then $p_g = 0, q = 1$ and S is minimal non-ruled by Thm.VI.13. Also, $12K \sim 0$ by Prop.VI.15. Hence, $P_{12} = 1$. Conversely, assume $p_g = 0, q = 1$ and S is minimal, we want to show S is bielliptic. Again, by Thm.VI.13, $S = {}^{B \times F}/_{G}$ with B and F are irrational smooth, $g({}^{F}/_{G}) = 0$ and either $1.g(B) = 1$ or $2.g(F) = 1$. Now, it suffices to show $g(B) = g(F) = 1$ in both cases.

1. Assume $g(F) > 1$. Let $\mathcal{L}_{12} = K_{\mathbb{P}^1_k}^{\otimes 12}(\sum[12(1 - 1/e_P)]P)$, then $0 = deg(\mathcal{L}_{12}) = -24 + \sum[12(1 - 1/e_P)]$. Note $-2n + \sum[n(1 - 1/e_P)] = 2g(F) - 2 \geq 2$ by Hurwitz's theorem. Thus, $r \geq 3$.
   (a) If $r \geq 4$, then $r = 4$ and $e_P = 2 \forall P$. So, $-2n + \sum[n(1 - 1/e_P)] = 0 \rightarrow\leftarrow$
   (b) If $r = 3$, then assume $e_1 \leq e_2 \leq e_3$. Thus, $3/e_3 \leq \sum 1/e_P < 1$.
      i. If $e_1 \geq 3$, then $e_1 = e_2 = e_3 = 3 \rightarrow\leftarrow$.

ii. If $e_1 = 2$, then $1/e_2 + 1/e_3 < 1/2 \Rightarrow e_2 \geq 3$.
To be more specific, $(e_2, e_3) = (4, 5), (5, 5), (3, 7), (3, 8), ..., (3, 11)$, which is impossible since $n \leq 4g(F) + 4$.

2. Assume $g(B) > 1$, then there exists at least one ramified point P. Note $P_k(S) = h^0({}^{B}/_{G}, K(\sum[k(1 - 1/e_P)]P) =: D_k)$. Thus, $1 = P_{12}(S) = h^0(D_{12}) = deg(D_{12}) \geq 6 \rightarrow\leftarrow$.

Remark (by Shuang-Yen Lee). We give another solution for the case 1(b)ii. without using the fact $n \leq 4g(F) + 4$.

Proof. Since G is a subgroup of translations, G is abelian, so $G = \prod \mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}$. Let $H_q = \prod_{p_i=q} \mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}$ and let $H_p' = \prod_{q \neq p} H_q$. Now, we factor $F \to F/G$ into $\pi_p : F \to F/H_p'$ and $\pi_p' : F/H_p' \to F/G$. Let P_1, P_2, P_3 be the branch points with ramification index e_1, e_2, e_3, respectively. Then P_i will be a branch point of π_p' if $p \mid e_i$. Hurwitz formula gives

$$2g(F/H_p') - 2 = \deg \pi_p' \cdot (2g(F/G) - 2) + \sum_{p|e_i} \deg \pi_p' \left(1 - \frac{1}{e_i}\right) = \deg \pi_p' \cdot \left(\sum_{p|e_i} \left(1 - \frac{1}{e_i}\right) - 2\right) < 0$$

when $p \mid e_i$ for some i. Thus, $g(F/H_p') = 0$ and then $\deg \pi_p' = (-2)/(-2 + \sum_{p|e_i}(1 - 1/e_i))$. For each case

$$(e_1, e_2, e_3) = (2, 4, 5), (2, 5, 5), (2, 3, 7), (2, 3, 8), (2, 3, 9), (2, 3, 10), (2, 3, 11),$$

we take $p = 5, 2, 2, 3, 2, 3, 2$, respectively, and we get

$$\deg \pi_p' = \frac{5}{3}, \frac{4}{3}, \frac{4}{3}, \frac{3}{2}, \frac{4}{3}, \frac{3}{2}, \frac{4}{3},$$

respectively, a contradiction. □

Exercise 4 (by Yu-Ting Huang)**.**

Suppose we have the surjective fibration $p : S \to B$. By taking Stein factorization, we may assume B is connected. We first consider $g(B) > 0$.
In general, we have the formula

$$0 = \chi_{top}(S) = \chi_{top}(B)\chi_{top}(F_\eta) + \sum_{s \in \Sigma}(\chi_{top}(F_s) - \chi_{top}(F_\eta)),$$

where Σ is the set of points over which p is not smooth. And $2g(F_\eta) - 2 = F_\eta^2 + F_\eta.K \geq 0$, so $g(F_\eta) \geq 1$
For the case that there is no singular fiber,

$$0 = \chi_{top}(B)\chi_{top}(F_\eta) = 4(1 - g(B))(1 - g(F_\eta)).$$

Then whether $g(F_\eta) = 1$, or $g(B) = 1$ and $g(F_\eta) \geq 1$. By proposition VI.8, we can conclude that $S \simeq B \times F_\eta/G$, where one of B and F_η is elliptic.
For the case that there exists a singular fiber, we have $\chi_{top}(B)\chi_{top}(F_\eta) = 4(1 - g(B))(1 - g(F_\eta)) \geq 0$. Then

$\sum_{s\in\Sigma}(\chi_{top}(F_s) - \chi_{top}(F_\eta)) \leq 0$.
For $s \in \sigma$, write $F_s = \sum_i n_i C_i$, where C_i are irreducible. We will show that $\chi_{top}(F_s) \geq \chi_{top}(F_\eta)$.

$$\begin{aligned}\chi_{top}(F_s) &\geq 2\chi(\mathscr{O}_{\cup_i C_i}) = -((\sum_i C_i)^2 + \sum_i C_i.K) \geq -\sum_i C_i.K \geq -\sum_i n_i C_i.K \\ &= F_s.K = F_\eta.K = 2\chi(\mathscr{O}_{F_\eta}) = 2\chi_{top}(F_\eta).\end{aligned}$$

Now, we can conclude that $\chi_{top}(F_s) = \chi_{top}(F_\eta)$, for we have the inequality of two sides. Then all inequalities above turn into equalities. Hence, $F_s = nC$ for some irreducible smooth curve C with $C.K = 0$ (Actually, the above process is similar to proposition VI.6). This implies that $g(C) = 1 + C.K = 1$ and $g(F_\eta) = 1$. Then also by proposition VI.8, the result follows.
(I haven't figured out the case $g(B) = 0$ and $g(F) > 1$, or this case will not happen??) (In that case, proposition VI.8 is even unapplicable.)

VII Kodaira dimension

Exercise 1 (by Pei-Hsuan Chang)**.**

For a divisor D on V. Define

$$L(D) = H^0(V, \mathscr{O}(D)) = \{f \in K(V) \mid (f) + D \geq 0\} \cup \{0\}$$

$$Q(D) = \text{subfield of } K(V) \text{ generated by } L(D),$$

$$Q = \text{sufield of } K(V) \text{ generated by all } Q(D), \text{ for some } D \in |nK|.$$

Notice that $R(V) := \oplus_{n\geq 0} H^0(V, \mathscr{O}(nK) = \oplus_{n\geq 0} L(nK)t^n$, and let n be the smallest number such that $L(nK) \neq 0$. Then notice that for $f \in L(nK)$, f is algebraic over Q, but t^n is not algebraic over Q, so ft^n is not algebraic over Q. Since $ft^n \in \operatorname{Frac} R(V)$, $\operatorname{Frac} R(V)$ is not algebraic over Q. However $\operatorname{Frac} R(V)$ is algebraic over $Q(t)$, so

$$\operatorname{tr.deg} \operatorname{Frac} R(V) = \operatorname{tr.deg} Q + 1.$$

Our goal is to show $\operatorname{tr.deg} Q = \kappa(V)$. For n such that $|nK| \neq \varnothing$, take $D \in |nK|$, then we have

$$L(D) \subseteq L(2D) \subseteq L(3D) \subseteq \cdots \subseteq K(V).$$

So,

$$Q(D) \subseteq Q(2D) \subseteq Q(3D) \subseteq \cdots \subseteq K(V).$$

Since $K(V)$ is finitely generated over k, $\cup_{n\geq 0} Q(nD)$ is also finitely generated over k. Thus, $\exists m \in \mathbb{N}$ such that $Q(mD) = Q((m+1)D) = \ldots$. Since this holds for any n such that $|nK| \neq \varnothing$, so $Q = Q(mD)$ for m large enough. Now, notice that for a effective divisor E, $Q(E) = K(\operatorname{Im}(\varphi_{|E|})$. Hence,

$$\operatorname{tr.deg} Q = \operatorname{tr.deg} Q(mD) = \dim(\operatorname{Im}(\varphi_{|E|})) = \kappa(V).$$

This complete the proof.

Exercise 2 (by Tzu-Yang Tsai)**.**

We know that $\omega_{V\times W} = K_X = p_1^*\omega_V + p_2^*\omega_W$, where $V \xleftarrow{p_1} X = V \times W \xrightarrow{p_2} W$ are projection.

Also, $H^0(V \times W, \mathcal{O}_{V\times W}(nK_{V\times W})) = H^0(V, \mathcal{O}_V(nK_V)) \oplus H^0(W, \mathcal{O}_W(nK_W))\, \forall n \in \mathbb{N}$.
Thereby the map $\phi_n K_{V\times W} : V \times W \dashrightarrow \mathbb{P}^{N_{V\times W}}$ can be factor as:

$$\phi_{nK_{V\times W}} : V \times W \overset{(\phi_{nK_V}, \phi_{nK_W})}{\dashrightarrow} \mathbb{P}^{N_V} \times \mathbb{P}^{N_W} \overset{\text{Segre's embedding}}{\hookrightarrow} \mathbb{P}^{N_{V\times W}}$$

Thus $\dim(Im(\phi_{nK_{V\times W}})) = \dim(Im(\phi_{nK_V})) + \dim(Im(\phi_{nK_W}))\, \forall n \in \mathbb{N}$
Let $a, b \in \mathbb{N}$ s.t. $\dim(Im(\phi_{aK_V})) = \kappa(V), \dim(Im(\phi_{bK_W})) = \kappa(W)$, then

$$\kappa(V) + \kappa(W) \geq \kappa(V \times W) \geq \dim(Im(\phi_{abK_{V\times W}})) = \dim(Im(\phi_{abK_V})) + \dim(Im(\phi_{abK_W})) = \kappa(V) + \kappa(W)$$

Therefore $\kappa(V) + \kappa(W) = \kappa(V \times W)$

Exercise 3 (by Yu–Chi Hou)**.**

Given any surjective morphism $f : X \to Y$ between varieties, one has $K(Y) \subset K(X)$. Also, the induced map on global section of pluricanonical bundle

$$f^* : H^0(Y, rK_Y) \hookrightarrow H^0(X, rK_X)$$

is injective, for any $r \geq 0$. In view of Exercise 1 in this chapter, we see that pluricanonical ring of Y is a subring of pluricanonical ring of X and thus $\kappa(Y) \leq \kappa(X)$.

Now, observe that f is étale if and only if it is flat and $\Omega_{X/Y} = 0$. Thus, the exact sequence of Kähler differential gives

$$f^*\omega_X \to \omega_Y \to 0.$$

Since X and Y are smooth varieties, $f^*\omega_X$ and ω_Y are both invertible sheaves, $f^*\omega_X \cong \omega_Y$ and thus

$$H^0(X, rK_X) \cong H^0(Y, rK_Y).$$

This of course shows that $\kappa(X) = \kappa(Y)$.

VIII Surfaces with $\kappa = 0$

Exercise 1 (by Yi-Heng Tsai)**.**

Since $K^2 + \mathcal{X}_{top}(S) = \mathcal{X}(\mathcal{O}_S) = 0$ and $K^2, \mathcal{X}_{top}(S) \geq 0$, we have $K^2 = \mathcal{X}_{top}(S) = 0$. By Ex.VI.4, $S = {}^{B\times F}/_G$ with $g(B) = 1$. Thus, $g({}^B/_G) = 0, 1$ and $B \times F \to S$ is étale.

1. $(g({}^B/_G) = 0)$ Note $q = h^0(\Omega_S) = g({}^B/_G) + g({}^F/_G)$, so $g({}^F/_G) = 2$. Also, $P_2 = h^0({}^F/_G, \mathcal{L}_2)$ where $\mathcal{L}_2 = \omega^2_{F/G}(\sum_P [2(1 - 1/e_P)]P)$. Apply Riemann-Roch theorem, we have $P_2 \geq 3 + \sum[2(1 - 1/e_P)] > 1$.

2. $(g({}^B/_G) = 1)$ Similarly, $g({}^F/_G) = 1$. If $g(F) > 1$, $r \geq 3$ by Hurwitz's theorem. In this case, $P_2 \geq \sum[2(1 - 1/e_P)] \geq 3 > 1$. On the other hand, if $g(F) = 1$, then $P_n = 1 \forall n \geq 1$. In this case, $\kappa(S) = 0$, which implies S is an Abelian surface.

Exercise 10 (by Chi-Kang Chang)**.**

For $g = 2k - 1$ case, consider the double cover $f : S \to \mathbb{P}^1 \times \mathbb{P}^1$ which is branch on a (4,4) curve, hten we have $K_S = f^*(K_{\mathbb{P}^1 \times \mathbb{P}^1} + R$ with R be the ramified locus with coeffienent 1. Thus $K_S = f^*(-2h_1 - 2h_2) + R = 0$.

Now since $h_1 + kh_2$ is very ample on $\mathbb{P}^1 \times \mathbb{P}^1$, then since f is finite, we have $f^*|h_1 + kh_2|$ is ample and base point free. Thus the general element $C \in f^*|h_1 + kh_2|$ is smooth (and hence reduced). To show the irreducibility, consider the exact sequence $0 \to \mathscr{O}_S(-C) \to \mathscr{O}_S \to \mathscr{O}_C \to 0$, this induces the cohomology sequence

$$0 \to H^0(S, -C) \to H^0(S, 0) \to H^0(C, 0) \to H^1(S, -C)...$$

Since by Kodaira vanishing $H^1(S, -C) = 0$, by the above sequence we conclude $H^0(C, 0) = \mathbb{C}$, thus C is connected, hence irreducible by the smoothness. And then we have $C^2 = (\deg(f))(h_1 + kh_2)^2 = 4k = 2g - 2$, thus $g(C) = 2k + 1$.

Next we show that S is K3. Now we consider the exact sequence $0 \to \mathscr{O}_S \to \mathscr{O}_S(C) \to \mathscr{O}_C \to 0(C)$, this induces the cohomology sequence

$$0 \to H^0(S, 0) \to H^0(S, C) \to H^0(C, C|_C) \to H^1(S, 0) \to H^1(S, C) \to H^1(C, C|_C)...$$

Again by Kodaira vanishing, $\chi(\mathscr{O}_S(C)) = h^0(\mathscr{O}_S(C) = g - q + 1$ by surfece Riemann-Roch. On the other hand, again by surface Riemann-Roch and Kodaira vanishing

$$\begin{aligned} h^0(\mathscr{O}_S(C)) &= h^0(\mathbb{P}^1 \times \mathbb{P}^1, h_1 + kh_2) \\ &= \chi(\mathbb{P}^1 \times \mathbb{P}^1, h_1 + kh_2) = v2k + 2 = g + 1. \end{aligned}$$

Thus $q = 0$ and hence S is K3. Finally, since $h^0(\mathscr{O}_S(C)) = g + 1$, we know that $\phi_{|C|}$ is a 2 to 1 map sends S to $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^g$, completes our proof.

For $g = 2k$ case, let $\mathbb{F}_1$ be the ruled surface $\mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-1))$. Then take the double cover of $\mathbb{F}_1$ branch over a curve linearly equivalent to $4C_0 + 6f$, then repeat the similar computation of $g = 2k + 1$ case again, we get our consequence.

Exercise 12 (by Yi-Tsung Wang)**.**

For S being a K3 surface, consider the exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

It gives the long exact sequence

$$0 \to H^1(S, \mathcal{O}) \to \operatorname{Pic}(S) = H^1(S, \mathcal{O}^*) \xrightarrow{\alpha} H^2(S, \mathbb{Z}) \xrightarrow{\beta} H^2(S, \mathcal{O}) = H^0(S, \mathcal{O}) = \mathbb{C}$$

For $x \in (H^2(S, \mathbb{Z}))_{\mathrm{tor}}$, $\beta(x) \in \mathbb{C}_{\mathrm{tor}} = \{0\}$, that is, $\beta(x) = 0$. Hence there exists $L \in \operatorname{Pic}(S)$ such that $\alpha(L) = x$. Suppose $mx = 0$, then $\alpha(mL) = 0$. Since α is injective, we have $mL = 0$. In particular, $L \equiv 0$. By Riemann-Roch theorem, $h^0(L) + h^0(-L) \geq 2$, we see that either $h^0(L) \geq 1$ or $h^0(-L) \geq 1$. No matter which the case holds, we have $L \sim 0$, and then $x = 0$, that is, $(H^2(S, \mathbb{Z}))_{\mathrm{tor}} = 0$. By mixed variance universally coefficient theorem,

$$H^2(S, \mathbb{Z}) = \operatorname{Ext}^1_{\mathbb{Z}}(H_1(S, \mathbb{Z}), \mathbb{Z}) \oplus \operatorname{Hom}_{\mathbb{Z}}(H_2(S, \mathbb{Z}), \mathbb{Z})$$

Write $H_1(S, \mathbb{Z}) = \mathbb{Z}^r \oplus (H_1(S, \mathbb{Z}))_{\mathrm{tor}}$, then we have

$$\operatorname{Ext}^1_{\mathbb{Z}}(H_1(S, \mathbb{Z}), \mathbb{Z}) = \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}^r, \mathbb{Z}) \times \operatorname{Ext}^1_{\mathbb{Z}}((H_1(S, \mathbb{Z}))_{\mathrm{tor}}, \mathbb{Z}) = (H_1(S, \mathbb{Z}))_{\mathrm{tor}}$$

Therefore $0 = (H^2(S, \mathbb{Z}))_{\mathrm{tor}} = (H_1(S, \mathbb{Z}))_{\mathrm{tor}}$. By Poincaré duality and Hodge decomposition theorem, we have

$$H_1(S, \mathbb{Z}) \otimes \mathbb{C} = H_1(S, \mathbb{C}) = H^3(S, \mathbb{C})^\vee \quad \text{and} \quad H^3(S, \mathbb{C}) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3} = 0$$

Thus $H_1(S, \mathbb{Z}) \otimes \mathbb{C} = 0$, which says $r = 0$. Therefore we conclude that $H_1(S, \mathbb{Z}) = 0$.
For X being an Enrique surface, we also have the long exact sequence

$$0 \to \operatorname{Pic}(X) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}) = H^0(X, K) = 0$$

Hence $\operatorname{Pic}(X) \cong H^2(X, \mathbb{Z})$. For $L \in (\operatorname{Pic}(X))_{\mathrm{tor}}$, let $mL = 0$ for some $m \in \mathbb{N}_{n \geq 2}$, we have $h^0(L) = h^0(-(m-1)L) = 0$. By Riemann-Roch theorem, $h^0(K - L) \geq 1$, so $K - L \geq 0$, and $-2L \geq 0$. Since $L \equiv 0$, we get $-2L \sim 0$. Note that $2K \sim 0$, then $K - L \sim -K + L$, and then $-(K - L) \geq 0$, thus $K - L \sim 0$, that is, $K \sim L$. Since $p_g = 0$, we have $K \not\sim 0$. Therefore $(\operatorname{Pic}(X))_{\mathrm{tor}}$ (and then $(H^2(X, \mathbb{Z}))_{\mathrm{tor}}$) is $\mathbb{Z}/2\mathbb{Z}$ generated by $[K]$.

IX Surfaces with $\kappa = 1$

Exercise 2 (by Pei-Hsuan Chang)**.**

Recall Let E be an Euclidean space with a positive definite symmetric billnear form (,). Define $\sigma_\alpha := \beta - \dfrac{2(\beta,\alpha)}{(\alpha,\alpha)}$, and $<\beta,\alpha>:= \dfrac{2(\beta,\alpha)}{(\alpha,\alpha)}$.

Definition. Φ is called a *root system* in E if

1. Φ is finite, span E and does not contain 0.
2. If $\alpha \in \Phi$, then the only multiple of α in Φ are exactly $\pm\alpha$.
3. If $\alpha \in \Phi$, then $\sigma_\alpha(\Phi) = \Phi$.
4. If $\alpha, \beta \in \Phi$, then $<\beta,\alpha>\in \mathbb{Z}$.

Definition. We can choose a *set of positive roots* $\Phi^+ \subset \Phi$. This is a subset of Φ such that

1. $\forall \alpha \in \Phi$, exactly one of the roots $\alpha, -\alpha$ is contain in Φ^+.
2. $\forall \alpha, \beta \in \Phi^+$ such that if $\alpha + \beta \in \Phi$, then $\alpha + \beta \in \Phi^+$.

An element of Φ^+ is said to be simple root of it cannot be written as sum of two elements in Φ^+.

Solution. By adjunction formula,

$$02g(F) - 2 = F.(F + K_S) = F.K_S = (\sum n_i C_i).K_S = \sum n_i (2g(C_i) - 2 - C_i^2).$$

Notice that $C_i^2 < 0$, $\forall i$, so "$2g(C_i) - 2 - C_i^2 < 0 \Rightarrow g(C_i) = 0, C_i^2 = -1$". Since S is minimal, we get

$$2g(C_i) - 2 - C_i^2 \geq 0.$$

Thus,

$$0 = C_i.K_S = 2g(C_i) - 2 - C_i^2,\ \forall i.$$

Hence, $g(C_i) = 0, C_i^2 = -2,\ \forall i$. Also, Corollary VIII.4 says that

$$0 \geq (C_i + C_j)^2 = -4 + 2C_i.C_j,$$

so we know that $C_i.C_j \leq 2$ and "$C_i.C_j = 2 \Leftrightarrow F = m(C_i + C_j)$ for some $m \in \mathbb{Q}$". So there are two cases:

$$(1)\ F = m(C_1 + C_2) \qquad (2)\ C_i.C_j = 0 \text{ or } 1,\ \forall i \neq j.$$

For the second case, let M' be the $\mathbb{Z}$-modules generated by $\{C_i\}$ in Pic S. M' is free, since if $\sum m_iC_i \sim 0$, then $(\sum m_iC_i)^2 = 0$. By Corollary VIII.4 again, $\sum m_iC_i = rF$, for some $r \in \mathbb{Q}$, so $rF \sim 0 \Rightarrow r = 0 \Rightarrow m_i = 0$, $\forall i$. Now, define $M = M'/\mathbb{Z}[F]$. The intersection pairing induce a well-defined symmetric billnear form on M, since $C_i.F = 0$ and $F^2 = 0$. Now, let $(a, b) = -\frac{1}{2}a.b$, so

$$(C_i, C_j) = \begin{cases} 1 & \text{, if } i = j \\ -\frac{1}{2}\text{or}0 & \text{, if } i \neq j \end{cases}$$

Again, by Corollary VIII.4, $\forall a \in M$, $(a, a) = -\frac{1}{2}a^2 > 0$, so $(\, , \,)$ is positive definite.

Now, let $\Phi := \{r \in M | (r, r) = -\frac{1}{2}r^2 = 1\}$. It is easy to check Φ is a root system. Also, let $\Phi^+ := \{r \in \Phi | r = \sum m_iC_i \text{ with } m_i \geq 0, \forall i\}$. It is clearly a set of positive roots, and $\{C_i\}$ are all simple roots. Finally, we only need to check that Φ is of type A_n, D_n, E_n. Since

$$< C_i, C_j >< C_j, C_i >= \frac{2(C_i, C_j)}{(C_j, C_j)}\frac{2(C_j, C_i)}{(C_i, C_i)} = 0\text{or}1,$$

any two points in the Dynkin diagram are connected by at most one line. By the classification of Lie algebra, Φ cannot of the type B_n, C_n, G_2. Thus, Φ must of type A_n, D_n, E_n.

Exercise 4 (by Shuang-Yen Lee)**.**

S is Enriques implies $p_g = q = 0$, $2K \sim 0$. By R-R,

$$h^0(K + E) - h^1(K + E) + h^2(K + E) = \frac{1}{2}(K + E).E + 1 = 1,$$

so $h^0(K + E) \geq 1$ since $h^2(K + E) = h^0(-E) = 0$, and hence $|K + E| \neq \varnothing$.

Let $E' \in |K + E|$, then $E.E' = E(K + E) = 0$ and $E \not\sim E'$ implies $E \cap E' = \varnothing$. The exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(E) \longrightarrow \mathcal{O}_E(E) \longrightarrow 0$$

implies $h^0(E) = 1 + h^0(E|_E)$, so $h^0(E) = 1$ or 2 since $E^2 = 0$.

If $h^0(E) = 1$, by the exact sequence

$$0 \longrightarrow \mathcal{O}_S(-E - E') \longrightarrow \mathcal{O}_S(-E') \longrightarrow \mathcal{O}_E(-E') \longrightarrow 0,$$

we get $0 \to \mathbb{C} = h^0(-E'|_E) \to h^1(-E - E')$ exact, implies $h^1(-E - E') \geq 1$ and by R-R,

$$h^0(-E - E') - h^1(-E - E') + h^2(-E - E') = \frac{1}{2}(-E - E')(-E - E' - K) + 1 = 1,$$

so $h^0(2E) \geq 2$. The exact sequence

$$0 \longrightarrow \mathcal{O}_S(E) \longrightarrow \mathcal{O}_S(2E) \longrightarrow \mathcal{O}_E(2E) \longrightarrow 0$$

implies that $h^0(2E) \leq h^0(E) + h^0(2E|_E) = 2$ so $h^0(2E) = 2$ and hence $\dim |2E| = 1$. Since $2E \sim 2E'$ and $2E \cap 2E' = \varnothing$, $|2E| : S \to \mathbb{P}^1$ is base-point-free, then $|2E| = A \cup B$, where $A = \{D \in |2E| \mid D \text{ is smooth } \}$ and $B = |2E| \setminus A$ is finite. For $D \in A$, write $D = \sum C_i$, then $D^2 = 0$ implies that $\sum C_i^2 = 0$ since $C_i \cap C_j = \varnothing$ for $i \neq j$. If there's i such that $C_i^2 < 0$, then $C_i.(2E) = C_i.D = C_i < 0$ implies $C_i.E < 0$, but E is numerically effective, a contradiction. So $C_i^2 = 0$ for all i, implies $g(C_i) = 1$.

By Stein factorization theorem, $|2E| : S \to \mathbb{P}^1$ can be factored into $\pi : S \to C$ and a finite morphism $C \to \mathbb{P}^1$. Then $\pi(D) = \{P_1, \ldots, P_n\}$ with $C_i = \pi^{-1}(P_i)$. Then π is an elliptic fibration, by Ex. IX. 1, we have $0 = q(S) = g(C)$ or $g(C) + 1$, which means $C \cong \mathbb{P}^1$. Then $C_i \sim C_j$, this gives $D \sim nF$, but general fiber is reduced, and hence $n = 1$. So $|2E|$ is a pencil of elliptic curves.

If $h^0(E) = 2$, then $h^0(E|_E) = 1$, or $E|_E = \mathcal{O}_E$. Let $D \in |E|$ and suppose that $D \neq E$, then $D \cap E = \varnothing$, which means $|E|$ is base-point-free, by s same arguement as above, we have $|E|$ is a pencil of elliptic curves.

X Surface of General Types

Exercise 1 (by Yi-Tsung Wang)**.**

Since $K^2>0$, we may assume that $p_g\geq 3$. Write $|K|=|C|+V$, where $|C|$ is the mobile part and V is the fixed part. Then we have $C^2\geq 0$ and $C.V\geq 0$, and then $2-2g_a(C)=(K+C).C\geq 0$. Hence

$$\begin{aligned}
K^2&=\frac{1}{2}\left(K\left(C+V\right)+\left(C+V\right)K\right)\\
&\geq\frac{1}{2}\left(K.C+K^2\right)\\
&=p_a\left(C\right)-1\\
&=h^1\left(C,\mathcal{O}_C\right)-1\\
&=h^0\left(C,\Omega^1_C\right)-1\\
&=h^0\left(C,\mathcal{O}_X\left(K_X+C\right)|_C\right)-1\\
&=h^0\left(C,2C+V\right)-1\\
&\geq h^0\left(C,2C\right)-1
\end{aligned}$$

For $1,s_1,\dots,s_{n-1}\in\Gamma\left(C,\mathcal{O}_C\left(C\right)\right)$, note that $1,s_1,\dots,s_{n-1},s_1^2,s_1s_2,\dots,s_1s_{n-1}\in\Gamma\left(C,\mathcal{O}\left(2C\right)\right)$ are linearly independent sections, hence $h^0\left(C,2C\right)\geq 2h^0\left(C,\mathcal{O}_C\left(C\right)\right)-1$. Now consider the exact sequence

$$0\to\mathcal{O}_X\to\mathcal{O}_X\left(C\right)\to\mathcal{O}_X\left(C\right)|_C\to 0$$

It gives that $h^0\left(C,\mathcal{O}_C\left(C\right)\right)\geq h^0\left(\mathcal{O}_X\left(C\right)\right)-1$. Hence we conclude that $K^2\geq 2h^0\left(C,\mathcal{O}_C\left(C\right)\right)-2=2p_g-4$.

Exercise 2 (by Ping-Hsun Chuang)**.**

Proof. Since $H_1\left(S,\mathbb{Z}\right)$ is finitely generated with free rank $q=0$, we have $H_1\left(S,\mathbb{Z}\right)$ is finite. Consider the abelian universal cover of S, $S^{\mathrm{ab}}\xrightarrow{p}S$. Note that p is étale since it is a covering space. Moreover, p is of degree $|H_1\left(S,\mathbb{Z}\right)|$ since $\pi_1\left(S^{\mathrm{ab}}/S\right)=\left[\pi_1\left(S\right),\pi_1\left(S\right)\right]$ and $H_1\left(S,\mathbb{Z}\right)=\pi_1\left(S\right)/\left[\pi_1\left(S\right),\pi_1\left(S\right)\right]$. Say $|H_1\left(S,\mathbb{Z}\right)|=d$. Then, we have $K^2_{S^{\mathrm{ab}}}=K^2_S=d$ and $\chi\left(S^{\mathrm{ab}}\right)=d\chi\left(S\right)=d$. By Noether inequality, we have $p_g\geq\dfrac{K^2+4}{2}$. Note that $\pi^*K_S=K_{S^{\mathrm{ab}}}$ implies $K_{S^{\mathrm{ab}}}$ is nef, that is, S^{ab} is minimal. Then, we have

$$d=\chi\left(S^{\mathrm{ab}}\right)=1-q\left(S^{\mathrm{ab}}\right)+p_g\left(S^{\mathrm{ab}}\right)\leq 1+p_g\left(S^{\mathrm{ab}}\right)\leq 1+\frac{K^2_{S^{\mathrm{ab}}}+4}{2}=1+\frac{4+d}{2}.$$

Hence, $d\leq 6$.

Now, we need to exclude the case $d=6$. For $d=6$, we have $q\left(S^{\mathrm{ab}}\right)=0$, $K^2_{S^{\mathrm{ab}}}=d=6$, and Noether equality. If the equality holds in Noether inequality, then write $K=C+V$, where C is moving part and V is fixed part. Then, $C.V=0$ and $K.V=0$ (See proof in class.) Moreover, using the fact that the general $C\in|K|$ are hyperelliptic curve, we may find an automorphism on C such that it has fixed point. (Here, we need $g\left(C\right)\geq 2$, this holds since now $g\left(C\right)=\dfrac{C\left(C+K\right)}{2}+1=K^2+1=7$.) However, the automorphism of S^{ab} cannot have fixed point except identity since it is the covering space. This isa contradiction. Hence, we proved $|H_1\left(S,\mathbb{Z}\right)|\leq 5$.

However, I cannot conclude $|H_1\left(S,\mathbb{Z}\right)|=5$ implies S is Godeaux. □

Exercise 3 (by Yu-Chi Hou)**.**

Consider $G := (\mathbb{Z}/2\mathbb{Z})^3$ acting on $\mathbb{P}^6$ by

$$\begin{aligned} e_1 &: [x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \to [-x_0 : x_1 : x_2 : x_3 : -x_4 : -x_5 : -x_6] \\ e_2 &: [x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \to [x_0 : -x_1 : x_2 : -x_3 : x_4 : -x_5 : -x_6] \\ e_3 &: [x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \to [x_0 : x_1 : -x_2 : -x_3 : -x_4 : x_5 : -x_6] \end{aligned}$$

From the construction of the action, one see that any point which is fixed by e_i has at least three coordinates vanishing for each $i = 1, 2, 3$. Thus, consider

$$S' = Z(\sum_{i=0}^{6} a_i x_i^2, \sum_{i=0}^{6} b_i x_i^2, \sum_{i=0}^{6} c_i x_i^2, \sum_{i=0}^{6} d_i x_i^2),$$

for any a_i, b_i, c_i, d_i's such that any maximal minors of the matrix

$$\Lambda = \begin{pmatrix} a_0 & \cdots & a_6 \\ b_0 & \cdots & b_6 \\ c_0 & \dots & c_6 \\ d_0 & \dots & d_6 \end{pmatrix}$$

are non–zero. The condition for the minors shows that S' is a non–singular complete intersection in $\mathbb{P}^6$. Moreover, it has degree $2^4 = 16$ and $K_{S'} \sim (-7H + 8H)|_{S'} = H|_{S'}$. Obviously, S' is $G-$invariant. If e_i acts on S' trivially, say e_1 for instance, then $x_1 = x_2 = x_3 = 0$. Hence, the equation of S' is given by

$$a_0x_0^2+a_4x_4^2+a_5x_5^2+a_6x_6^2 = b_0x_0^2+b_4x_4^2+b_5x_5^2+b_6x_6^2 = c_0x_0^2+c_4x_4^2+c_5x_5^2+c_6x_6^2 = d_0x_0^2+d_4x_4^2+d_5x_5^2+d_6x_6^2 = 0$$

In matrix notation, this gives

$$\begin{pmatrix} a_0 & a_4 & a_5 & a_6 \\ b_0 & b_4 & b_5 & b_6 \\ c_0 & c_4 & c_5 & c_6 \\ d_0 & d_4 & d_5 & d_6 \end{pmatrix} \begin{pmatrix} x_0^2 \\ x_4^2 \\ x_5^2 \\ x_6^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since the maximal minors of Λ are non–zero, the matrix from the left–hand side is invertible. Hence, it only has trivial solution $x_0^2 = x_4^2 = x_5^2 = x_6^2 = 0$ and thus $x_0 = x_4 = x_5 = x_6 = 0$. However, $[0 : \cdots : 0] \notin \mathbb{P}^6$. Similarly, we can repeat the argument for e_2 and e_3. Thus, G acts on S' freely.

Since G acts on S' freely and hence $S' \to S = S'/G$ is a deg 8 unramified covering. Since S' is complete intersection and hence the irregularity $q(S') = 0 = q(S)$. Also, $p_g(S') = 7$ since $K_{S'} \sim H|_{S'}$, where H is the hyperplane on $\mathbb{P}^6$. Hence, $\chi(\mathcal{O}_{S'}) = 1 - q(S') + p_g(S') = 8$ and $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_{S'})/8 = 1$. Finally, $K_S^2 = K_{S'}^2/8 = 16/8 = 2$.

In sum, we obtain a surface which $p_g = 0 = q$ and $K^2 = 2$. This is a surface of general type, which again provides an example showing that Castelnuovo's rationality criterion is sharp.

Exercise 4 (by Po-Sheng Wu)**.**

Let $C_1 = C_2 = C$ and $S = (C_1 \times C_2)/G$. We choose $\phi(a,b) = (a-2b, a-4b)$. Note that the action of $(a,b) \in G$ on C_1 has fixed points iff $a=0$ or $b=0$ or $a=b$, while acting on C_2 has fixed points iff $a-2b=0$ or $a-4b=0$ or $2a-6b=0$, thus G acts freely on $C_1 \times C_2$. Note that $q = g(C_1/G) + g(C_2/G) = 0$ since $2g(C) - 2 = n(2g(C/G) - 2) + \deg R$ and $g(C) = 6, n = 25, R = 3 \cdot 20$, and $\chi(\mathcal{O}_S) = \frac{1}{25}\chi(\mathcal{O}_{C\times C}) = \frac{1}{25}\chi(\mathcal{O}_C)^2 = \frac{1}{25}(1-g(C))^2 = 1$, so $p_g = 0$. $K_S^2 = \frac{1}{25}K_{C\times C} = \frac{1}{25}(p_1^*(\omega_{C_1}) + p_2^*(\omega_{C_2}))^2 = \frac{1}{25} \cdot 2 \cdot 10 \cdot 10 = 8$.

To give another example, consider C the complete intersection of $x^3+y^3+z^3+w^3=0$ and $xy=zw$ in $\mathbb{P}^3$, which by adjunction formula has genus 4. $(a,b) \in G = (\mathbb{Z}/3)^2$ acts on C by $(a,b)(x,y,z,w) = (\omega^a x, \omega^{-a}y, \omega^b z, \omega^{-b}w)$, $\omega^3 = 1$, $S = (C\times C)/G$ with action $g(x,y) = (gx, \phi(g)y)$ with $\phi(a,b) = (a+b, a-b)$. Similarly we have $q = 0$, $\chi(\mathcal{O}_S) = \frac{1}{9}\cdot(-3)^2 = 1 \Rightarrow p_g = 0$ and $K_X^2 = \frac{1}{9}\cdot 2\cdot 6\cdot 6 = 8$.