Witten genus and Exotic Spheres

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This report consists of two main parts. First, we will prove that the Witten genus is modular. Second, we will count a certain kind of exotic spheres. That is, we will count the differentiable structures of S^{4k-1}. For the second part, we first use that Fourier series of a Witten genus is integral to derive the divisibility of signatures. Then since the signature characterizes when two homotopy spheres are diffeomorphic, we can therefore differentiate two exotic spheres.

Contents

1 Preliminary

Definition 1.1 (Cobordism)**.** Two closed oriented manifolds M^n and N^n are called cobordant $M \sim N$ in case that $M + (-N) = \partial W$, where W is an $n+1$ dimensional compact oriented manifold. Here $+$ denotes disjoint union, and $-N$ is the manifold N with reversed orientation. The relation is an equivalence relation. Reflexive: $V + (-V) = \partial(V \times I)$ where $I = [0, 1]$. Symmetry: OK. Transitive: Glue two manifolds along diffeomorphic boundary components.

Definition 1.2. Let Ω^n denote the set of equivalence classes of n dimensional, compact, oriented smooth manifold w.r.t. $\sim$. The $(\Omega^n, +)$ is an finitely generated abelian group. And the Cartesian product induces a map $\Omega^n \times \Omega^m \to \Omega^{n+m}, (A, B) \mapsto A \times B$. Thus $\Omega = \sum_{n=0}^{\infty} \Omega^n$ is a graded commutative ring with identity, the so called cobordism ring. The equivalence class of a point is the unit.

Theorem 1.3 (Thom)**.** $\Omega^n \otimes \mathbb{Q} = 0$ for $4 \not| n$ and Ω^{4k} is a finitely generated abelian group with rank equal to the number of partitions of k.

Definition 1.4 (Chern class)**.** For each complex vector bundle E over a manifold X, there exists Chern classes $c_i(E)$ with

1. $c_i(E) \in H^{2i}(X, \mathbb{Z})$, $c_0(E) = 1$, and the total Chern class is $c(E) := \sum_{i=1}^{\infty} c_i(E) \in H^*(X, \mathbb{Z})$.
2. $c_i(f^*E) = f^*c_i(E)$.
3. $c(E \oplus F) = c(E) \cdot c(F)$.
4. $c(H) = 1 - g$. H refers to the tautological bundle over complex projective n space $\mathbb{P}^n$ and $g \in H^2(\mathbb{P}^n, \mathbb{Z})$ denotes the generating element of the cohomology ring of $\mathbb{P}^n$.

Definition 1.5 (Pontryagin class)**.** Let E be a real vector bundle over X. The Pontryagin class $p_i(E)$ are defined to be $p_i(E) = (-1)^i c_{2i}(E \otimes \mathbb{C}) \in H^{4i}(X, \mathbb{Z})$.

Definition 1.6 (Almost complex manifold)**.** A smooth manifold is said to have an almost complex structure if its tangent bundle is derived from a complex bundle, where a complex bundle is a vector bundle whose fibers are complex vector spaces.

Definition 1.7 (Chern numbers, Pontryagin numbers)**.** Let X be a compact, oriented, almost complex manifold of dimension $2n$ and $(i_1, \ldots, i_r)$ be a partition of n. Then the Chern number corresponding to this partition is $\left(\prod_{j=1}^{r} c_{i_j}(X)\right)[X]$.

Similarly, the Pontryagin number for a compact, oriented, manifold of dimension $4n$ is defined to be $\prod_{j=1}^{r} p_{i_j}(X)[X]$.

Theorem 1.8 (Thom)**.** Two manifolds represent the same class in $\Omega \otimes \mathbb{Q}$ precisely when all their Pontryagin numbers coincide.

Definition 1.9 (Genus)**.** Let R be an integral domain over $\mathbb{Q}$. Then a genus is a ring homomorphism $\varphi : \Omega \otimes \mathbb{Q} \to R$ with $\varphi(1) = 1$.

Definition 1.10 (Genus corresponding to a power series Q)**.** Let $Q(x) = 1 + a_2x^2 + a_4x^4 + \cdots$ be an even power series $a_i \in R$. We assign a weight two for each variable x_i $1 \leq i \leq n$. Then $\prod_{i=1}^{n} Q(x_i)$ is symmetric in x_i^2.

$$\prod_{i=1}^{n} Q(x_i) = 1 + a_2 \sum_{i=1}^{n} x_i^2 + \cdots$$

The term of weight $4r$ can thus be expressed as a homogeneous polynomial $K_r(p_1, \ldots, p_r)$ of weight $4r$ in the elementary symmetric functions p_j of the x_i^2. That is,

$$\prod_{i=1}^{n} Q(x_i) = 1 + K_1(p_1) + K_2(p_1, p_2) + \cdots + K_n(p_1, \cdots, p_n) + K_{n+1}(p_1, \cdots, p_n, 0) + \cdots$$

Now we can define the genus associated to the power series $Q(x)$. Let M a compact, oriented, smooth manifold of dimension $4n$. The genus φ_Q corresponding to a power series Q is defined to be

$$\varphi_Q(M) := K_n(p_1, \ldots, p_n)[M] \in R$$

with $p_i = p_i(M) \in H^{4i}(M, \mathbb{Z})$. To make our notation neat in calculation, we also define that

$$K(TM) := K(p_1, \ldots, p_n) := 1 + K_1(p_1) + K_2(p_1, p_2) + \cdots$$

and $K(M) := K(TM)[M]$, we have $\varphi_Q(M) = K(M)$. Note that $K_i(p_1, \cdots, p_i)[M]$ is defined to be zero if it cannot evaluate at the fundamental cycle. One can verify that φ_Q is a well-defined genus.

2 Witten genus is modular

Let $L \subset \mathbb{C}$ be a lattice, and recall the Weierestrass σ-function for the lattice L:

$$\sigma_L(x) = x \cdot \prod_{\omega \in L \setminus \{0\}} \left(\left(1 - \frac{x}{\omega}\right) \exp\left(\frac{x}{\omega} + \frac{x^2}{2\omega^2}\right)\right) \tag{1}$$

The infinite product converges uniformly on compact sets, due to the exponential convergence factors. The function $\sigma_L(x)$ is odd, so that $Q(x) = x/\sigma_L(x) = 1 + a_2x^2 + a_4x^4 + \cdots$ is even.

Lemma 2.1. For $\tau \in \mathbb{H}$, $\sigma(\tau, x)$ has the two following expressions:

$$\sigma(\tau, x) = \exp(G_2(\tau)x^2) \cdot \left(e^{\pi i x} - e^{-\pi i x}\right) \prod_{n=1}^{\infty} \frac{(1 - e^{2\pi i x}q^n)(1 - e^{2\pi i x}q^n)}{(1 - q^n)^2} \tag{2}$$

$$= x \exp(-\sum_{k=2}^{\infty} \frac{2}{(2k)!} G_{2k}(\tau) \cdot x^{2k}) \tag{3}$$

where $q = e^{2\pi i \tau}$

Lemma 2.2. $Q(\tau, x) = \exp\left(\sum_{k=2}^{\infty} \frac{2}{(2k)!} G_{2k}(\tau) \cdot x^{2k}\right)$

Proof. By lemma 1. $\square$

Definition 2.3. For a compact, oriented differentiable manifold X^{4k}, the Witten genus associated to the characteristic power series $Q(\tau, x) = \dfrac{x}{\sigma(\tau, x)}$ is

$$\varphi_{W,\tau}(X) = \left(\prod_{i=1}^{2k} \frac{x_i}{\sigma(\tau, x_i)}\right)[X]$$

Theorem 2.4. For a $4k$-dimensional compact, oriented, differentiable manifold X with $p_1(X) = w_2(X) = 0$, the Witten genus φ_W is a modular form of weight $2k$ with integral coefficients.

Proof. First note that $\sigma(\tau, x)$ is a Jacobi form of weight -1 and index 0, that is, $\sigma(A\tau, \frac{x}{c\tau+d}) = \frac{1}{c\tau+d}\sigma(\tau, x)$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$:

$$\begin{aligned}
\sigma(A\tau, \frac{x}{c\tau+d}) &= \frac{x}{c\tau+d} \prod_{\omega\in\mathbb{Z}\frac{a\tau+b}{c\tau+d}+\mathbb{Z}} \left(1 - \frac{x}{c\tau+d}\frac{1}{\omega}\right) \exp\left(\frac{x}{c\tau+d}\frac{c\tau+d}{\omega} + \frac{1}{2}\frac{x^2}{(c\tau+d)^2}\frac{1}{\omega^2}\right) \\
&= \frac{1}{c\tau+d}x \prod_{\omega\in\mathbb{Z}(a\tau+b)+\mathbb{Z}(c\tau+d)} \left(1 - \frac{x}{c\tau+d}\frac{c\tau+d}{\omega}\right) \exp\left(\frac{x}{c\tau+d}\frac{c\tau+d}{\omega} + \frac{1}{2}\frac{x^2}{(c\tau+d)^2}\frac{(c\tau+d)^2}{\omega^2}\right) \\
&= \frac{1}{c\tau+d}x \cdot \prod_{\omega\in L\setminus\{0\}} \left(\left(1-\frac{x}{\omega}\right)\exp\left(\frac{x}{\omega}+\frac{x^2}{2\omega^2}\right)\right) = \frac{1}{c\tau+d}\sigma(\tau, x)
\end{aligned}$$

where the last equality holds because $\mathbb{Z}(a\tau+b) + \mathbb{Z}(c\tau+d) = \mathbb{Z}\tau + \mathbb{Z}$. $\subset$: OK. $\supset$: $(a\tau+b)d - (c\tau+d)b = (ad-bc)\tau = \tau$, and $(a\tau+b)(-c) + (c\tau+d)a = ad-bc = 1$.

Now we prove that $\varphi_{W,\tau}(X)$ is a modular form of weight $2k$

$$\begin{aligned}
\varphi_{W,A\tau}(X) &= \prod_{i=1}^{2k} Q(A\tau, x_i)[X] = \prod_{i=1}^{2k} Q(A\tau, \frac{x_i}{c\tau+d})[X](c\tau+d)^{2k} \\
&= \prod_{i=1}^{2k} \frac{x_i}{c\tau+d}\frac{1}{\sigma(A\tau, \frac{x_i}{c\tau+d})}[X](c\tau+d)^{2k} \\
&= \prod_{i=1}^{2k} \frac{x_i}{c\tau+d}\frac{c\tau+d}{\sigma(\tau, x_i)}[X](c\tau+d)^{2k} = (c\tau+d)^{2k}\varphi_{W,\tau}(X)
\end{aligned}$$

where the second inequality follows from that

$$\begin{aligned}
\prod_{i=1}^{2k} Q(\tau, \frac{x_i}{c\tau+d})[X] &= 1 + K_1(\frac{p_1}{(c\tau+d)^2}) + \cdots + K_k(\frac{p_1}{(c\tau+d)^2}, \frac{p_2}{(c\tau+d)^4}, \ldots, \frac{p_k}{(c\tau+d)^{2k}})[X] \\
&= K_k(\frac{p_1}{(c\tau+d)^2}, \frac{p_2}{(c\tau+d)^4}, \ldots, \frac{p_k}{(c\tau+d)^{2k}})[X] \\
&= \left(\frac{1}{c\tau+d}\right)^{2k} K_k(p_1, \ldots, p_k)[X] = \left(\frac{1}{c\tau+d}\right)^{2k} \prod_{i=1}^{2k} Q(\tau, x_i)[X]
\end{aligned}$$

Holomorphy of $\varphi_{W,\tau}(X)$: by lemma 2, $Q(x) = \exp\left(\sum_{m=2}^{\infty} \frac{2}{(2m)!}G_{2m}(\tau)\cdot x^{2m}\right)$. Using this and definition for a genus associated to a characteristic polynomial, we see that $\varphi_{W,\tau}(X)$ is a sum of products of G_{2m} and is of course homogeneous of weight $2k$. Thus $\varphi_{W,\tau}(X)$ is holomorphic, and is holomorphic at infinity.

Finally, $p_1 = 0$ gives that the Witten genus is of the form $\varphi_W(X) = q^{-\frac{4k}{24}}\hat{A}(X, \otimes_{n=1}^{\infty} S_{q^n}(TX\otimes\mathbb{C}))\cdot\Delta^{\frac{4k}{24}}$. The vanishing of the second Stiefel-Whitney class $w_2(X) = 0$ means that X is a spin manifold. And for a spin manifold, the $\hat{A}$-genus is the index of the Dirac operator and is therefore integral. On the other hand, $q^{-\frac{4k}{24}}\Delta^{\frac{4k}{24}} = \prod_{n=1}^{\infty}(1-q^n)^{4k}$ is also integral. The theorem now follows. □

3 Exotic spheres

In order to see the connection between Witten genus and exotic spheres, we introduce the following method which help us determine the $\varphi_W(X)$ when all but top Pontryagin numbers vanish.

We can similarly define complex genera using a power series $Q(x) = 1 + b_1 x + \cdots$ not necessarily even. Now let $c(X) = 1 + c_1(X) + \cdots + c_n(X) = (1 + x_1) \cdots (1 + x_n)$. Then the genus associated to the power seies Q, is given by $\varphi(X) = \left(\prod_{i=1}^{n} Q(x_i) \right) [X]$. Similar in the real case, one can define a sequence of polynomials $K_r(c_1, \ldots, c_r)$. Here the polynomial K_r is homogeneous of degree r.

$$\varphi(X) = K_n(c_1, \ldots, c_n)[X]$$

The c_n term in K_n is linear, thus the coefficient of c_n is a fixed polynomial s_n in the coefficients b_j of the power series Q. Since s_n only depends on terms of degree at most n, formally

$$1 + b_1 x + \cdots + b_n x^n = \prod_{i=1}^{n} (1 + \beta_i x)$$

Thus

$$\begin{aligned}
\prod_{j=1}^{n} Q(x_j) &\equiv \prod_{j=1}^{n} \prod_{i=1}^{n} (1 + \beta_i x_j) \pmod{x^{n+1}} \\
&\equiv \prod_{i=1}^{n} \prod_{j=1}^{n} (1 + \beta_i x_j) \pmod{x^{n+1}} \\
&\equiv \prod_{j=1}^{n} (1 + \beta_i c_1 + \cdots + \beta_i^n c_n) \pmod{x^{n+1}}
\end{aligned}$$

Putting $c_1 = \cdots = c_{n-1} = 0$. We therefore obtain $s_n(b_1, \ldots, b_n) = \beta_1^n + \cdots + \beta_n^n$. The expression is symmetric in β_i, and can be written as a polynomial in b_j.

Lemma 3.1 (Cauchy). Now write $f(x) = x/Q(x)$. Then

$$x \frac{f'(x)}{f(x)} = \sum_{j=0}^{\infty} (-1)^j s_j \cdot x^j$$

Proof.

$$\begin{aligned}
f(x) &\equiv \frac{x}{(1+\beta_1 x)\cdots(1+\beta_n x)} \pmod{x^{n+2}} \\
\Rightarrow \frac{f'(x)}{f(x)} &\equiv \frac{1}{x} - \frac{\beta_1}{1+\beta_1 x} - \cdots - \frac{\beta_n}{1+\beta_n} \pmod{x^n} \\
\Rightarrow x\frac{f'(x)}{f(x)} &\equiv 1 - \frac{\beta_1 x}{1+\beta_1 x} - \cdots - \frac{\beta_n x}{1+\beta_n} \pmod{x^{n+1}} \\
&\equiv 1 + \frac{-\beta_1 x}{1-(-\beta_1 x)} - \cdots - \frac{-\beta_n x}{1-(-\beta_n)} \pmod{x^{n+1}} \\
&\equiv 1 + \sum_{j=1}^{\infty}(-\beta_1 x)^j + \cdots + \sum_{j=1}^{\infty}(-\beta_n x)^j \pmod{x^{n+1}} \\
&\equiv 1 + \sum_{j=1}^{\infty}(-1)^j(\beta_1^j + \cdots + \beta_n^j)x^j \pmod{x^{n+1}}
\end{aligned}$$

□

If X is a manifold for which all Chern classes but the highest vanish, then its genus is given by $\varphi(X) = s_n \cdot c_n[X]$.

Lemma 3.2. s_n can be directly read off from the logarithm of the power series Q. That is, $Q(x) = \exp\left(\sum_{j=1}^{\infty}(-1)^{j+1}\frac{s_j}{j}x^j\right)$

Proof.

$$\begin{aligned}
f(x) = \exp(\ln(f(x))) &= \exp\left(\int \frac{f'(x)}{f(x)}dx\right) \\
&= \exp\left(\int\left(\frac{1}{x}\sum_{j=0}^{\infty}(-1)^j s_j \cdot x^j\right)dx\right) \\
&= \exp\left(\ln(x) + \sum_{j=1}^{\infty}(-1)^j\frac{s_j}{j}x^j\right) \\
&= x\exp\left(\sum_{j=1}^{\infty}(-1)^j\frac{s_j}{j}x^j\right)
\end{aligned}$$

Therefore, $Q(x) = \exp\left(\sum_{j=1}^{\infty}(-1)^{j+1}\frac{s_j}{j}x^j\right)$ □

Lemma 3.3. Let M be a $4k$-dimensional compact, oriented, smooth manifold for which all composite Pontryagin numbers vanish, that is, for $p = 1 + p_1 + \cdots + p_k$ we have

$$(p_{i_1}\cdots p_{i_r})[M] = 0 \text{ for all } i_1 + \cdots + i_r = k \text{ with } r > 1$$

The remaining one is $p_k[M]$. Also, we require that $p_1 = 0$. $\varphi_W(M) = \hat{A}(M) \cdot E_{2k}$.

Proof. Notice that we can proceed similarly as above argument, therefore from

$$\frac{x}{\sigma_L(x)} = \exp\left(\sum_{r=2}^{\infty} \frac{-B_{2r}}{2r\cdot(2r)!}E_{2r}(\tau)\cdot x^{2r}\right)$$

$\varphi_W(M)$ is a constant multiple of E_{2k}. Note that the constant terms of E_{2k} and $\varphi_W(M)$ are 1 and $\hat{A}(M)$ respectively, therefore, $\varphi_W(M) = \hat{A}(M)\cdot E_{2k}$. □

Theorem 3.4 (Divisibility property of signature of almost-parallelizable manifolds)**.** Let M be an almost-parallelizable manifold. The signature $\operatorname{sign}(M) \equiv 0 \pmod{a_k 2^{2k+1}(2^{2k-1}-1)\cdot \text{numerator of } \left(\frac{B_{2k}}{4k}\right)}$, where $a_k = 2$ if k is odd, and $a_k = 1$ if k is even.

Proof. We use the fact that $\operatorname{sign}(M) = -2^{2k+1}(2^{2k-1}-1)\cdot\hat{A}(M)$. Now if M is a spin manifold, the $\hat{A}$-genera are integral, and

$$\varphi_W(M) = \hat{A}(M)\cdot E_{2k}$$

is integral. We thus obtain from

$$E_{2k}(\tau) = 1 + \frac{4k}{-B_{2k}}\cdot\sum_{n=1}^{\infty}\sigma_{2k-1}(n)\cdot q^n$$

for $n = 1$, the divisibility condition:

$$\hat{A}(M) \equiv 0 \pmod{\text{numerator of the reduced representation of}(B_{2k}/4k)}$$

we therefore have

$$\operatorname{sign}(M) \equiv 0 \pmod{2^{2k+1}(2^{2k-1}-1)\cdot \text{numerator}\left(\frac{B_{2k}}{4k}\right)}$$

Actually, we can obtain a small improvement. For a spin manifold M, the $\hat{A}$-genus $\hat{A}(M)$ is always even if the dimension of M is divisible by 4 but not by 8. Putting $a_k = 1$ if k is even and $a_k = 2$ if k is odd, then

$$\operatorname{sign}(M) \equiv 0 \pmod{a_k 2^{2k+1}(2^{2k-1}-1)\cdot \text{numerator}\left(\frac{B_{2k}}{4k}\right)}$$

An almost-parallelizable manifold M^{4k} has $w_2 = 0$, thus the argument works for almost-parallelizable manifolds. □

Now we have come to our final section. The divisibility criterion of signature is used in the construction of "an" exotic sphere V^{4k-1} $k \geq 2$. And with this exotic sphere, we obtain more exotic spheres by means of connected sum. This is due to a theorem that characterizes the diffeomorphic classes by signature. (Theorem 3.9)

Definition 3.5. A smooth manifold V^k is said to be a homotopy sphere if V^k is homotopic equivalent to a sphere S^k.

Theorem 3.6 (Smale)**.** Each homotopy sphere V^k with $k \geq 5$ is homeomorphic to the sphere S^k.

Definition 3.7. Let bP_{4k} be the set of diffeomorphism classes of homotopy spheres Σ^{4k-1} which bound parallelizable manifolds M. ($\partial M = \Sigma$) This is a group with connected sum as group operation. Notice that by Smale h-cobordism theorem below, the bP_{4k} is a subgroup of all smooth structures on S^n.

Definition 3.8. M and N are said to be h-cobordant if $M_1 - M_2 = \partial W$ and M_1, M_2 are both deformation retracts of W. It is an equivalence relation.

Theorem 3.9 (Smale h-cobordism theorem)**.** Let M and N be simply-connected closed n-manifolds and X a simply-connected h-cobordism between them. If $n \geq 5$, then X is diffeomorphic to $M \times [0,1]$. In particular, M and N are diffeomorphic.

Theorem 3.10. Let Σ_1, Σ_2 be two $4k-1$ homotopy spheres, $\partial M_i = \Sigma_i$, with M_i parallelizable. Then Σ_1 is h-cobordant to Σ_2 if and only if $\sigma(M_1) \equiv \sigma(M_2) \pmod{\left(a_k 2^{k+1}(2^{k+1}-1)\cdot \text{numerator}(\frac{B_{2k}}{4k})\right)}$.

This theorem together with theorem 3.9 gives an elegant method to determine whether two homotopy spheres, as boundaries of paralleilzable manifolds, are diffeomorphic.

Now if we have constructed our Milnor sphere V^{4k-1}, which is a boundary of a manifold having signature 8, then V^{4k-1} and its connected sums constitute the cyclic subgroup bP_{4k}. Therefore the order of the cyclic subgroup bP_{4k} is exactly $\frac{1}{8}\left(a_k 2^{k+1}(2^{k+1}-1)\cdot \text{numerator}(\frac{B_{2k}}{4k})\right)$.

Theorem 3.11. bP_{4k} is generated by the Milnor sphere V^{4k-1} and the order of the group is

$$n_k = \frac{1}{8}\left(a_k 2^{k+1}(2^{k+1}-1)\cdot \text{numerator}(\frac{B_{2k}}{4k})\right)$$

Proof. (Sketch) Let $D(S^m) = \{x \in TS^m \| x| \leq 1\}$ be the disc bundle. We construct our manifold $X^{2m}_{E_8}$ according to E_8: For each vertex, one takes a copy of $D(S^m)$ and joins together two of these copies if there is an edge in the graph between the corresponding vertices. In general, this construction yields an oriented manifold X^{2m} with boundary.

Let $s_1, \ldots, s_8 \in H_{2k}(X, \mathbb{Z})$ intersect as indicated in E_8. While a self intersection $s_i s_i$ is the zeroes of a vector field that has only finitely many zeros and this equals Euler characteristic by Poincare Hopf theorem. And we know that $\chi(S^m) = 0$ if m is odd and $\chi(S^m) = 2$ if m is even. The resulting manifold has boundary $\partial X^{4k}_{E_8} = V^{4k-1}$ that is a homotopy sphere. And by Smale theorem, this is homeomorphic to S^{4k-1} for $k \geq 2$. Now if V^{2m-1} were diffeomorphic to S^{2m-1} then $X^{2m}_{E_8} \cup_{S^{2m-1}} D^{2m}$ would be almost parallelizable. However, it has signature 8 for $m = 2k$ because we define the manifold according to E_8, which has signature 8. Thus

$$8 \equiv 0 \pmod{a_k 2^{2k+1}(2^{2k-1}-1)\cdot \text{numerator}\left(\frac{B_{2k}}{4k}\right)}$$

where $a_k 2^{2k+1}(2^{2k-1}-1)\cdot \text{numerator}\left(\frac{B_{2k}}{4k}\right) = 8\cdot 28$ for $k=2$, a contradiction. Thus by direct construction, we see that bP_{4k} has at least $n_k = \frac{1}{8}\left(a_k 2^{k+1}(2^{k+1}-1)\cdot \text{numerator}(\frac{B_{2k}}{4k})\right)$ elements. While the reference says that this method gives all the bP_{4k}. $\square$

4 References

1. Manifolds and Modular Forms by Hizebruch. 2. A report on Hizebruch signature formula and Milnor's exotic seven spheres by Chin-Lung Wang.