Modular Forms Via Theta Functions and A Cubic Equation

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1 Goal

We focus on the cubic equation

$$C : x^3 = d,$$

where $d \in \mathbb{N}$ is cubic free. For prime p, define

$$a_p(C) = \#\{\text{solutions of } C \bmod p\} - 1$$

Then

$$a_p(C) = \begin{cases} 2 & \text{, if } p \equiv 1 \pmod 3 \text{ and } d \text{ is a cube mod } p; \\ -1 & \text{, if } p \equiv 1 \pmod 3 \text{ and } d \text{ is not a cube mod } p; \\ 0 & \text{, if } p \equiv 2 \pmod 3 \text{ or } p \mid 3d. \end{cases}$$

Our goal is to find a modular form θ_χ, whose Fourier coefficient coincides with $a_p(C)$. Moreover, we'll see it is actually a normalized eigenform. Lastly, we'll see its image of Galois representation being S_3.

2 Construct θ_χ

For $z \in \mathbb{C}$, denote

$$\mathbf{e}(z) = e^{2\pi i z}.$$

Let $A = \mathbb{Z}[\mu_3], B = \frac{1}{\sqrt{-3}}A$, and $\mathrm{Tr}(z) = z + z^*$ for $z \in \mathbb{C}$, where z^* is the complex conjugate of z.
For $N \in \mathbb{N}, \bar{u} \in \frac{1}{3}A/NA, \tau \in \mathcal{H}$, define the holomorphic function

$$\theta^{\bar{u}}(\tau, N) = \sum_{n \in A} \mathbf{e}\left(N|\frac{u}{N} + n|^2 \tau\right).$$

Lemma 1. Let $d, N \in \mathbb{N}, \bar{u} \in B/NA$, then we have:

$$\theta^{\bar{u}}(\tau + 1, N) = \mathbf{e}\left(\frac{|u|^2}{N}\right)\theta^{\bar{u}}(\tau, N) \tag{1}$$

$$\theta^{\bar{u}}(\tau, N) = \sum_{\substack{\bar{v} \in B/dNA \\ \bar{v} \equiv \bar{u}(NA)}} \theta^{\bar{v}}(d\tau, dN) \tag{2}$$

$$\theta^{\bar{u}}\left(\frac{-1}{\tau}, N\right) = \frac{-i\tau}{N\sqrt{3}} \sum_{\bar{w} \in B/NA} \mathbf{e}\left(-\frac{\mathrm{Tr}(uw^*)}{N}\right)\theta^{\bar{w}}(\tau, N). \tag{3}$$

Proof. (1) Notice that $N|\frac{u}{N}+n|^2 \equiv \frac{|u|^2}{N}$ in $\mathbb{Z}$, so

$$\theta^{\bar{u}}(\tau+1,N) = \sum_{n\in A} \mathbf{e}(N|\frac{u}{N}+n|^2)\,\mathbf{e}\left(N|\frac{u}{N}+n|^2\tau\right) = \mathbf{e}\left(\frac{|u|^2}{N}\right)\theta^{\bar{u}}(\tau,N)$$

(2) Observe that

$$\theta^{\bar{u}}(\tau,N) = \sum_{n\in A} \mathbf{e}\left(N|\frac{u}{N}+n|^2\tau\right) = \sum_{n\in A} \mathbf{e}\left(Nd|\frac{\frac{u}{N}+n}{d}|^2\tau d\right)$$

If we let $n = v + md$ for some $v, m \in A$, then

$$\theta^{\bar{u}}(\tau,N) = \sum_{\substack{\bar{v}\in A/dA \\ m\in A}} \mathbf{e}\left(Nd|\frac{u+Nv}{Nd}+m|^2\tau d\right) = \sum_{\bar{v}\in A/dA} \theta^{\overline{u+Nv}}(d\tau, dN).$$

(3) Note that $B/NA \subset \frac{1}{3}A/NA \cong A/3NA$. Then if we identify A as $\mathbb{Z}^2$, for $v \in B$, $3v \in \sqrt{-3}A \subset A$ can also be recognized as an element in $\mathbb{Z}^2$.
Let $f = e^{-\pi|x|^2}$. For $\bar{v} \in (\mathbb{Z}/N\mathbb{Z})^2, \gamma \in \mathrm{GL}_2(\mathbb{R})$, define

$$\theta_0^{\bar{v}}(\gamma) = \sum_{n\in\mathbb{Z}^2} f((v/N+n)\gamma).$$

Also, recall the Fourier transformation of a function $F : (\mathbb{Z}/N\mathbb{Z})^2 \to \mathbb{C}$ is defined as

$$\hat{F}(\bar{v}) = \frac{1}{N}\sum_{\bar{w}\in(\mathbb{Z}/N\mathbb{Z})^2} F(\bar{w})\mu_N^{-\langle w,vS\rangle},$$

where $S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. We compute, for $r > 0$,

$$r\theta_0^{\bar{v}}(\gamma r) = r\sum_{n\in\mathbb{Z}^2} f((v/N+n)\gamma r) = r\sum_{n\in\mathbb{Z}^2}\phi(v/N+n),$$

where $\phi(x) = f(x\gamma r)$. Then $\hat{\phi}(x) = r^{-2}f(x(\gamma^{-1})^T r^{-1})$. The Poisson summation formula thus gives

$$\begin{aligned}
r\theta_0^{\bar{v}}(\gamma r) &= r^{-1}\sum_{n\in\mathbb{Z}^2} f(nS(\gamma^{-1})^T r^{-1})e^{2\pi i\langle nS, v/N\rangle} \\
&= r^{-1}\sum_{n\in\mathbb{Z}^2} f(n\gamma r^{-1})\mu_N^{\langle n,-vS\rangle} \\
&= r^{-1}\sum_{\bar{w}\in(\mathbb{Z}/N\mathbb{Z})^2}\ \sum_{\substack{n\in\mathbb{Z}^2 \\ n\equiv w(N)}} f(n\gamma r^{-1})\mu_N^{\langle n,-vS\rangle} \\
&= r^{-1}\sum_{\bar{w}\in(\mathbb{Z}/N\mathbb{Z})^2} \theta_0^{\bar{w}}(\gamma N r^{-1})\mu_N^{\langle n,-vS\rangle} = Nr^{-1}\hat{\theta}_0^{\bar{v}}(\gamma N r^{-1}).
\end{aligned}$$

Back to our problem, let $\gamma \in \mathrm{SL}_2(\mathbb{R})$ be the positive square root of $\frac{1}{\sqrt{3}}\left[\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right]$, then observe

$$\theta_0^{\overline{3v}}\left(\gamma\sqrt{3N}\sqrt{t/\sqrt{3}}\right) = \sum_{n\in\mathbb{Z}^2} e^{-\pi\sqrt{3}N|(3v/3N+n)\gamma|^2 t} = \sum_{n\in\mathbb{Z}^2} e^{-2\pi N|(v/N+n)|^2 t} = \theta^{\bar{v}}(it,N).$$

Then the above formula by Poisson summation formula indicates

$$\theta^{\bar{v}}\left(\frac{-1}{i\tau}, N\right) = \frac{t}{N\sqrt{3}} \sum_{\bar{u} \in \frac{1}{3}A/NA} \mathbf{e}\left(-\frac{3(vu^*)_2}{N}\right) \theta^{\bar{u}}(3it, N),$$

where $vu^* = (vu^*)_1 + (vu^*)_2\mu_3$. Extend the function of $t \in \mathbb{R}^+$ analytically to $-i\tau$ for $\tau \in \mathcal{H}$, we get

$$\theta^{\bar{v}}\left(\frac{-1}{\tau}, N\right) = \frac{-i\tau}{N\sqrt{3}} \sum_{\bar{u} \in \frac{1}{3}A/NA} \mathbf{e}\left(-\frac{(\sqrt{-3}v(\sqrt{-3}u)^*)_2}{N}\right) \theta^{\bar{u}}(3\tau, N)$$

Notice that for any $\bar{w} \in B/NA$, $\sum_{n \in A} \mathbf{e}\left(N \mid \frac{w/N+n}{\sqrt{-3}} \mid^2 3\tau\right)$, so

$$\theta^{\bar{w}}(\tau, N) = \sum_{\substack{\bar{r} \in A/\sqrt{-3}A \\ m \in A}} \mathbf{e}\left(N \mid \frac{(w+Nr)/\sqrt{-3}}{N} + m \mid^2 3\tau\right) = \sum_{\bar{r} \in A/\sqrt{-3}A} \theta^{\overline{(w+Nr)/\sqrt{-3}}}(3\tau, N)$$

Then noting that $(\sqrt{-3}vw^*)_2 = \mathrm{Tr}(vw^*)$, substituting this formula to the above equation completes the proof. □

We now present a useful lemma computing the sum of exponential:

Lemma 2. For $b, d \in \mathbb{Z}, (3b, d) = 1$, let $\phi_{b,d} = \sum_{\bar{v} \in A/dA} \mathbf{e}\left(\frac{b|v|^2}{d}\right)$, then $\phi_{b,d} = d\left(\frac{d}{3}\right)$

Proof. First consider the case d is a prime $\neq 3$. If $d = 2$, $\phi_{b,d} = \mathbf{e}(b \cdot |0|^2/2) + \mathbf{e}(b \cdot |1|^2/2) + \mathbf{e}(b \cdot |\mu_3|^2/2) + \mathbf{e}(b \cdot |1+\mu_3|^2/2) = -2 = 2 \cdot \left(\frac{2}{3}\right)$ since b is odd.
For $d > 4$, there is an isomorphism $\mathbb{Z}[\sqrt{-3}]/d\mathbb{Z}[\sqrt{-3}]$ to A/dA, so

$$\phi_{b,d} = \sum_{r_1, r_2 \in \mathbb{Z}/d\mathbb{Z}} \mathbf{e}\left(\frac{b(r_1^2 + 3r_2^2)}{d}\right)$$

If $d \nmid m$, note that the number of $r \in \mathbb{Z}/d\mathbb{Z}$ such that $s \equiv r^2 \pmod{d}$ is $1 + \left(\frac{s}{d}\right)$ for $s \in \mathbb{Z}/d\mathbb{Z}$, then we have

$$\sum_{r \in \mathbb{Z}/d\mathbb{Z}} \mathbf{e}\left(\frac{mr^2}{d}\right) = \sum_{r \in \mathbb{Z}/d\mathbb{Z}} \left(1 + \left(\frac{ms}{d}\right)\right) \mathbf{e}\left(\frac{s}{d}\right) = \left(\frac{m}{d}\right) g(\chi),$$

where $\chi = \left(\frac{\bullet}{d}\right)$. But $g(\chi)^2 = \chi(-1)|g(\chi)|^2 = \left(\frac{-1}{d}\right) d$, we conclude

$$\phi_{b,d} = \left(\frac{b}{d}\right) \sqrt{\left(\frac{-1}{d}\right) d} \cdot \left(\frac{3b}{d}\right) \sqrt{\left(\frac{-1}{d}\right) d} = \left(\frac{-3}{d}\right) d = \left(\frac{d}{3}\right) d$$

by the quadratic reciprocity.
Observe that for $x \in N^{-1}B$,

$$\sum_{\bar{w} \in A/NA} \mathbf{e}\left(\mathrm{Tr}(xw^*)\right) = \begin{cases} N^2 & x \in B, \\ 0 & x \notin B; \end{cases}$$

$$\sum_{\bar{v}\in B/NA} \mathbf{e}\left(\mathrm{Tr}(vx^*)\right) = \begin{cases} 3N^2 & x \in A, \\ 0 & x \notin A. \end{cases} \tag{4}$$

We now return to the original statement of lemma. If $d = p^t$ for some prime $p \neq 3$ and $t > 1$, we apply induction on t:

$$\begin{aligned} \phi_{b,p^t} &= \sum_{\bar{v}\in A/p^{t-1}A} \sum_{\bar{w}\in A/pA} \mathbf{e}\left(\frac{b|v+p^{t-1}w|^2}{p^t}\right) \\ &= \sum_{\bar{v}\in A/p^{t-1}A} \mathbf{e}\left(\frac{b|v|^2}{p^t}\right) \sum_{\bar{v}\in A/dA} \mathbf{e}\left(\frac{\mathrm{Tr}(bvw^*)}{p^t}\right) = \phi_{b,p^{t-2}}p^2 = \left(\frac{p^t}{3}\right)p^t. \end{aligned}$$

If $d = d_1d_2 > 0$ for some $d_1, d_2 \in \mathbb{N}$ such that $(d_1, d_2) = 1$, consider the bijection from $A/d_1A \times A/d_2A$ to A/dA by $(\bar{u_1}, \bar{u_2}) \mapsto \overline{d_2u_1 + d_1u_2}$. Then

$$\phi_{b,d} = \sum_{\bar{v}\in A/dA} \mathbf{e}\left(\frac{b|v|^2}{d_1d_2}\right) = \sum_{\bar{u_1}\in A/d_1A} \mathbf{e}\left(\frac{b|u_1|^2}{d_1}\right) \sum_{\bar{u_2}\in A/d_2A} \mathbf{e}\left(\frac{b|u_2|^2}{d_2}\right) = \phi_{bd_2,d_1}\phi_{bd_1,d_2}$$

Thus, for general $d = \prod_i p_i^{\alpha_i} > 0$, $\phi_{b,d} = \prod_i \left(\frac{p_i^{\alpha_i}}{3}\right) p_i^{\alpha_i} = \left(\frac{d}{3}\right) d$ via induction to the number of prime factor. Lastly, for $d < 0, \phi_{b,d} = \phi_{-b,-d} = \left(\frac{-d}{3}\right)(-d)$ by $\left(\frac{-1}{3}\right) = -1$. ◻

For $N \in \mathbb{N}$, define $\Gamma_0(3N, N) = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid b \equiv 0 \pmod N, c \equiv 0 \pmod{3N}\}$.

Proposition 2.1. For $N \in \mathbb{N}, \bar{u} \in A/NA, \gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(3N, N)$,

$$(\theta^{\bar{u}}[\gamma]_1)(\tau, N) = \left(\frac{d}{3}\right)\theta^{\overline{au}}(\tau, N),$$

where $\left(\frac{d}{3}\right)$ is the Legendre symbol.

Proof. May assume $d > 0$ since $\theta^{\overline{au}} = \theta^{\overline{-au}}$. Observe that $\frac{a\tau+b}{c\tau+d} = \frac{1}{d}\left(\frac{1}{d/\tau+c} + b\right)$, then by lemma,

$$\begin{aligned} \theta^{\bar{u}}(\gamma(\tau), N) &= \sum_{\substack{\bar{v}\in B/dNA \\ \bar{v}\equiv\bar{u}(NA)}} \theta^{\bar{u}}\left(\frac{1}{d/\tau+c} + b, dN\right) \\ &= \sum_{\substack{\bar{v}\in B/dNA \\ \bar{v}\equiv\bar{u}(NA)}} \mathbf{e}\left(\frac{b|v|^2}{dN}\right)\theta^{\bar{v}}\left(-\frac{1}{-d/\tau-c}, dN\right) \\ &= \frac{i(d/\tau+c)}{dN\sqrt{3}} \sum_{\substack{\bar{v},\bar{w}\in B/dNA \\ \bar{v}\equiv\bar{u}(NA)}} \mathbf{e}\left(\frac{b|v|^2 - \mathrm{Tr}(vw^*)}{dN}\right)\theta^{\bar{w}}\left(-\frac{d}{\tau} - c, dN\right) \\ &= \frac{i(c\tau+d)}{dN\tau\sqrt{3}} \sum_{\substack{\bar{v},\bar{w}\in B/dNA \\ \bar{v}\equiv\bar{u}(NA)}} \mathbf{e}\left(\frac{b|v|^2 - \mathrm{Tr}(vw^*) - c|w|^2}{dN}\right)\theta^{\bar{w}}\left(-\frac{d}{\tau}, dN\right) \end{aligned}$$

Observe the sum above fixing w. Notice that $cw \in NA$, we have

$$\sum_{\substack{\bar{v}\in B/dNA\\ \bar{v}\equiv\bar{u}(NA)}} \mathbf{e}\left(\frac{b|v|^2 - \mathrm{Tr}(vw^*) - c|w|^2}{dN}\right) = \sum_{\substack{\bar{v}\in B/dNA\\ \bar{v}\equiv\bar{u}(NA)}} \mathbf{e}\left(\frac{b|v-cw|^2 - \mathrm{Tr}((v-cw)w^*) - c|w|^2}{dN}\right)$$
$$= \mathbf{e}\left(-\frac{\mathrm{Tr}(auw^*)}{N}\right) \sum_{\substack{\bar{v}\in B/dNA\\ \bar{v}\equiv\bar{u}(NA)}} \mathbf{e}\left(\frac{b|v|^2}{dN}\right),$$

since $ad - bc = 1$. Notice that the summand depends only on $v \pmod{dA}$ since $N \mid b$, and the fact that $(d, N) = 1$, we see from Lemma 2

$$\sum_{\substack{\bar{v}\in B/dNA\\ \bar{v}\equiv\bar{u}(NA)}} \mathbf{e}\left(\frac{b|v|^2}{dN}\right) = \sum_{\bar{v}\in A/dA} \mathbf{e}\left(\frac{b|v|^2}{dN}\right) = \left(\frac{d}{3}\right) d.$$

Back to the original formula, the above induces

$$\theta^{\bar{u}}(\gamma(\tau), N) = \frac{i(c\tau+d)}{N\tau\sqrt{3}}\left(\frac{d}{3}\right) \sum_{\bar{w}\in B/dNA} \mathbf{e}\left(-\frac{\mathrm{Tr}(auw^*)}{N}\right)\theta^{\bar{w}}\left(-\frac{d}{\tau}, dN\right)$$
$$= \frac{i(c\tau+d)}{N\tau\sqrt{3}}\left(\frac{d}{3}\right) \sum_{\bar{v}\in B/NA} \mathbf{e}\left(-\frac{\mathrm{Tr}(auv^*)}{N}\right)\theta^{\bar{v}}\left(-\frac{1}{\tau}, N\right)$$
$$= \frac{c\tau+d}{3N^2}\left(\frac{d}{3}\right) \sum_{\bar{v},\bar{w}\in B/NA} \mathbf{e}\left(-\frac{\mathrm{Tr}(vw^* + auv^*)}{N}\right)\theta^{\bar{v}}(\tau, N)$$

By equation (4), $\sum_{\bar{v}\in B/NA} \mathbf{e}\left(-\frac{\mathrm{Tr}(vw^*+auv^*)}{N}\right) = \begin{cases} 3N^2 & \text{if } \bar{w} = -\overline{au}, \\ 0 & \text{if } \bar{w} \neq -\overline{au}. \end{cases}$ Therefore,

$$(\theta^{\bar{u}}[\gamma]_1)(\tau, N) = (c\tau+d)^{-1}\theta^{\bar{u}}(\gamma(\tau), N) = \left(\frac{d}{3}\right)\theta^{\overline{au}}(\tau, N).$$

□

Observe that if $\delta = \left[\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right]$, then $\delta\Gamma_0(3N^2)\delta^{-1} = \Gamma_0(3N, N)$. This inspires the following construction:

Theorem 2.2. Let $N \in \mathbb{N}, \chi : (A/NA)^* \to \mathbb{C}^*$ be a character extended multiplicatively to A. Define

$$\theta_\chi(\tau) = \frac{1}{6}\sum_{\bar{u}\in A/NA} \chi(u)\theta^{\bar{u}}(N\tau, N).$$

Then $\theta_\chi \in \mathcal{M}_1(3N^2, \psi)$, where $\psi = \chi \cdot \left(\frac{\bullet}{3}\right)$.

Proof. Observe that for $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(3N^2)$,

$$(\theta^{\bar{u}}[\delta\gamma]_1)(\tau, N) = (\theta^{\bar{u}}[(\delta\gamma\delta^{-1})\delta]_1)(\tau, N) = \left(\frac{d}{3}\right)(\theta^{\overline{au}}[\delta]_1)(\tau, N),$$

since $\delta\gamma\delta^{-1} \in \Gamma_0(3N, N)$ and $d_{\delta\gamma\delta^{-1}} = d$. Thereby,

$$\begin{aligned}\theta_\chi[\gamma]_1(\tau) &= \frac{1}{6}\sum_{\bar{u}\in A/NA} \chi(u)\theta^{\overline{u}}[\gamma]_1(N\tau, N) \\ &= \frac{1}{6}\sum_{\bar{u}\in A/NA} \chi(u)\theta^{\overline{u}}[\delta\gamma]_1(\tau, N) \\ &= \frac{1}{6}\sum_{\bar{u}\in A/NA} \chi(u)\left(\frac{d}{3}\right)\theta^{\overline{au}}[\delta]_1(\tau, N) \\ &= \frac{1}{6}\left(\frac{d}{3}\right)\chi(a)^{-1}\sum_{\bar{u}\in A/NA} \chi(u)\theta^{\overline{u}}(N\tau, N) = \left(\frac{d}{3}\right)\chi(d)\theta_\chi(\tau).\end{aligned}$$

It's left to check the growth of the Fourier coefficient. In fact,

$$\theta_\chi(\tau) = \frac{1}{6}\sum_{n\in A}\chi(n)\,\mathbf{e}(|n|^2\tau) = \sum_{m=0}^{\infty} a_m(\theta_\chi)q^m.$$

So the Fourier coefficient $a_m(\theta_\chi) = \frac{1}{6}\sum_{\substack{n\in A \\ |n|^2=m}} \chi(n) = \mathcal{O}(m)$, which indicates the last necessary condition of modular form. □

The unit group of A is $A^* = \{\pm 1, \pm\mu_3, \pm\mu_3^2\}$. By the formula Fourier coefficient, $\theta_\chi = 0$ unless $\chi\mid_{A^*}$ trivial, so we'll assume this in the following.

Let $\pi \in A$ be a prime such that $|\pi|^2 \neq 3$. For any α relatively prime to π, the analog of Fermat's Little Theorem states that

$$\alpha^{|\pi|^2-1} \equiv 1 \pmod{\pi}$$

Thus,

$$\alpha^{\frac{|\pi|^2-1}{3}} \equiv \mu_3^k \pmod{\pi}$$

for some $k = 0, 1, 2$. Therefore, we define the cubic residue character χ_π of α modulo π by $\chi_\pi(\alpha) = \mu_3^k$. Additionally, $\chi_\pi(\alpha) = 0$ if $\pi \mid \alpha$. We have the following properties (without proof, ref. Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, Chapter 9):

Proposition 2.3. Let $\pi, \pi' \in A$ be primes such that $|\pi|^2 \neq |\pi'|^2 \neq 3$. For any α_1, α_2 relatively prime to π,

1. $\chi_\pi(\alpha) = 1$ if and only if $x^3 \equiv \alpha \pmod{\pi}$ has a solution.

2. $\chi_\pi(\alpha_1\alpha_2) = \chi_\pi(\alpha_1)\chi_\pi(\alpha_2)$.

3. If $\alpha_1 \equiv \alpha_2 \pmod{\pi}$, then $\chi_\pi(\alpha_1) = \chi_\pi(\alpha_2)$.

4. (Cubic Reciprocity) $\chi_\pi(\pi') = \chi_{\pi'}(\pi)$.

Then the theorem follows:

Theorem 2.4. Let $d \in \mathbb{N}$ be a cubic free integer, $N = 3\prod_{p|d} p$. Then there exist a character $\chi : (A/NA)^* \to \{1, \mu_3, \mu_3^2\}$, such that the multiplicative extension of χ to A is trivial on A^* and on primes $p \nmid N$, while on elements π of A such that $\pi\bar{\pi}$ is a prime $p \nmid N$ it is trivial if and only if d is a cube modulo p.

Corollary. With the character χ defined in the theorem, $a_p(\theta_\chi) = a_p(C)$, for all $p \in \mathcal{P}$.

3 Normalized Eigenform

Let $\mathcal{P}$ be the set of all primes in $\mathbb{N}$. Recall Theorem 5.9.2.:

Theorem 3.1. Let $f \in \mathcal{M}_k(N, \chi), f(\tau) = \sum_{m=0}^{\infty} a_m(f)q^m$, then the following are equivalent:

1. f is a normalized eigenform.

2. $L(s, f)$ has an Euler product expansion

$$L(s, f) = \prod_{p \in \mathcal{P}} (1 - a_p p^{-s} + \chi(p)p^{k-1-2s})^{-1}.$$

Note that A is a principal ideal domain, so every maximal ideal is generated by an irreducible element $t = a + b\mu_3$ of A. Then $|t|^2 = a^2 - ab + b^2$ is either p or p^2 for some prime p.
Notice that $a^2 - ab + b^2 = \frac{1}{4}((2a - b)^2 + 3b^2)$, so $\exists t \in A$ such that $|t|^2 = p$ for $p \neq 3$ is equivalent to $\left(\frac{-3}{p}\right) = 1$, which is again equivalent to $3 \mid p - 1$. With this discussion, we may separate p into three cases:

1. For $p = 3$, $|t|^2 = a^2 - ab + b^2 = 3$ gives solutions $(a, b) = \pm(1, 2), \pm(2, 1), \pm(1, -1)$, which induce $t = \pm\sqrt{-3}, \frac{\pm 3 \pm \sqrt{-3}}{2}$. In fact, this solution set is just $\sqrt{-3}A^*$, so every t generates the same maximal ideal $\langle\sqrt{-3}\rangle$. Also,

$$a_3(\theta_\chi) = \frac{1}{6} \sum_{\substack{n \in A \\ |n|^2 = 3}} \chi(n) = \chi(\sqrt{-3}), \psi(3) = 0.$$

2. For $3 \mid p - 1$, there exists $\pi_p \in A$ such that $|\pi_p|^2 = p$, and $\langle\pi_p\rangle, \langle\bar{\pi}_p\rangle$ are both maximal ideal. Also,

$$a_p(\theta_\chi) = \frac{1}{6} \sum_{\substack{n \in A \\ |n|^2 = p}} \chi(n) = \chi(\pi_p) + \chi(\bar{\pi}_p),$$

since $\{n \in A \mid |n|^2 = p\} = \pi_p A^* \sqcup \bar{\pi}_p A^*$. In addition,

$$\psi(p) = \chi(p) \left(\frac{p}{3}\right) = \chi(p).$$

3. For $3 \mid p - 2$, there isn't any $t \in A$ such that $|t|^2 = p$. But we still get $\langle p\rangle$ as a maximal ideal. Also,

$$a_p(\theta_\chi) = 0, \psi(p) = \chi(p) \left(\frac{p}{3}\right) = -\chi(p).$$

Let $\mathcal{S} = \{\sqrt{-3}\} \sqcup \{\pi_p, \bar{\pi}_p \mid p \equiv 1 \pmod 3\} \sqcup \{p \mid p \equiv 2 \pmod 3\}$ be the set of all generators of maximal ideals of A, then $\forall x \in A$, x can be uniquely written as $u \prod_{\pi \in S} \pi^{a_\pi}$ for some $u \in A^*, a_\pi \in \mathbb{N}_0, a_\pi = 0$ for all but finitely many π.
By the definition, we compute

$$L(s, \theta_\chi) = \frac{1}{6} \sum_{n \in A/\{0\}} \chi(n)|n|^{-2s} = \prod_{\pi \in \mathcal{S}} (1 - \chi(\pi)|\pi|^{-2s})^{-1} = \prod_p L_p(s, \theta_\chi).$$

1. For $p = 3$,
$$L_p(s, \theta_\chi) = (1 - \chi(\sqrt{-3})|\sqrt{-3}|^{-2s})^{-1} = (1 - \chi(\sqrt{-3})3^{-s})^{-1};$$

2. For $3 \mid p - 1$,
$$\begin{aligned} L_p(s, \theta_\chi) &= (1 - \chi(\pi_p)|\pi_p|^{-2s})^{-1}(1 - \chi(\bar{\pi}_p)|\bar{\pi}_p|^{-2s})^{-1} \\ &= ((1 - \chi(\pi_p)p^{-s})(1 - \chi(\bar{\pi}_p)p^{-s}))^{-1} \\ &= ((1 - (\chi(\pi_p) + \chi(\bar{\pi}_p))p^{-s} + \chi(p)p^{-2s})^{-1} \\ &= (1 - a_p(\theta_\chi)p^{-s} + \psi(p)p^{-2s})^{-1}; \end{aligned}$$

3. For $3 \mid p - 2$,
$$\begin{aligned} L_p(s, \theta_\chi) &= (1 - \chi(p)|p|^{-2s})^{-1} \\ &= (1 - a_p(\theta_\chi)p^{-s} + \psi(p)p^{-2s})^{-1}; \end{aligned}$$

In either case, the second condition in the Theorem 5.9.2. is verified. Therefore, we may conclude

Theorem 3.2. For any $N \in \mathbb{N}$ and a character $\chi : (A/NA)^* \to \mathbb{C}^*$ extended multiplicatively to A, θ_χ is a normalized eigenform.

Remark. One can prove that θ_χ is actually a cusp form when $d > 1$.

4 Galois Representation

Embed S_3 into $\mathrm{GL}_2(\mathbb{Z})$ by

$$(1\,2\,3) \mapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, (2\,3) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Let $F = \mathbb{Q}[d^{\frac{1}{3}}, \mu_3]$, then the above gives a representation $\rho : \mathrm{Gal}(F/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Z})$. Then

$$\mathrm{Tr}\,\rho(\mathrm{Frob}_p) = \begin{cases} 2 & \text{if } p \equiv 1 \pmod 3 \text{ and } d \text{ is a cube mod } p; \\ -1 & \text{if } p \equiv 1 \pmod 3 \text{ and } d \text{ is not a cube mod } p \\ 0 & \text{if } p \equiv 2 \pmod 3). \end{cases}$$

matches the formula of $a_p(\theta_\chi)$. Also, the determinant of ρ is defined on conjugacy classes over unramified primes,

$$\det \rho(\mathrm{Frob}_p) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{3}; \\ -1 & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

This coincides with $\psi(p)$. So the Galois group representation ρ, as described by its trace and determinant on Frobenius elements, arises from the modular form θ_χ.