Algebraic Geometry II Homework
Appendix A: Intersection Theory

A course by prof. Chin-Lung Wang

2020 Spring

Exercise 6 (by Yi-Heng Tsai)**.**

Let $i : X \to X \times X$ be the diagonal map. Consider the exact sequence $0 \to \mathcal{T}_X \to \mathcal{T}_{X\times X}|_X \to \mathcal{N} \to 0$, then we have $c(\mathcal{T}_X).c(\mathcal{N}) = c(\mathcal{T}_{X\times X}|_X) = c(\mathcal{T}_X)^2$. Hence, $c_n(\mathcal{T}_X) = c_n(\mathcal{N}) = i^*i_*(1_X) = \Delta^2$ by (C7).

Exercise 7 (by Yu–Chi Hou)**.**

By Hirzebruch–Riemann–Roch, for any non–singualar projective $n-$fold X, any locally free sheaf $\mathcal{E}$ on X, one has

$$\chi(\mathcal{E}) = \deg(\mathrm{ch}(\mathcal{E}).\mathrm{td}(\mathcal{T}_X))_n, \tag{HRR}$$

where $\mathcal{T}_X$ is the tangent sheaf of X. Now, we consider the case $n = 3$.

First, for $\mathcal{E} = \mathcal{O}_X$, one has

$$p_a(X) = (-1)^3(\chi(\mathcal{O}_x - 1) = 1 - \chi(\mathcal{O}_X)$$

and

$$c(\mathcal{O}_X) = 1 \in A^*(X).$$

Therefore, HRR gives

$$\chi(\mathcal{O}_X) = 1 - p_a(X) = \deg(\mathrm{td}(\mathcal{T}_X))_3.$$

Let $c_i := c_i(X)$ be the $i-$th Chern classes of X, for $i = 1, 2, 3$. Then one has

$$\mathrm{td}(\mathcal{T}_X) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2,$$

Thus, $\deg(\mathrm{td}(\mathcal{T}_X))_3 = c_1c_2/24$. This proves the first part.

For the second part, for any divisor D, take $\mathcal{E}$ to be the associated invertible sheaf $\mathcal{O}_X(D)$. Then it has Chern polynomial $c_t(\mathcal{O}_X(D)) = 1 + tD$ and thus its Chern character is given by

$$\mathrm{ch}(\mathcal{O}_X(D)) = 1 + D + \frac{1}{2}D^2 + \frac{1}{6}D^3.$$

By HRR,

$$\chi(\mathcal{O}_X(D)) = \deg((1 + D + \frac{1}{2}D^2 + \frac{1}{6}D^3))(1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2))_3$$

By direct computing, the degree 3 component of right hand side is given by

$$\frac{1}{24}c_1c_2 + \frac{1}{12}D(c_1^2 + c_2) + \frac{1}{4}D^2c_1 + \frac{1}{6}D^3. \tag{1}$$

Since $c_1(X) = -c_1(\omega_X) = -K$ and $c_1c_2/24 = \chi(\mathcal{O}_X) = 1 - p_a(X)$ by previous part, (1) becomes

$$1 - p_a(X) + \frac{1}{12}(D.K^2 - 3D^2.K + 2D^3) + \frac{1}{12}D.c_2 = \frac{1}{12}D.(D-K).(2D-K) + \frac{1}{12}D.c_2$$

This proves the second assertion.

Exercise 8 (by Yu–Chi Hou)**.**

As in exercise 8, for $X = \mathbb{P}^3$,

$$\mathrm{td}(\mathcal{T}_X) = 1 + \frac{1}{2}c_1(X) + \frac{1}{12}(c_1^2(X) + c_2(X)) + \frac{1}{24}c_1(X)c_2(X),$$

On the other hand, $\mathcal{E}$ is a rank 2 locally free sheaf with Chern classes c_1, c_2

$$\mathrm{ch}(\mathcal{E}) = 2 + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2)$$

By (HRR),

$$\chi(\mathcal{E}) = \deg_3(\mathrm{td}(\mathcal{T}_X)\mathrm{ch}(\mathcal{E}))_3$$

Hence,

$$\chi(\mathcal{E}) = \frac{1}{12}c_1(X)c_2(X) + \frac{1}{12}c_1.(c_1^2(X) + c_2(X)) + \frac{1}{4}c_1(x).(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2).$$

Then the left hand side is an integer, but right hand side is a priori a rational number only. Since $X = \mathbb{P}^3$, by Euler sequence, $c_k(X) = \binom{4}{k}h^k$, where $h \in A^1(X)$ is the hyperplane class. Therefore, $c_1(X) = 4h$, $c_2(X) = 6h^2$, $c_3(X) = 4h^3 = 4$. As a result,

$$\chi(\mathcal{E}) = 2 + \frac{11c_1.h^2}{6} + h.(c_1^2 - 2c_2) + \frac{c_1^3}{6} - \frac{c_1.c_2}{2}.$$

Exercise 9 (by Yi-Heng Tsai)**.**

(a) The goal is to verify $d^2 - 10d - 5H.K - 2K^2 + 12 + 12p_a = 0$. By the definition of rational cubic scroll, we have $K \equiv -2C_0 - 3f$ and $H \equiv C_0 + 2f$. Thus, $d^2 - 10d - 5H.K - 2K^2 + 12 + 12p_a = 9 - 30 + 25 - 16 + 12 + 0 = 0$.

(b) By the definition of K3 surface, we have $K = 0$ and $p_a = 1$. Thus the formula in (4.1.3) becomes $d^2 - 10d + 24 = 0$, which implies $d = 4, 6$.

(c) Again, by the definition of abelian surface, the formula in (4.1.3) becomes $d^2 - 10d = 0$, which implies $d = 10$.

(d) Assume X_e can be embedded in $\mathbb{P}^4$ by the very ample divisor $H \equiv aC_0 + bf$. Then $5 = h^0(\mathcal{L}(H)) = h^0(\pi_*\mathcal{L}(H)) = h^0((\mathcal{O} \oplus \mathcal{O}(-e))^{\otimes a} \otimes \mathcal{O}(b)) = \oplus_{i=1}^{a} C_i^a \mathcal{O}(b - ie) = (\sum_{i=1}^{a} C_i^a (b - ie + 1))$. Note that we have $a > 0, b > ae$ and $e \geq 0$. Thus, $(a, b, e) = (1, 2, 1)$. Indeed, by the above facts, $a = 1, 2, 3, 4$. When $a = 1$,we have $5 = 2b - e + 2$, which implies $(a, b, e) = (1, 2, 1)$. When $a = 2$,we have $5 = 4b - 4e + 4$, which is impossible. The rest cases($a = 3, 4$) admit no solution similarly. Combining with (a), the rational ruled surface $X_e, e \geq 0$ which admitting an embedding in $\mathbb{P}^4$ is the rational cubic scroll in $\mathbb{P}^4$.

Exercise 10 (by Yi-Heng Tsai)**.**

Suppose we have such embedding, then consider the exact sequence $0 \to \mathcal{T}_X \to \mathcal{T}_{\mathbb{P}^5}|_X \to \mathcal{N}_{Y/X} \to 0$ where $i : X \to Y = \mathbb{P}^5$ is the embedding. Thus, $c(\mathcal{T}_{\mathbb{P}^5}|_X) = c(\mathcal{T}_X).c(\mathcal{N}_{Y/X})$ i.e. $1 + 6x + 15x^2 + 20x^3 = 1.(1 + c_1(\mathcal{N}_{Y/X}) + c_2(\mathcal{N}_{Y/X})$ with x the pullback of $\mathcal{O}(1)$. Therefore we get $20x^3 = 0$, which is impossible.

Algebraic Geometry II Homework
Appendix B: Transcendental Methods

A course by prof. Chin-Lung Wang

2020 Spring

Exercise 1 (by Yu–Chi Hou)**.**

Suppose $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\} = X^h$ for some scheme X over $\mathbb{C}$. Then X must be $1-$dimensional non–singular algebraic variety over $\mathbb{C}$. Since X is quasi–projective, we can take normalization Y of closure of X with respect to the embedding. Hence, we can write

$$X = Y \setminus \{P_1, \dots, P_n\},$$

for some $n \geq 1$. Also, Y is non–singular complete (and hence projective) curve. By Chow's theorem, its associated analytic space $Y^h = Y$ is a compact Riemann surface of genus g. If $g \geq 1$, then Y is not simply connected and hence X^h which is puntured Riemann surface is not simply connected as well. However, $\mathbb{D}$ is contractible and hence simply conected. Therefore, one must have $Y = \mathbb{P}^1$. If $n > 2$, then X^h is homeomorphic to $n-1$ puntured complex plane, which is again not simply connected. If $n = 1$, then $X^h \cong \mathbb{C}$. However, by Liouville theorem, there cannot be any non–constant holomorphic map from $\mathbb{C}$ to $\mathbb{D}$. Thus, $\mathbb{C}$ cannot be biholomorphic to $\mathbb{D}$. This gives the contradiction.

Exercise 2 (by Tzu-Yang Chou)**.**

Assume that there exists an desired ideal sheaf $\mathscr{I}$. Then since it is over affine line, it corresponds to an ideal of $\mathbb{C}[z]$, which is principal. But its generator can not have infinitely many zeros since it's a polynomial, which leads to a contradiction. On the other hand, by complex analysis, we can construct a holomorphic function f with prescribed zeros $z_1, z_2, ...$ and then the sheaf corresponding to the $\mathbb{C}[z]$-module generated by f is our desired coherent sheaf.

Algebraic Geometry II Homework
Appendix C: The Weil Conjectures

A course by prof. Chin-Lung Wang

2020 Spring

Exercise 1 (by Yu–Chi Hou)**.**

Let $N_r^{(i)}$ be the number of $\mathbb{F}_{q^r} := k_r-$rational points of $\bar{X}_i = X_i \times_k \bar{k}$. Since $X = \coprod_i X_i$, $N_r = \sum_i N_r^{(i)}$, for all $r \in \mathbb{N}$ and for all i. Hence, the Zeta function for X is given by

$$Z(X;t) = \exp\left(\sum_{r=1}^{\infty} N_r \frac{t^r}{r}\right) = \exp\left(\sum_{r=1}^{\infty}\left(\sum_i N_r^{(i)}\right)\frac{t^r}{r}\right) = \exp\left(\sum_i \sum_{r=1}^{\infty} N_r^{(i)} \frac{t^r}{r}\right)$$

Thus,

$$Z(X;t) = \prod_i \exp\left(\sum_{r=1}^{\infty} N_r^{(i)} \frac{t^r}{r}\right) = \prod_i Z(X_i;t).$$

Exercise 2 (by Po-Sheng Wu)**.**

For the projective space of dimension n we have $N_r = \dfrac{q^{r(n+1)}-1}{q^r-1} = (1+q^r+\cdots+q^{rn})$, so

$$Z(\mathbb{P}_n, t) = \exp(\sum_{r=1}(1+q^r+\cdots+q^{rn})\frac{t^r}{r}) = \exp(\sum_{i=0}^{n} -\log(1-q^i t)) = \frac{1}{(1-t)(1-qt)^2\ldots(1-q^n t)}$$

Exercise 3 (by Yu–Chi Hou)**.**

Observe that $N_r(X \times \mathbb{A}^1) = N_r(X) \times N_r(\mathbb{A}^1) = q^r N_r(X)$. Hence, the Zeta function

$$Z(X \times \mathbb{A}^1; t) = \exp\left(\sum_{r=1}^{\infty} N_r(X \times \mathbb{A}^1)\frac{t^r}{r}\right) = \exp\left(\sum_{r=1}^{\infty} N_r(X)\frac{(qt)^r}{r}\right) = Z(X;qt).$$

Exercise 4 (by Yu–Chi Hou)**.**

Consider

$$\zeta_X(s) := \prod_{x \in X_{\mathrm{cl}}} (1 - N(x)^{-s})^{-1},$$

where X_{cl} is the set of closed points of X, $N(x) = |k(x)|$. Since X is defined over $k = \mathbb{F}_q$, $x \in X_{\mathrm{cl}}$ if and only if $k(x)/k$ is a finite algebraic extension and hence $N(x) = q^{[k(x):k]}$. Let $D_r := \{x \in X_{\mathrm{cl}} : [k(x):k] = r\}$ and $d_r := |D_r|$, then for $x \in D_r$, $N(x) = q^r$. Thus, we can write $\zeta_X(s)$ into

$$\zeta_X(s) = \prod_{r \geq 1} (1 - q^{-rs})^{-d_r}$$

Next, observe that for $k_l = \mathbb{F}_{q^l}$, we claim the following:

Claim.

$$N_l := |\overline{X}(k_l)| = \sum_{r|l} r d_r$$

Assuming the claim for now, we show that $\zeta_X(s) = Z(X; q^{-s})$. Taking logarithm on $Z(X; q^{-s})$,

$$\log Z(X; q^{-s}) = \sum_{l \geq 1} N_l \frac{q^{-sl}}{l} = \sum_{l \geq 1} \left(\sum_{r|l} r d_r \right) \frac{q^{-ls}}{l} = \sum_{r \geq 1} d_r \sum_{k \geq 1} \frac{q^{-rks}}{k} = -\sum_{r \geq 1} d_r \log(1 - q^{-rs}).$$

On the other hand, taking logarithm on $\zeta_X(s)$,

$$\log \zeta_X(s) = -\sum_{r \geq 1} d_r \log(1 - q^{-rs})$$

Hence, this porves the assertion.

Proof of Claim. Recall that a k_l-rational point of $\overline{X}$ is a morphism of scheme $\mathrm{Spec}(k_l) \to \overline{X}$ and thus

$$\overline{X}(k_l) = \mathrm{Hom}_{\mathrm{Spec}\, k}(\mathrm{Spec}(k_l), \overline{X})$$

By Hartshorne Ex.II.2.7 and the fact that x is a k_l-rational points if and only if $k(x)/k$ is finite algebraic and thus $x \in X_{\mathrm{cl}}$,

$$\overline{X}(k_l) = \mathrm{Hom}_{\mathrm{Spec}\, k}(\mathrm{Spec}(k_l), \overline{X}) = \coprod_{x \in X_{\mathrm{cl}}} \mathrm{Hom}_{k-\mathrm{alg}}(k(x), k_l) = \coprod_{x \in D_r,\, r|l} \mathrm{Hom}_{k-\mathrm{alg}}(k(x), k_l)$$

Now, for $x \in D_r$, then $\mathrm{Hom}_{k-\mathrm{alg}}(k(x), k_l)$ has a transitive action by $\mathrm{Gal}(\mathbb{F}_{q^l}/\mathbb{F}_q) \cong \mathbb{Z}/l\mathbb{Z}$. Moreover, the stabilizer of each element is just $\mathrm{Gal}(\mathbb{F}_{q^l}/\mathbb{F}_{q^r})$. Thus,

$$N_l = |\overline{X}(k_l)| = \sum_{x \in D_r,\, r|l} |\mathrm{Hom}_{k-\mathrm{alg}}(k(x), k_l| = \sum_{x \in D_r,\, r|l} \frac{|\mathrm{Gal}(\mathrm{F_{q^l}}/\mathrm{F_q})|}{|\mathrm{Gal}(\mathbb{F}_{q^l}/\mathbb{F}_{q^r})|} = \sum_{x \in D_r,\, r|l} r = \sum_{r|l} r d_r.$$

$\square$

Exercise 5 (by Yu–Chi Hou)**.**

By Weil conjecture, we know that the zeta function for X is given by

$$Z(X;t) = \frac{(1-\alpha_1 t)\cdots(1-\alpha_{2g}t)}{(1-t)(1-qt)},$$

Write $N_r = 1 - a_r + q^r$, where $\alpha_1, \dots, \alpha_{2g}$ are algebraic integers with $|\alpha_i| = q^{1/2}$. By Exercise 7, $a_r = \sum_{i=1}^{2g} \alpha_i^r$. Thus, for $r > 2g$, $a_r = \sum_{i=1}^{2g} \alpha_i^r$ is the symmetric polynomial in $\alpha_1, \dots, \alpha_{2g}$, which can be generated by the power sum symmetric polynomial $a_1, \dots, a_{2g}$. Hence, knowing $N_1, \dots, N_{2g}$ suffices to determine N_r for all $r \geq 1$. It remains to show that $N_1, \dots, N_g$ already determine $N_{g+1}, \dots, N_{2g}$.

In view of the proof of Theorem 4.4 in Hartshorne Appendix C, one finds that in the case of curve, $B_1(C) = 2g$. The functional equation is given by

$$Z(X; \frac{1}{qt}) = q^{1-g} t^{2-2g} Z(X;t)$$

A little unwinding shows (cf. calculation in Exercise 7c)

$$(qt - \alpha_1)\cdots(qt - \alpha_{2g}) = q^g (1-\alpha_1 t)\cdots(1-\alpha_{2g}t).$$

This shows that the set (counted multiplicities) $\{\alpha_1, \dots, \alpha_{2g}\}$ is invariant under $x \mapsto q/x$. Moreover, since $P(t) = (1-\alpha_1 t)\cdots(1-\alpha_{2g}t)$ is a rational polynomial in t, the roots of $P(t)$ must come in conjugate pairs. By Riemann hypothesis, $|\alpha_i| = q^{1/2}$, $\overline{\alpha_i} = |\alpha_i|^2/\alpha_i = q/\alpha_i$. Combining these observations, we may rename indices of α_i so that $\alpha_{2g-i+1} = q/\alpha_i$, for $i = 1, \dots, g$. Thus, we can write $P(t)$ as

$$P(t) = (1-\alpha_1 t)(1-\alpha_2 t)\cdots(1-\alpha_g t)(1-\frac{q}{\alpha_g}t)\cdots(1-\frac{q}{\alpha_1}t) = c_{2g}t^{2g} + c_{2g-1}t^{2g-1} + \cdots + c_1 t + c_0,$$

where $c_{2g} = q^g$, $c_0 = 1$. By above observation, consider

$$f(t) = \prod_{i=1}^{2g}(t-\alpha_i) = t^{2g} + c_1 t^{2g-1} + \cdots + c_{2g-1}t + c_{2g},$$

then $t^{2g} f(q/t)$ has the same roots as $f(t)$. In other words,

$$t^{2g} f(q/t) = q^g + c_1 q^{2g-1} t + \cdots + c_{2g} t^{2g} = q^g c_0 t^{2g} + q^g c_1 t^{2g-1} + \cdots + q^g c_{2g-1} t + c_{2g} q^g = q^g f(t).$$

Then by comparing coefficients, one finds that

$$c_{g+l} = q^l c_{g-l}, \quad l = 0, \dots, g.$$

Also, $c_i = (-1)^i \sigma_i$, where σ_i is the elementary symmetric polynomial of degree i in $\alpha_1, \dots, \alpha_{2g}$. Thus, $\sigma_{g+l} = q^l \sigma_{g-l}$. By Newton's identity,

$$\sigma_k = \frac{1}{k}\left(\sum_{i=1}^{k-1}(-1)^{i-1}\sigma_{k-i}a_i\right), \quad k = 1, \dots, g.$$

Hence, $\sigma_1, \dots, \sigma_g$ is determined by $a_1, \dots, a_g$. On the other hand,

$$a_k = (-1)^{k-1} k\sigma_k + \sum_{i=1}^{k-1} (-1)^{k-1+i} \sigma_{k-i} a_i, \quad k = g+1, \dots, 2g.$$

shows that $a_{g+1}, \dots, a_{2g}$ can be determined inductive by $a_1, \dots, a_g$ and σ_i. From $\sigma_{g+l} = q^l \sigma_{g-l}$, we conclude that $a_{g+1}, \dots, a_{2g}$ can be determined by $a_1, \dots, a_g$. Therefore, $N_i = 1 + q^i - a_i$ is determined by $N_1, \dots, N_g$, for $i = g+1, \dots, 2g$.

Exercise 6 (by Po-Sheng Wu)**.**

By IV.Ex 4.16(c), we have $N_r = 1 - a_r + q^r = 1 - (f^r + \hat{f}^r) + q^r$ (identity in $\mathbb{Q}(f) \subset \mathrm{End}^0(E)$), so

$$\begin{aligned} Z(t) &= \exp(\sum_{r=1} (1 - (f^r + \hat{f}^r) + q^r) \frac{t^r}{r}) \\ &= \exp(-\log(1-t) + \log(1+ft) + \log(1+\bar{f}t) - \log(1+qt)) \\ &= \frac{(1-ft)(1-\bar{f}t)}{(1-t)(1-qt)} = \frac{1 - at + qt^2}{(1-t)(1-qt)} \end{aligned}$$

$E = 0$, thus $Z(1/(qt)) = \dfrac{1 - a/(qt) + 1/(qt^2)}{(1-1/(qt))(1-1/t)} = \dfrac{qt^2 - at + 1}{(qt-1)(t-1)} = Z(t)$. Since $|a| \le 2q^{1/2}$, the two roots of $1 - ta + qt^2$ are either both $\pm q^{1/2}$ or non-real, hence are conjugate with each other and has absolute value $q^{1/2}$.

Exercise 7 (by Yu-Chi Hou)**.**

(a) For a curve C of genus g defined over $\mathbb{F}_q$, the Zeta function is given by

$$Z(C; t) = \frac{(1-\alpha_1 t)\cdots(1-\alpha_{2g} t)}{(1-t)(1-qt)},$$

where $\alpha_1, \cdots, \alpha_{2g}$ are all algebraic integers. Taking logarithm on both sides,

$$\sum_{r=1}^{\infty} N_r \frac{t^r}{r} = \log Z(C; t) = \sum_{i=1}^{2g} \log(1-\alpha_i t) - \log(1-t) - \log(1-qt).$$

Using the Taylor expansion $\log(1-x) = -\sum_{r=1}^{\infty} \frac{x^r}{r}$ to expand right hand side, one has

$$\sum_{r=1}^{\infty} N_r \frac{t^r}{r} = \sum_{r=1}^{\infty} \frac{(1 + q^r - \sum_{i=1}^{2g} \alpha_i^r) t^r}{r}.$$

By comparing coefficients and writing $N_r = 1 + q^r - a_r$, we obtain

$$a_r = \sum_{i=1}^{2g} \alpha_i^r.$$

(b) If $|\alpha_i| \leq \sqrt{q}$, then

$$|a_r| = \left|\sum_{i=1}^{2g} \alpha_i^r\right| \leq \sum_{i=1}^{2g} |\alpha_i|^r \leq 2g\sqrt{q^r}, \quad \text{for all } r.$$

Conversely, if $|a_r| \leq 2g\sqrt{q^r}$ for all r, then consider the generating function

$$\sum_{r=1}^{\infty} a_r t^r = \sum_{r=1}^{\infty}\sum_{i=1}^{2g} \alpha_i^r t^r = \sum_{i=1}^{2g}\sum_{r\geq 1} (\alpha_i t)^r = \sum_{i=1}^{2g} \frac{\alpha_i t}{1-\alpha_i t}. \tag{A}$$

Notice that

$$\sum_{r=1}^{\infty} |a_r|\,|t|^r \leq \sum_{r=1}^{\infty} 2g(q^{1/2}|t|)^r = \frac{2gq^{1/2}|t|}{1-q^{1/2}|t|}.$$

Thus, for $|t| < q^{-1/2}$, the generating function converges absolutely and hence the expression in (A) is legitimate. Moreover, (A) shows that the generating function has poles at $t = 1/\alpha_i$. But above estimates shows that those poles cannot occur in the disk $|t| < q^{-1/2}$. As a result, we obtain that for all $i = 1, \ldots, 2g$, $|1/\alpha_i| \geq q^{-1/2}$ and hence $|\alpha_i| \leq \sqrt{q}$.

(c) By functional equation,

$$Z(C; 1/qt) = \pm q^{E/2} t^E Z(C;t).$$

Since $E = c_2(C) = 2 - 2g$ and C is a curve (cf. Discussion in Exercise 5),

$$Z(C; \frac{1}{qt}) = q^{1-g} t^{2-2g} Z(C;t)$$

That is,

$$\frac{(1-\frac{\alpha_1}{qt})\cdots(1-\frac{\alpha_{2g}}{qt})}{(1-\frac{1}{qt})(1-\frac{1}{t})} = q^{1-g}t^{2-2g}\frac{(1-\alpha_1 t)\cdots(1-\alpha_{2g}t)}{(1-t)(1-qt)}$$

and thus

$$\frac{(t-\frac{\alpha_1}{q})\cdots(t-\frac{\alpha_{2g}}{q})}{(qt-1)(t-1)} = q^{-g}\frac{(1-\alpha_1 t)\cdots(1-\alpha_{2g}t)}{(1-t)(1-qt)}.$$

This gives

$$(t-\alpha_1/q)\cdots(t-\alpha_{2g}/q) = q^{-g}(1-\alpha_1 t)\cdots(1-\alpha_{2g}t).$$

This shows that $|\alpha_1\cdots\alpha_{2g}| = q^g$. Thus, if there exists $j \in \{1, 2, \ldots, 2g\}$ such that $|\alpha_j| < \sqrt{q}$, then $|\alpha_1\cdots\alpha_{2g}| < q^g$, a contradiction. In conclusion, one must have $|\alpha_i| = \sqrt{q}$, for all i.