Thm 1: Let $f(\tau) = \sum_{n\geq 0} a_n q^n \in M_k(N,\varepsilon)$ (resp. $S_k(N,\varepsilon)$)

$\omega(N) := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$.

$f[\gamma]_k = (\det\gamma)^{\frac{k}{2}} j(\gamma,\tau)^{-k} f(\gamma(\tau))$.

χ: a character of conductor m, $(m,N)=1$.

Then $f_\chi(\tau) := \sum_{n\geq 0} \chi(n) a_n q^n \in M_k(m^2N, \varepsilon\chi^2)$ (resp. $S_k(m^2N, \varepsilon\chi^2)$)

and $f_\chi[\omega(Nm^2)]_k = C_\chi \left(f[\omega(N)]_k\right)_{\overline{\chi}}$, where $C_\chi = \dfrac{\varepsilon(m)\chi(-N)g(\chi)}{g(\overline{\chi})} = \dfrac{\varepsilon(m)\chi(N)g(\chi)^2}{m}$.

Pf:

Define an operator $L_\chi := \frac{1}{m}\sum_{x,y\in \mathbb{Z}/m\mathbb{Z}} \chi(x) e^{-2\pi i xy/m} \begin{pmatrix} m & y \\ 0 & m \end{pmatrix}_k$, then

$$L_\chi f = \frac{1}{m}\sum_{x,y\in\mathbb{Z}/m\mathbb{Z}} \chi(x) e^{\frac{-2\pi i xy}{m}} m^k m^{-k} \sum_{n\geq 0} a_n \mu_m^{yn} q^n$$

$$= \frac{1}{m}\sum_{n\geq 0} a_n q^n \sum_{x,y\in \mathbb{Z}/m\mathbb{Z}} \chi(x) \mu_m^{y(n-x)}$$

$$= \sum_{n\geq 0} \chi(n) a_n q^n = f_\chi.$$

Note that $\forall\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(m^2N)$,

$$f_\chi \circ [\gamma]_k = m^{-1}\sum_{x,y\in\mathbb{Z}/m\mathbb{Z}} \chi(x) e^{\frac{-2\pi i xy}{m}} f\circ\left[\begin{matrix} m & y \\ 0 & m \end{matrix}\right]_k \circ \begin{pmatrix} a & b \\ c & d \end{pmatrix}_k$$

$$= m^{-1}\sum_{x,y\in\mathbb{Z}/m\mathbb{Z}} \chi(x) e^{\frac{-2\pi i xy}{m}} \varepsilon(d)\, f\circ\begin{pmatrix} m & d^2 y \\ 0 & m\end{pmatrix}_k$$

$(x,y) \mapsto (d^2x, d^{-2}y)$

$$= (\chi^2\varepsilon)(d) f_\chi.$$

$$\begin{pmatrix} m & y \\ 0 & m \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \frac{1}{m} & -\frac{d^2 y}{m^2} \\ 0 & \frac{1}{m} \end{pmatrix}$$

$$= \begin{pmatrix} ma+yc & mb+yd \\ mc & md \end{pmatrix}\begin{pmatrix} \frac{1}{m} & -\frac{d^2 y}{m^2} \\ 0 & \frac{1}{m}\end{pmatrix}$$

$$= \begin{pmatrix} a+\frac{c}{m}y & -\frac{cd^2}{m^2}y^2 + b + \frac{1-ad}{m}dy \\ c & d-\frac{c}{m}d^2 y \end{pmatrix} \in \Gamma_0(N)$$

Also, $|a_n(f)| = |a_n(f_\chi)|$ shows that $f_\chi \in M_k(m^2N, \chi^2\varepsilon)$ (resp. $S_k(m^2N, \chi^2\varepsilon)$).

Now, for $(u,m)=1 \rightsquigarrow (uN, m)=1 \rightsquigarrow \exists\ \begin{matrix} u^* \\ n \end{matrix} \in \mathbb{N}$ s.t. $\gamma(m,u) := \begin{pmatrix} m & -u^* \\ -Nu & n \end{pmatrix} \in \Gamma_0(N)$.

Define $\alpha(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{C})$ $\forall x\in\mathbb{C}$. Then

$$\alpha\left(\tfrac{u}{m}\right)\omega(Nm^2) = m\cdot \omega(N)\,\gamma(m,u)\,\alpha\left(\tfrac{u^*}{m}\right).$$

Note that $f_\chi(\tau) = \frac{1}{m}\sum_{x,y\in\mathbb{Z}/m\mathbb{Z}} \chi(x)\, e^{-\frac{2\pi i xy}{m}} f(\tau+\tfrac{y}{m}) = \frac{g(\chi)}{m}\sum_{y\in\mathbb{Z}/m\mathbb{Z}} \bar\chi(y)\, f[\alpha(\tfrac{y}{m})]_k$. — (*).

$$\leadsto f_\chi[\omega(Nm^2)]_k = \frac{g(\chi)}{m}\sum_{u\in\mathbb{Z}/m\mathbb{Z}} \bar\chi(u)\,\bar\varepsilon(n)\, \underbrace{f[\omega_N]_k}_{\in M_k(N,\bar\varepsilon)}\, [\alpha(\tfrac{u^*}{m})]_k$$

$$\begin{pmatrix} nm\equiv 1\ (N) \\ Nuu^*\equiv -1\ (m)\end{pmatrix} \quad = \frac{g(\chi)}{m}\sum_{u^*\in\mathbb{Z}/m\mathbb{Z}} \chi(-N)\,\chi(u^*)\,\varepsilon(m)\,(f[\omega_N]_k)\,[\alpha(\tfrac{u^*}{m})]_k = C_\chi(f[\omega_N]_k)_{\bar\chi}.$$

$\square$

Rmk: (*) holds for any $f(\tau)=\sum_{n\geq 0} a_n q^n$, $a_n = O(n^\nu)$.

Recall: $f\in M_k(N,\chi)$, $S_k(N,\chi)$ $\longrightarrow$ $a_n(f) = O(n^{k-1})$, $O(n^{k/2})$. $L(s;f) := \sum_{n\geq 1}\frac{a_n}{n^s}$ is analytic on $Re(s) > \max\{k, \frac{k}{2}+1\}$.

$\Lambda_N(s;f) := \left(\frac{2\pi}{\sqrt{N}}\right)^{-s}\Gamma(s)\,L(s;f)$ ---- $Re(s) > \max\{k,\frac{k}{2}+1\}$.

Thm2 (Hecke) f, g have order $\nu > 0$. $\left(\text{i.e. } \begin{matrix} f(\tau)=\sum_{n\geq 0} a_n e^{2\pi i n\tau} \\ g(\tau)=\sum_{n\geq 0} b_n e^{2\pi i n\tau}\end{matrix} \text{ are holo on } \mathbb{H} \text{ and } \begin{matrix} a_n = O(n^\nu) \\ b_n = O(n^\nu)\end{matrix}\right)$

$k, N\in\mathbb{N}$.

TFAE.

(A): $g = f\circ[\omega(N)]_k$, $\omega(N)=\begin{pmatrix}0 & -1\\ N & 0\end{pmatrix}$ and $g\circ[\omega(N)]_k = (\det\omega(N))^{\frac{k}{2}} j(\omega(N),\tau)^{-k} g(\omega(N)(\tau))$.

(B): $\Lambda_N(s;f)$, $\Lambda_N(s;g)$ are mero on $\mathbb{C}$ and satisfy the FE $\Lambda_N(s;f) = i^k\Lambda_N(k-s;g)$.

And the function $\Lambda_N(s;f) + \frac{a_0}{s} + \frac{i^k b_0}{k-s}$ is EBV.

(entire and bounded on any vertical strip.)

Pf.

$\boxed{A\Rightarrow B}$ $a_n = O(n^\nu) \Rightarrow \sum_{n\geq 0}|a_n| e^{-\frac{2\pi n t}{\sqrt{N}}}$ and $\sum_{n\geq 1}\int_0^\infty |a_n| t^\sigma e^{-\frac{2\pi n t}{\sqrt{N}}}\frac{dt}{t} = \sum_{n\geq 1} \underbrace{a_n\left(\frac{\sqrt{N}}{2\pi n}\right)^\sigma}_{O(n^{\nu-\sigma})}\int_0^\infty r^\sigma e^{-r}\frac{dr}{r}$

converge, for $\sigma > \nu+1$.

Hence, for $\operatorname{Re} s > \nu+1$,

$$\Lambda_N(s;f) = \left(\frac{2\pi}{\sqrt{N}}\right)^{-s}\Gamma(s)\,L(s;f)$$

$$= \sum_{n\geq 1}\frac{a_n}{n^s}\left(\frac{2\pi}{\sqrt{N}}\right)^{-s}\int_0^\infty e^{-t}t^s\,\frac{dt}{t}$$

$$= \sum_{n\geq 1}\int_0^\infty a_n t^s e^{-\frac{2\pi n}{\sqrt{N}}t}\,\frac{dt}{t}$$

need convergence

$$= \int_0^\infty \sum_{n\geq 1} a_n t^s e^{-\frac{2\pi n}{\sqrt{N}}t}\,\frac{dt}{t}$$

$$= \int_0^\infty t^s\left(f\left(\frac{it}{\sqrt{N}}\right)-a_0\right)\frac{dt}{t}$$

$$= -\frac{a_0}{s} + \int_1^\infty t^{-s} f\left(\frac{i}{\sqrt{N}t}\right)\frac{dt}{t} + \int_1^\infty t^s\left(f\left(\frac{it}{\sqrt{N}}\right)-a_0\right)\frac{dt}{t}$$

Since $f\left(\frac{i}{\sqrt{N}t}\right) = (it)^k g\left(\frac{it}{\sqrt{N}}\right)$, for $\operatorname{Re} s > \max\{\nu+1, k\}$, we have

$$\Lambda_N(s;f) = -\frac{a_0}{s} + i^k\int_1^\infty t^{k-s} g\left(\frac{it}{\sqrt{N}}\right)\frac{dt}{t} + \int_1^\infty t^s\left(f\left(\frac{it}{\sqrt{N}}\right)-a_0\right)\frac{dt}{t}$$

$$= -\frac{a_0}{s} - \frac{i^k b_0}{k-s} + \underbrace{i^k\int_1^\infty t^{k-s}\left(g\left(\frac{it}{\sqrt{N}}\right)-b_0\right)\frac{dt}{t} + \int_1^\infty t^s\left(f\left(\frac{it}{\sqrt{N}}\right)-a_0\right)\frac{dt}{t}}_{(1)}$$

By the definition of f and g,

$$(f(it)-a_0) = O(e^{-2\pi t}) \quad \text{and} \quad g(it)-b_0 = O(e^{-2\pi t}) \quad \text{as } t\to\infty.$$

(1) is EBV. $\rightsquigarrow$ $\Lambda_N(s;f) + \frac{a_0}{s} + \frac{i^k b_0}{k-s}$ is EBV.

Similarly, since $g = f[W_N]_k \iff (-1)^k f = g[W_N]_k$, we have

$$\Lambda_N(k-s;g) = -\frac{b_0}{k-s} - \frac{(-i)^k a_0}{s} + \int_1^\infty t^s\left(f\left(\frac{it}{\sqrt{N}}\right)-a_0\right)\frac{dt}{t} + (-i)^k\int_1^\infty t^{k-s}\left(g\left(\frac{it}{\sqrt{N}}\right)-b_0\right)\frac{dt}{t}.$$

$$\Longrightarrow \Lambda_N(s;f) = i^k\,\Lambda_N(k-s;g).$$

$\boxed{B\Rightarrow A}$ First, $\Gamma(s)$ is the Mellin transform of e^{-t}, by Mellin inverse formula,

$$e^{-t} = \frac{1}{2\pi i}\int_{Re(s)=\sigma} \Gamma(s)\, t^{-s}\, ds \qquad \forall \sigma>0.$$

$O(e^{-\frac{\pi y}{2}})$

$$\Rightarrow f(iy) = \sum_{n\geq 0} a_n e^{-2\pi n y} = \sum_{n\geq 1} a_n \frac{1}{2\pi i}\int_{Re(s)=\sigma} (2\pi n)^{-s}\, \underset{\shortparallel}{\Gamma(s)}\, y^{-s}\, ds + a_0 \qquad \forall y>0.$$

For $\sigma > \nu+1$, $L(s;f) = \sum_{n\geq 1} a_n n^{-s}$ is uniformly converges on $Re(s)=\sigma$.

By Stirling's estimate, $\Gamma(\sigma+i\tau) \sim \sqrt{2\pi}\, \tau^{\sigma-\frac{1}{2}} e^{-\frac{\pi|\tau|}{2}}$ as $|\tau|\mapsto\infty$.

$\Rightarrow \Lambda_N(s;f) = \left(\frac{2\pi}{\sqrt N}\right)^{-s}\Gamma(s)\, L(s;f)$ is integrable on $Re(s)=\sigma>\nu+1$.

$\sum\int = \int\sum$

$$\Rightarrow f(iy) = \frac{1}{2\pi i}\int_{Re(s)=\sigma} (\sqrt N y)^{-s}\Lambda_N(s;f)\, ds + a_0 \qquad \text{for } \sigma>\nu+1.$$

$\because a_n = O(n^\nu)$, $\therefore L(s;f)$ is bounded on $Re\, s > \nu+1$.

By Stirling's estimate, $\Lambda_N(s;f) = \left(\frac{2\pi}{\sqrt N}\right)^{-s}\Gamma(s)\, L(s;f) = O(|Im\, s|^{-\mu})$ as $|Im\, \tau|\to\infty$, for any $\mu>>0$ on $Re\, s = \alpha > \nu+1$. (Similarly, since $\Lambda_N(s;f) = \Lambda_N(k-s;g)\, i^k$, choose $k-\beta>\nu+1$, $\Lambda_N(s;f) = i^k\Lambda_N(k-s;g) = O(|Im\, s|^{-\mu})$ as $|Im\, s|\to\infty$ on $Re\, s=\beta$.

By assumption, $\Lambda_N(s;f) + \frac{a_0}{s} + \frac{i^k b_0}{k-s}$ is bounded on $\beta\leq Re\, s\leq\alpha$.)

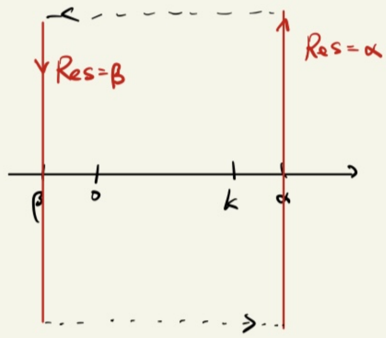
By Phragmen-Lindelöf lemma (apply to), for any $\mu>0$,

$$\Lambda_N(s;f) = O(|Im\, s|^{-\mu}) \text{ as } |Im\, s|\to\infty \text{ uniformly on } \beta\leq Re\, s\leq\alpha.$$

Now, let $\alpha>k$ and $\beta<0$. By Cauchy integral formula,

$$\frac{1}{2\pi i}\int_{Re\, s=\alpha}(\sqrt N y)^{-s}\Lambda_N(s;f)\, ds = \frac{1}{2\pi i}\int_{Re\, s=\beta}(\sqrt N y)^{-s}\Lambda_N(s;f)\, ds - a_0 + (\sqrt N y)^{-k} b_0\, i^k \qquad (\mu >> \alpha)$$

$$\Rightarrow f(iy) = \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=\beta} (\sqrt{N}y)^{-s}\Lambda_N(s;f)\,ds + (\sqrt{N}y)^{-k} b_0\, i^k$$



$$= \frac{i^k}{2\pi i}\int_{\mathrm{Re}\, s=\beta} (\sqrt{N}y)^{-s}\Lambda_N(k-s;g)\,ds + (\sqrt{N}y)^{-k} b_0$$

$$= \frac{i^k}{2\pi i}\int_{\mathrm{Re}\, s'=k-\beta} (\sqrt{N}y)^{s'-k}\Lambda_N(s';g)\,ds' + (\sqrt{N}y)^{-k} b_0$$

$$= (-i\sqrt{N}y)^{-k}\left[\frac{1}{2\pi i}\int_{\mathrm{Re}\, s'=k-\beta} \left(\sqrt{N}\cdot\frac{1}{Ny}\right)^{-s'}\Lambda_N(s';g)\,ds' + b_0\right]$$

$$= (-i\sqrt{N}y)^{-k}\, g\!\left(\frac{i}{Ny}\right) = (\sqrt{N}\cdot -iy)^{-k}\, g\!\left(\frac{-1}{Nyi}\right) \quad \text{for } y>0.$$

$$\Rightarrow g = f[W_N]_k \quad \text{on } \mathrm{Re}\,\tau = 0 \underset{\text{holo}}{\Rightarrow} g = f[W_N]_k \quad \text{on } \mathbb{H}. \quad \square$$

<u>Cor 1</u>: $f\in S_k(N,\varepsilon) \Rightarrow \Lambda_N(s;f)$ is entire and $\Lambda_N(s;f) = i^k\Lambda_N(k-s;f[W_N]_k)$.

<u>P.f.</u> $f\in S_k(N,\varepsilon) \Rightarrow f[W_N]_k\in S_k(N,\bar\varepsilon)$ and $a_0 = b_0 = 0$. □

<u>Cor 2</u>: f has order $\nu>0$ and $k\in\mathbb{N}$.

$f\in M_k(SL_2(\mathbb{Z})) \Leftrightarrow \Lambda_1(s;f)$ is mero on $\mathbb{C}$ and satisfies $\Lambda_1(s;f) = i^k\Lambda_1(k-s;f)$

and $\Lambda_1(s;f) + \frac{a_0}{s} + \frac{i^k a_0}{k-s}$ is EBV.

Moreover, $a_0 = 0 \rightsquigarrow f\in S_k(SL_2(\mathbb{Z}))$.

<u>P.f.</u> $N=1$ in Theorem 2.

<u>Def</u>: $f(\tau) = \sum_{n\ge 0} a_n q^n$, $N\in\mathbb{N}$, χ: a character with conductor m.

Then $f_\chi(\tau) := \sum_{n\ge 0} a_n\chi(n) q^n$.

$L(s;f,\chi) := \sum_{n\ge 0}\frac{a_n\chi(n)}{n^s}$.

$\Lambda_N(s;f,\chi) := \left(\frac{2\pi}{m\sqrt{N}}\right)^{-s}\Gamma(s)\,L(s;f,\chi) = \Lambda_{Nm^2}(s;f_\chi)$.

Coro 3: f, g have order $\nu \geq 0$, χ: a character of conductor $m > 1$, $(m, N) = 1$.

Then TFAE.

(A_χ) $f_\chi [\omega(Nm^2)]_k = C_\chi \, g_{\bar{\chi}}$.

(B_χ) $\Lambda_N(s; f, \chi)$ is EBV and satisfies $\Lambda_N(s; f, \chi) = i^k C_\chi \Lambda_N(k-s; g, \bar{\chi})$.

for a constant C_χ.

Pf: $(f, g, N) \longmapsto (f_\chi, C_\chi g_{\bar{\chi}}, Nm^2)$, and note that $a_0(f_\chi) = b_0(g_{\bar{\chi}}) = 0$. □

Thm 3: $f \in M_k(N, \varepsilon)$, χ: conductor $m > 1$, $(m, N) = 1$.

Then $\Lambda_N(s; f, \chi)$ is EBV and satisfies $\Lambda_N(s; f, \chi) = i^k C_\chi \Lambda_N(k-s; f[\omega(N)]_k, \bar{\chi})$

where $$C_\chi = \frac{\varepsilon(m)\chi(-N) g(\chi)}{g(\bar{\chi})} = \frac{\varepsilon(m)\chi(N) g(\chi)^2}{m}$$

Pf: Thm 1 + Coro 3. □

Recall: $(m, uN) = 1$, define $r(m, u) := \begin{pmatrix} m & -u \\ -u^* N & n \end{pmatrix} \in \Gamma_0(N)$ for some $u^*, n \in \mathbb{N}$.

$$\alpha\left(\frac{u^*}{m}\right) \omega(Nm^2) = m\,\omega(N)\, r(m, u)\, \alpha\left(\frac{u}{m}\right), \quad \begin{array}{l} Nuu^* \equiv -1 \ (m) \\ mn \equiv 1 \ (N) \end{array}$$

Thm 4: (Weil)

$M \subseteq \mathcal{M} := \{4, 3, 5, 7, 11, 13, \cdots\}$ satisfies $M \cap \{a + nb : n \in \mathbb{Z}\} \neq \phi$ $\forall (a, b) = 1$.

$k, N \in \mathbb{N}$, ε: a character mod N, $\varepsilon(-1) = (-1)^k$, f, g have order $\nu \geq 0$.

If the following two condition are satisfied:

1.) $\Lambda_N(s; f)$ and $\Lambda_N(s; g)$ satisfy condition (B) in Thm 2.

2.) $\forall \chi$: character with conductor $m_\chi \in M$, $\Lambda_N(s; f, \chi)$ and $\Lambda_N(s; g, \bar{\chi})$ satisfy the condition (B_χ) in Coro3 with $C_\chi = \varepsilon(m)\chi(-N)\dfrac{g(\chi)}{g(\bar\chi)}$.

Then $f \in M_k(N,\varepsilon)$, $g \in M_k(N,\bar\varepsilon)$ and $g = f[\omega_N]_k$.

Moreover, if $L(s;f)$ is absly convergent at $s=k-\delta$ for some $\delta>0$, then f and g are cusps forms.

Pf:

Lemma1: $m\in\mathfrak{M}$ and $m\nmid N$. ε: a char mod N with $\varepsilon(-1)=(-1)^k$.

$g = f[\omega(N)]_k$ has order $\nu>0$.

Suppose that (A_χ) holds $\forall\chi$: primitive character mod $\cancel{N}$ m, with

$C_\chi = \dfrac{\varepsilon(m)\chi(-N)g(\chi)}{g(\bar\chi)}$. Then $g[(\varepsilon(m)-\gamma(m,u))\alpha(\frac{u}{m})]_k$ is indep of $u\in(\mathbb{Z}/m\mathbb{Z})^\times$

and the choice of $\gamma(m,u)$. ($f[\rho]_k = \sum_\ell a_\ell f[\alpha_\ell]_k$ $\forall \rho = \sum_\ell a_\ell \alpha_\ell \in \mathbb{C}[GL_2(\mathbb{R})]$)

Pf: $(*)$ and A_χ imply

$$\Rightarrow C_\chi g(\chi)^{-1}\sum_{u\in(\mathbb{Z}/m\mathbb{Z})^\times}\chi(u)g[\alpha(\tfrac{u}{m})]_k = C_\chi g_{\bar\chi} = f_\chi[\omega(Nm^2)]_k = g(\bar\chi)^{-1}\sum_{u\in(\mathbb{Z}/m\mathbb{Z})^\times}\bar\chi(u) f[\alpha(\tfrac{u}{m})\cdot\omega(Nm^2)]_k$$

$$\rightsquigarrow \varepsilon(m)\chi(-N)\sum_u \chi(u)g[\alpha(\tfrac{u}{m})]_k = \sum_u \bar\chi(u) f[\alpha(\tfrac{u}{m})\omega(Nm^2)]_k.$$

$$f[\alpha(\tfrac{u}{m})\omega(Nm^2)]_k = f[\omega(N)\gamma(m,u^*)\alpha(\tfrac{u^*}{m})]_k = g[\gamma(m,u^*)\alpha(\tfrac{u^*}{m})]_k$$

$$= \sum_u \bar\chi(u)\, g[\gamma(m,u^*)\alpha(\tfrac{u^*}{m})]_k \quad (Nuu^* \equiv -1\ (m))$$

$$= \chi(-N)\sum_u \chi(u) g[\gamma(m,u)\alpha(\tfrac{u}{m})]_k.$$

Note that L.H.S. is indep of the choice of $\gamma(m,u)$, thus the R.H.S. is also.

$$\Rightarrow \sum_u \chi(u)\, g[(\varepsilon(m)-\gamma(m,u))\alpha(\tfrac{u}{m})]_k = 0 \quad \forall\chi: \text{primitive mod } m.$$

Since $\{\chi$: primitive mod $m\}$ is linear indep, $g[(\varepsilon(m)-\gamma(m,u))\alpha(\frac{u}{m})]_k$ is indep of

$\# = m-2.$

$u\in(\mathbb{Z}/m\mathbb{Z})^\times$ and $\gamma(m,u)$. $\square$

Lemma 2: Suppose that f and g satisfy Lemma 1 for $m,n \in \mathfrak{M}$ and $(mn, N)=1$,

then $g[\gamma]_k = \underbrace{\varepsilon(m)}_{=\overline{\varepsilon(n)}}\, g$ for all $\gamma = \begin{pmatrix} m & -u \\ -u^*N & n \end{pmatrix} \in \Gamma_0(N)$.

Pf: $\gamma := \begin{pmatrix} m & -u \\ -u^*N & n \end{pmatrix}$, $\gamma' := \begin{pmatrix} m & u \\ u^*N & n \end{pmatrix}$ $\xRightarrow{\text{Lemma 1}}$ $g\left[\left[\varepsilon(m) - \gamma(m,u)\right]\alpha\left(\frac{u}{m}\right)\right]_k = g\left[\left[\varepsilon(m) - \gamma(m,-u)\right]\alpha\left(\frac{-u}{m}\right)\right]_k$

$$\Longrightarrow g[\varepsilon(m)-\gamma]_k = g\left[(\varepsilon(m)-\gamma')\,\alpha\left(\tfrac{-2u}{m}\right)\right]_k \quad \text{---} (1)$$

Since $\gamma^{-1} = \begin{pmatrix} n & u \\ u^*N & m \end{pmatrix}$, by Lemma 2 again, we have

$$(\gamma')^{-1} = \begin{pmatrix} n & -u \\ -u^*N & m \end{pmatrix}$$

$$g\left[\varepsilon(n)-(\gamma')^{-1}\right]_k = g\left[(\varepsilon(n)-\gamma^{-1})\,\alpha\left(\tfrac{-2u}{n}\right)\right] \quad \text{---} (2).$$

$\because$ $\varepsilon(n)\varepsilon(m)=1$, $\therefore$ $\varepsilon(n)-(\gamma')^{-1} = -\varepsilon(n)\left(\varepsilon(m)-\gamma'\right)(\gamma')^{-1}$

$$\left(\varepsilon(n)-\gamma^{-1}\right)\alpha\left(\tfrac{-2u}{n}\right) = -\varepsilon(n)\left(\varepsilon(m)-\gamma\right)\gamma^{-1}\alpha\left(\tfrac{-2u}{n}\right)$$

$$(2) \Rightarrow g\left[\varepsilon(m)-\gamma'\right]_k = g\left[(\varepsilon(m)-\gamma)\,\gamma^{-1}\alpha\left(\tfrac{-2u}{n}\right)\gamma'\right]_k$$

$$\xRightarrow{(1)} g\left[(\varepsilon(m)-\gamma)\left(1-\underbrace{\gamma^{-1}\alpha\left(\tfrac{-2u}{n}\right)\gamma'\,\alpha\left(\tfrac{-2u}{m}\right)}_{=:\ \beta\ \in\ SL_2(\mathbb{R})}\right)\right]_k = 0$$

$h := g[(\varepsilon(m)-\gamma)]_k$, then $h[\beta]_k = h$.

Claim: $h \equiv 0$. ($\Rightarrow$ $g[\gamma]_k = \varepsilon(m) g$).

Pf: $\beta = \begin{pmatrix} n & u \\ Nu^* & m \end{pmatrix} \begin{pmatrix} 1 & \frac{-2u}{n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m & u \\ Nu^* & n \end{pmatrix} \begin{pmatrix} 1 & \frac{-2u}{m} \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} n & -u \\ Nu^* & \underset{\overset{\shortparallel}{\frac{2}{n}-m}}{\frac{mn-2Nuu^*}{n}} \end{pmatrix} \begin{pmatrix} m & -u \\ Nu^* & \underset{\overset{\shortparallel}{\frac{2}{m}-n}}{\frac{mn-2Nuu^*}{m}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\frac{2u}{m} \\ \frac{2Nu^*}{n} & \frac{4}{mn}-3 \end{pmatrix}$$

- the eigenvalue of ρ is not a root of unity: Since $\beta \in SL_2(\mathbb{Q})$, if not, then the eigenvalues of ρ are ± 1, $\pm i$, $\pm e^{\frac{\pi i}{3}}$, $\pm e^{\frac{2\pi i}{3}}$. Note that $\operatorname{tr}\beta = \frac{4}{mn} - 2 = 0, \pm 1, \pm 2$. $(m,n \geq 3)$. $\divideontimes$.
  $\det \beta = 1$.

- $\exists z_0 \in \mathbb{H}$ s.t. $\begin{cases} \operatorname{Re}(z_0) > 0. \\ \rho(z_0) = \beta \end{cases}$: Write $\beta = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL_2(\mathbb{Q})$,

$$\frac{Az+B}{Cz+D} = z \iff z = \frac{A-D+\sqrt{(A-D)^2+4BC}}{2C} = \frac{A-D+\sqrt{(A+D)^2-4}}{2C} \in \mathbb{H}$$

$$\iff \left|2 - \frac{4}{mn}\right| = |\operatorname{tr}(\beta)| < 2.$$

Also, $\operatorname{sign}(\operatorname{Re} z_0) = \operatorname{sign}\left(1 - \frac{1}{mn}\right) > 0$

$\rightarrow \exists z_0 \in \mathbb{H}$ s.t. $\rho(z_0) = z_0$, $\rightsquigarrow \frac{Az_0+B}{Cz_0+D} = \frac{z_0}{1}$, $\frac{A\bar{z}_0+B}{C\bar{z}_0+D} = \frac{\bar{z}_0}{1}$

$\rightarrow \begin{pmatrix} z_0 & \bar{z}_0 \\ 1 & 1 \end{pmatrix}^{-1} \beta \begin{pmatrix} z_0 & \bar{z}_0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda^l \neq 1$ $\forall l \in \mathbb{N}$.

Define $\rho = (z_0 - \bar{z}_0)^{\frac{1}{2}} \begin{pmatrix} z_0 & \bar{z}_0 \\ 1 & 1 \end{pmatrix}^{-1} = (z_0 - \bar{z}_0)^{\frac{1}{2}} \begin{pmatrix} 1 & -\bar{z}_0 \\ -1 & z_0 \end{pmatrix} \rightsquigarrow \rho\beta\rho^{-1} = \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$.

$p(w) = h[\rho^{-1}]_k(w)$ holomorphic on $K := \rho^{-1}(\mathbb{H}) \ni 0$. $\left(\rho(0) = \frac{0-\bar{z}_0}{0+z_0} = -e^{-2i\theta} \in \mathbb{H} \right.$ $z_0 = re^{i\theta}$, $\theta \in (0, \frac{\pi}{2})$)

By $h[\beta]_k = h$, we have $p[\rho\beta\rho^{-1}]_k = p$.

$\Rightarrow p(\lambda^2 w) = \lambda^{-k} p(w)$. Note that $0 \in K$, write

$p(w) = \sum_{n \geq 0} c_n w^n$ at $w=0 \Rightarrow \lambda^{2n} c_n = \lambda^{-k} c_n \quad \forall n$.

$\xRightarrow{\lambda^\ell \neq 1 \ \forall \ell \in \mathbb{N}} c_n \equiv 0 \quad \forall n$

$\Rightarrow p \equiv 0 \Rightarrow h \equiv 0$. $\square$

Proof of theorem of Weil:

1.) $\Rightarrow g = f[w_N]_k$. NTS. $\forall \gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$, $g[\gamma]_k = \bar{\varepsilon}(d) g$.

$c = 0 \Rightarrow a = d = \pm 1 \Rightarrow g[\gamma]_k = g$ since $\varepsilon(-1) = (-1)^k$ and g has fourier expansion.

$c \neq 0 \Rightarrow \begin{matrix} (a, cN) = 1 \\ (d, cN) = 1 \end{matrix} \Rightarrow \exists s, t$ s.t. $\begin{matrix} m := a + tcN \\ n := d + scN \end{matrix} \in M$. Then $\exists u^* \in \mathbb{Z}$ st.

$$\Rightarrow \begin{pmatrix} a & b \\ cN & d \end{pmatrix} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m & -u^* \\ -Nu & n \end{pmatrix} \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}$$

By Cro 3, we have (A_x). By lemma 1.2, we have

$$g[\gamma]_k = \varepsilon(m) g = \bar{\varepsilon}(n) g = \bar{\varepsilon}(d) g \Rightarrow g \in M_k(N, \bar{\varepsilon}) \text{. and } g = f[w_N]_k \Rightarrow f \in M_k(N, \varepsilon).$$

Suppose that $L(s; f)$ is absly converge at $s = k - \delta$. $\Rightarrow \sum_{n=1}^{\infty} |a_n| n^{-k+\delta}$ converge.

$\Rightarrow a_n = O(n^{k-\delta-1}) \xRightarrow[\text{Appendix B}]{} f(z) = O(|\mathrm{Im} z|^{-k+\delta}) \Longrightarrow f$ is a cusp form. $\square$

f not "contains" Eisenstein series.

Appendix:

- (3.2.8) (Stirling's estimate) $\Gamma(s) \sim \sqrt{2\pi}\,\tau^{\sigma-1/2}e^{-\pi|\tau|/2}$ $(s=\sigma+i\tau,\ |\tau|\to\infty)$, *uniformly on any vertical strip* $v_1 \leqq \sigma \leqq v_2$; , $v_1, v_2 > 0$

In fact, $\Gamma(z) = \sqrt{2\pi}\, z^{z-\frac{1}{2}} e^{-z}\left(1+O\left(|z|^{-\frac{1}{2}}\right)\right)$ $\qquad \left(\begin{aligned} z &= \sigma + i\tau \\ &= \sqrt{\sigma^2+\tau^2}\, e^{i\theta},\ \theta \to \pm\frac{\pi}{2} \end{aligned} \right.$

$$\sim \sqrt{2\pi}\,\tau^{\sigma-\frac{1}{2}} e^{-\theta\tau}$$

$$\sim \sqrt{2\pi}\,\tau^{\sigma-\frac{1}{2}} e^{-\frac{\pi}{2}|\tau|}$$

[Stein, complex analysis, Appendix A]

- Phragmen - Lindelöf

$F : \{s : v_1 \le \mathrm{Re}\, s \le v_2\} =: \Omega \longrightarrow \mathbb{C}$ holo.

$$\left\{ \begin{array}{l} |F(z)| = O\left(e^{|z|^\delta}\right) \text{ as } |z|\to\infty \text{ on } \Omega. \text{ for some } \delta>0 \\ |F(z)| = O\left(|z|^b\right) \text{ as } |z|\to\infty \text{ on } \mathrm{Re}\, s = v_1 \text{ or } v_2 \end{array} \right.$$

Then $|F(z)| = O(|z|^b)$ as $|z|\to\infty$ on Ω.

(Sketch)

Consider $\phi(z) = \dfrac{F(z)}{|z-v_1+1|^b}$, then w.l.o.g. we may assume $b=0$.

$$\Longrightarrow \left\{ \begin{array}{ll} |F(z)| \le L\, e^{|\mathrm{Im}\, z|^\delta} & \text{on } \Omega \\ |F(z)| \le M & \text{for } \mathrm{Re}\, z = v_1, v_2 \end{array} \right.$$

For $m = 4n+2 > \delta$ and $\mathrm{Re}(z^m) = -|\mathrm{Im}\, z|^m + O(|\mathrm{Im}\, z|^{m-1}) < N(m)$ on $\mathrm{Re}\, z = v_1$ or v_2,

$$\Longrightarrow |F(z)e^{\varepsilon z^m}| \le |F(z)|\, e^{-(|\mathrm{Im}\, z|^m - K|\mathrm{Im}\, z|^{m-1})\varepsilon}, \quad K = K(v_1, v_2).$$

$$\Rightarrow \quad \left|F(z)e^{\varepsilon z^m}\right| \le M e^{\varepsilon N(m)} \quad \text{on} \quad \mathrm{Re}\, z = v_1 \text{ or } v_2$$

$$\left|F(z)e^{\varepsilon z^m}\right| \le L e^{|\mathrm{Im}\, z|^\delta - \varepsilon |\mathrm{Im}\, z|^m - K|\mathrm{Im}\, z|^{m-1}\varepsilon} \longrightarrow 0 \quad \text{as} \quad |\mathrm{Im}\, z| \to \infty.$$

By Maximum principal $\Rightarrow$ $\left|F(z)e^{\varepsilon z^m}\right| \le M e^{\varepsilon N(m)}$ on Ω.

$\varepsilon \to 0 \Rightarrow |F(z)| \le M$ on Ω. $\square$.

- (Mellin inverse formula) $M(f)(s) := \int_0^\infty t^s f(t) \frac{dt}{t}$.

$$\Rightarrow f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M(f)(s)\, t^{-s}\, ds$$

(Sketch: $M(f)(c+2\pi\beta i) = \int_0^\infty f(t)\, t^c\, t^{2\pi\beta i} \frac{dt}{t} \overset{t=e^x}{=} \int_{\mathbb{R}} f(e^{-x}) e^{-cx} e^{-2\pi i \beta x}\, dx = \hat{g}(\beta)$,

$g(x) = f(e^{-x})\, e^{-cx}$, and we can apply the Fourier inverse transform.)

Reference:

1. Toshitsune Miyake, Modular Forms.

2. http://dsp-book.narod.ru/TAH/ch11.pdf