2023 SPRING - INTRODUCTION TO MODULAR FORMS

MIDTERM EXAM

A COURSE BY CHIN-LUNG WANG AT NTU

1. (a) Show that
$$\tau^{-2}E_2(-1/\tau) = E_2(\tau) + 12/(2\pi i\tau).$$
(b) Show that
$$\eta(\tau) := q_{24}\prod_{n=1}^{\infty}(1-q^n)$$
satisfies
$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$$
and conclude that $\eta^{24} \in \mathcal{S}_{12}(\mathrm{SL}_2(\mathbb{Z}))$.

2. (a) Show that for any congruence subgroup there are only finite many elliptic points. Also the isotropy groups are finite cyclic. What are they?
(b) Show that there are no elliptic points for $\Gamma_0(N)$ for N divisible by any prime $p \equiv -1 \pmod{12}$.

3. (a) Let Γ be a congruent subgroup. Derive the genus formula
$$g(X(\Gamma)) = 1 + \frac{d}{12} - \frac{\epsilon_2}{4} - \frac{\epsilon_3}{3} - \frac{\epsilon_\infty}{2}$$
where d is the degree of $X(\Gamma) \to X(1)$, and the dimension formula for $\mathcal{S}_k(\Gamma)$ for k even. (In terms of g, ϵ_2, ϵ_3 and ϵ_∞.)
(b) Show that
$$\mathcal{M}(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6]$$
and $\mathcal{S}(\mathrm{SL}_2(\mathbb{Z}))$ is the ideal $\langle \Delta \rangle$.

4. Let $N, k \geq 3$ with $\bar{v} \in (\mathbb{Z}/N\mathbb{Z})^2$ be a row vector of order N. Let
$$E_k^{\bar{v}}(\tau) := \sum_{(c,d)\equiv v\ (N);\,\gcd(c,d)=1} (c\tau + d)^{-k}.$$
(a) Show that
$$E_k^{\bar{v}}[\gamma]_k = E_k^{\overline{v\gamma}}, \qquad \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$
Moreover, the set $\{E_k^{\bar{v}}\}_{\bar{v}}$ with $v = (c, d)$ such that $\{-d/c\}_{\bar{v}}$ represents the set of cusps of $\Gamma(N)$ form a basis of $\mathcal{E}_k(\Gamma(N))$.
(b) Show that $E_k^{\bar{v}}$ is a linear combination of the unnormalized versions $G_k^{\bar{v}}$'s as
$$E_k^{\bar{v}} = \sum_{n\in(\mathbb{Z}/N\mathbb{Z})^\times} \zeta_+^n(k,\mu)\, G_k^{n^{-1}\bar{v}}.$$

Date: April 27, 2023, pm 6:00 – 9:00 at AMB 201. Total 120 points. You may work on each part independently.

5. (a) For $f = \sum_{m=0}^{\infty} a_m(f)q^m \in \mathcal{M}_k(N,\chi) \subset \mathcal{M}_k(\Gamma_1(N))$, show that

$$a_m(T_n f) = \sum_{d|(m.n)} \chi(d)\, d^{k-1}\, a_{mn/d^2}(f).$$

(b) For $f \in \mathcal{M}_k(\Gamma_0(N))$, show that $T_n f = \sum_{\gamma \in M_n} f[\gamma]_k$ where

$$M_n := \bigcup_{0<d|n,\,(n/d,N)=1} \bigcup_{j=0}^{d-1} \begin{bmatrix} n/d & j \\ 0 & d \end{bmatrix}.$$[1]

(c) Let $f \in \mathcal{M}_k(\Gamma_1(N))^{\text{new}}$ which is an eigenform for T_n and $\langle n \rangle$ for all n with $(n,N) = 1$. Show that the restriction $(n,N) = 1$ can be removed.

6. Let $f \in \mathcal{S}_k(\Gamma_1(N))$ with Mellin transform

$$g(s) := \int_0^{\infty} f(it)\, t^s\, \frac{dt}{t}$$

and $\Lambda(s) := N^{s/2} g(s)$.

(a) Show that

$$g(s) = (2\pi)^{-s}\Gamma(s)L(s,f)$$

for $\mathrm{Re}(s) > \frac{k}{2} + 1$.

(b) Let

$$(Wf)(\tau) := i^k N^{-k/2} \tau^{-k} f(-1/(N\tau)).$$

Show that W is idempotent. Moreover, if $Wf = \pm f$ then

$$\Lambda(s) = \pm\Lambda(k-s).$$

* You may replace one problem by presenting an essential topic in Ch.1–Ch.5 which you have well-prepared but not listed above.

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[1] It is in fact the coset representatives of

$$\coprod_{\det\alpha=n,\,\alpha\in M_2(\mathbb{Z})} \Gamma_0(N)\alpha\Gamma_0(N),$$

hence leads to an intrinsic definition of T_n using double cosets.