Given $p \nmid N$. $\quad S_1(N) = \{ [E, Q] \mid Q \text{ order } N \} \; / \mathbb{Q}$

p prime, $N \in \mathbb{N}$. $\quad \bar{S}_1(N) = \{ [\bar{E}, \bar{Q}] \mid \bar{E} / \bar{\mathbb{F}}_p, \; \bar{Q} \text{ order } N \}$

Let $\mathfrak{p}$ be a maximal ideal of $\bar{\mathbb{Z}}$ with $\mathfrak{p} \cap \mathbb{Z} = p$.

E has good reduction at $\mathfrak{p}$ $\Leftrightarrow$ $j(E) \in \bar{\mathbb{Z}}_{(\mathfrak{p})} \longrightarrow \bar{\mathbb{F}}_p \ni \overline{j(E)}$

$\rightsquigarrow \; S_1(N)'_{good} = \{ [E, Q] \in S_1(N) \mid E \text{ has good red. at } \mathfrak{p}, \; \overline{j(E)} \neq 0, 1728 \}$

$\downarrow \qquad\qquad\qquad\qquad \mathrm{Aut}(E) = \{\pm 1\}$ $\Updownarrow$

$\bar{S}_1(N)' = \{ [\bar{E}, \bar{Q}] \in \bar{S}_1(N) \mid j(\bar{E}) \neq 0, 1728 \}$

(surj.: lift the Weierstrass eq. from $\bar{\mathbb{F}}_p$ to $\bar{\mathbb{Z}}$ $\Rightarrow \begin{cases} \Delta \mapsto \bar{\Delta} \\ j \mapsto \bar{j} \end{cases}$ so $\exists E \mapsto \bar{E}$

$E[N] \twoheadrightarrow \bar{E}[N] \Rightarrow \exists Q \mapsto \bar{Q}$)

Thm (Igusa). If $p \nmid N$, then $X_1(N)$ has good reduction $\bar{X}_1(N)$ at $\mathfrak{p}$, and

$$\mathbb{F}_p(\bar{X}_1(N)) \simeq \mathbb{F}_p(\bar{j}, x(\bar{Q})), \qquad [\bar{E}_j, \bar{Q}] \in \bar{S}_1(N),$$

$\uparrow$ univ. ell. curve def. by

$$y^2 + xy = x^3 - \frac{36}{j - 1728} x - \frac{1}{j - 1728}$$

$$\Rightarrow \quad \begin{array}{ccccc} S_1(N)'_{good} & \longrightarrow & X_1(N)^{planar} & \longrightarrow & X_1(N) \\ \downarrow & & \downarrow & & \downarrow \\ \bar{S}_1(N)' & \longrightarrow & \bar{X}_1(N)^{planar} & \longrightarrow & \bar{X}_1(N) \end{array}.$$

(sketch of) pf: In fact, Igusa only proved the case $X(N) \longrightarrow \bar{X}(N)$.

Consider

$$\begin{array}{ccccc} A, B & \subseteq & R & \subseteq & K = \mathbb{Q}(j, j_N, x(\tfrac{1}{N})) \\ | & & | & & | \;\; (K(\mu_N) = K(N)) \\ \mathbb{Z}[j], \mathbb{Z}[1/j] & \subseteq & R_p & \subseteq & \mathbb{Q}(j) \end{array}$$

Want: Glue Spec$A$ and Spec$B$ into a model $/\mathbb{Z}$ that has a good reduction at p.

$$R_p = \left\{ \left| \frac{f}{g} \right|_p := \frac{\max |a_n(f)|_p}{\max |a_n(g)|_p} \leq 1 \right\} \longrightarrow \mathbb{F}_p(j)$$

(extended p-adic norm)

Lemma. R is an unramified valuation ring in K.

(sketch of) pf: Only do the case $p \neq 2$. Define λ by $j = \dfrac{2^8(\lambda^2-\lambda+1)^3}{\lambda^2(1-\lambda)^2}$.

$$\begin{array}{ccc} R_\lambda & \subseteq & K(\lambda) \\ \Big| & & \Big| \\ R_p & \subseteq & \mathbb{Q}(j) \end{array}$$

It suffices to prove that R_λ is unique and unramified

$$\Leftrightarrow [\kappa_\lambda := \kappa(R_\lambda) : \mathbb{F}_p(j)] = [K(\lambda) : \mathbb{Q}(j)] \quad (*)$$

Eq. for $\lambda \,/\, \mathbb{F}_p(j)$ is irred. $\Rightarrow [\mathbb{F}_p(\lambda) : \mathbb{F}_p(j)] = [\mathbb{Q}(\lambda) : \mathbb{Q}(j)]$

$$[\kappa_\lambda : \mathbb{F}_p(\lambda)] \leq [\underset{\mathbb{Q}(\lambda, j_N, x(T/N))}{\overset{\shortparallel}{K(\lambda)}} : \mathbb{Q}(\lambda)] = [\mathbb{F}_p(\lambda, j_N, x(T/N)) : \mathbb{F}_p(\lambda)] \leq [\kappa_\lambda : \mathbb{F}_p(\lambda)] \quad \square$$

another Igusa's thm., need $p \nmid N$. [cf. Fibre systems of Jacobian varieties III].

Consider
$$\begin{array}{ccc} R & \longrightarrow & \kappa = R/pR \\ A_m & \longrightarrow & \bar{A}_m \\ B_m & \longrightarrow & \bar{B}_m \end{array}$$

Let $A_m = \{ f \in A \mid j^{-m} f \text{ int. over } \mathbb{Q}[1/j] \}$

can be replaced by $\mathbb{Z}[1/j]$.

$$j^{-m} \Big\downarrow \wr \quad = A \cap j^m B$$
$$B_m = B \cap j^{-m} A$$

Let $\tilde{A}, \tilde{B}$ be the int closures of $\bar{A}, \bar{B}$ in κ, resp. $\tilde{A}_m = \tilde{A} \cap j^m \tilde{B}$.

Lemma. $\kappa \cap \bar{\mathbb{F}}_p = \mathbb{F}_p$ and $\bar{\mathbb{F}}_p \kappa = \bar{\mathbb{F}}_p(j, j_N, x(T/N)) = \bar{\mathbb{F}}_p(j, f_{1,0}, f_{0,1})$

(sketch of) pf: $\kappa_\lambda = \mathbb{F}_p(\lambda, j_N, x(T/N))$ and $\mathbb{F}_p(\mu_N, \lambda)$ are linearly disjoint $/ \mathbb{F}_p(\lambda)$.

$$\kappa_\lambda(\mu_N) \cap \bar{\mathbb{F}}_p = \mathbb{F}_p(\mu_N) \Rightarrow \kappa \cap \bar{\mathbb{F}}_p \subseteq \kappa_\lambda \cap \bar{\mathbb{F}}_p = \mathbb{F}_p. \quad \square$$

This shows (+ Igusa's thm 5) that K and κ have the same genus g.

$$\Rightarrow \operatorname{rk} A_m = H^0(X, \pi^* m(\infty)) \overset{m \gg 1}{=} m[K : \mathbb{Q}(j)] + 1 - g$$
$$\shortparallel \qquad\qquad\qquad\qquad$$
$$\dim_{\mathbb{F}_p} \bar{A}_m \cong A_m/pA_m \qquad = m[\kappa : \mathbb{F}_p(j)] + 1 - g = \dim_{\mathbb{F}_p} \tilde{A}_m$$

$$X \xrightarrow{\pi} \mathbb{P}^1$$

$\Rightarrow \bar{A}_m = \tilde{A}_m \;\; \forall m \gg 1 \Rightarrow \bar{A}$ (and $\bar{B}$) are int closed in κ.

Write $A_m = \bigoplus_{i=0}^{r} \mathbb{Z} f_i$, where $f_i = j^i \;\; \forall\, 0 \leq i \leq m$. $A = \mathbb{Z}[A_m]$ for $m \gg 1$.

$\rightsquigarrow \operatorname{Spec} A \hookrightarrow \mathbb{P}^r$

Similarly, we get $\operatorname{Spec} B \hookrightarrow \mathbb{P}^r$ (gluing by $A_m = j^m B_m$).

Kroneckerian model

Consider $C = \operatorname{Spec} A \cup \operatorname{Spec} B \subseteq \mathbb{P}^r_{\mathbb{Q}} \rightsquigarrow C_p = \operatorname{Spec} \bar{A} \cup \operatorname{Spec} \bar{B} \subseteq \mathbb{P}^r_{\mathbb{F}_p}$

nonsingular $\because A, B$ int closed $\qquad$ nonsingular $\because \bar{A}, \bar{B}$ int. closed $\quad \square$

For simplicity, write $D = \mathrm{Div}^\circ$, $J = \mathrm{Pic}^\circ$ $\rightsquigarrow$

$$\begin{tikzcd} D(S_1(N)'_{\text{good}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & J_1(N) \arrow[d] \\ D(\bar{S}_1(N)') \arrow[r] & \bar{J}_1(N) \end{tikzcd}$$

$$\langle d \rangle : X_1(N) \xrightarrow{\sim} X_1(N) \quad \Rightarrow \quad \begin{tikzcd} J_1(N) \arrow[r, "\langle d\rangle_*"] \arrow[d] & J_1(N) \arrow[d] \\ \bar{J}_1(N) \arrow[r, "\overline{\langle d\rangle}_*"] & \bar{J}_1(N) \end{tikzcd}$$

For T_p, consider the Néron model $\mathcal{J}_1(N)/\mathbb{Z}[\frac{1}{N}]$ of $J_1(N)$

Then
$$\begin{array}{l} J_1(N) \xrightarrow{T_p} J_1(N) \\ \quad \big\downarrow \text{ extend} \\ \mathcal{J}_1(N) \xrightarrow{T_p} \mathcal{J}_1(N) \\ \quad \big\downarrow \text{ reduce} \\ \bar{J}_1(N) \xrightarrow{\bar{T}_p} \bar{J}_1(N) \end{array} \quad \Rightarrow \quad \begin{tikzcd} J_1(N) \arrow[r, "T_p"] \arrow[d] & J_1(N) \arrow[d] \\ \bar{J}_1(N) \arrow[r, "\bar{T}_p"] & \bar{J}_1(N) \end{tikzcd}$$

($\mathrm{Pic}^\circ(X_1(N))_p \simeq \mathrm{Pic}^\circ(X_1(N)_p)$, cf. Néron models B-L-R Thm 9.5.1.)

On $S_1(N)$, $T_p[E, Q] = \sum_C [E/C, Q+C]$, where $|C| = p$, $C \cap \langle Q \rangle = 0$.

Lemma. $$[\overline{E/C}, \overline{Q+C}] = \begin{cases} [\bar{E}^{\sigma_p}, \bar{Q}^{\sigma_p}] & \text{if } C = C_0 := \ker(E[p] \to \bar{E}[p]), \\ [\bar{E}^{\sigma_p^{-1}}, [p]\bar{Q}^{\sigma_p^{-1}}] & \text{else.} \end{cases}$$

pf: Let
$$\begin{array}{ccccc} E & \xrightarrow{\varphi} & E' = E/C & \xrightarrow{\psi} & E \\ Q & \longmapsto & Q' & \longmapsto & [p]Q \end{array}$$

E has ordinary reduction at p:

If $C = C_0$, then consider
$$\begin{tikzcd} E'[p] \arrow[r, "\psi"] \arrow[d] & E[p] \arrow[d, two heads] \\ \bar{E}'[p] \arrow[r, "\bar{\psi}"] & \bar{E}[p] \end{tikzcd}$$
E ord. red. $\Rightarrow$ E' ord. red.
$\Rightarrow |\bar{E}'[p]| = p$

$$\left\{ \begin{array}{l} |\psi(E'[p])| = \dfrac{|E'[p]|}{|\ker \psi|} = p \\ \varphi\psi(E'[p]) = 0 \end{array} \right. \Rightarrow \psi(E'[p]) = \ker\varphi = C \Rightarrow \bar{\psi} = 0$$

$\Downarrow$

$$\ker(\bar{E}' \xrightarrow{\bar{\psi}} \bar{E}) \subseteq \ker \underbrace{\bar{\varphi}\bar{\psi}}_{[p]_{\bar{E}'}} \subseteq \bar{E}'[p] \Rightarrow \ker\bar{\psi} = \bar{E}'[p] = \ker [p]_{\bar{E}'}$$

$\Downarrow$ by taking #

$$\deg_s \bar{\psi} = \deg_s [p]_{\bar{E}'}$$

$\Rightarrow$ de $\bar{\varphi} = 1$. $\deg_i \bar{\varphi} = p$ $\Rightarrow$

$$\begin{tikzcd} \bar{E} \arrow[rr, "\varphi"] \arrow[dr, "\sigma_p"'] & & \bar{E}' & \bar{Q}' \\ & \bar{E}^{\sigma_p} \arrow[ur, "s", "i"'] & \bar{Q}^{\sigma_p} \arrow[ur, mapsto] & \end{tikzcd}$$

$\Rightarrow [\bar{E}', \bar{Q}'] = [\bar{E}^{\sigma_p}, \bar{Q}^{\sigma_p}]$.

If $C \neq C_0$, then consider

$$\begin{tikzcd} E[p] \arrow[r, "\varphi"] \arrow[d] & E'[p] \arrow[d] \\ \bar{E}[p] \arrow[r, "\bar{\varphi}"] & \bar{E}'[p] \end{tikzcd}$$

$\varphi(C_0) \subset E'[p]$ $\uparrow$ order p $\because C_0 \cap \underset{\ker\varphi}{\overset{}{C}} = 0$.

$$\left.\begin{array}{l} \varphi(C_0) \underset{\text{order } p}{\subset} C' = \ker\psi \Rightarrow \text{“}=\text{”}, \\ \overline{\varphi(C_0)} = 0 \Rightarrow \varphi(C_0) \subset C_0' \Rightarrow \text{“}\supseteq\text{”}. \end{array}\right\} \Rightarrow C' = C_0'.$$

Similar to the case $C = C_0$ (with $(E, Q, \varphi) \to (E', Q', \psi)$), we get

$$\begin{tikzcd} \bar{E}' \arrow[rr, "\bar{\psi}"] \arrow[dr, "\sigma_p"'] & & \bar{E} & {[p]\bar{Q}} \\ & (\bar{E}')^{\sigma_p} \arrow[ur, "s"'] & (\bar{Q}')^{\sigma_p} \arrow[ur, mapsto] & \end{tikzcd}$$

$([p] = \psi\varphi)$ $\Rightarrow [\bar{E}', \bar{Q}'] = [\bar{E}^{\sigma_p^{-1}}, [p]\bar{Q}^{\sigma_p^{-1}}]$.

E has supersingular reduction at p: We show that $[\bar{E}', \bar{Q}'] =$

$$[\bar{E}', \bar{Q}'] = [\bar{E}^{\sigma_p}, \bar{Q}^{\sigma_p}] = [\bar{E}^{\sigma_p^{-1}}, [p]\bar{Q}^{\sigma_p^{-1}}].$$

$$\ker\bar{\varphi} \subseteq \ker[p]_{\bar{E}} = 0 \Rightarrow \deg_i \bar{\varphi} = p \Rightarrow \deg_i \bar{\psi} = \frac{\deg_i [p]_{\bar{E}}}{\deg_i \bar{\varphi}} = p$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$[\bar{E}^{\sigma_p}, \bar{Q}^{\sigma_p}] = [\bar{E}', \bar{Q}'] = [\bar{E}^{\sigma_p^{-1}}, [p]\bar{Q}^{\sigma_p^{-1}}] \qquad \square$$

Hence, we get $\displaystyle\sum_C [\overline{E/C}, \overline{Q+C}] = [\bar{E}^{\sigma_p}, \bar{Q}^{\sigma_p}] + p\,[\bar{E}^{\sigma_p^{-1}}, [p]\bar{Q}^{\sigma_p^{-1}}]$

Note that $j \neq 0, 1728 \Rightarrow j$ of $\bar{E}_j^{\sigma_p}, \bar{E}_j^{\sigma_p^{-1}} \neq 0, \overset{12^3}{\overset{\|}{1728}}$, so

$$\begin{tikzcd} S_1(N)'_{good} \arrow[r, "T_p"] \arrow[d] & \mathrm{Div}(S_1(N)'_{good}) \arrow[d] \\ \bar{S}_1(N)' \arrow[r, "\sigma_p + p\langle p\rangle\sigma_p^{-1}"] & \mathrm{Div}(\bar{S}_1(N)') \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} T_p \curvearrowright D(S_1(N)'_{good}) \arrow[d] \\ \sigma_p + p\overline{\langle p\rangle}\sigma_p^{-1} \curvearrowright D(\bar{S}_1(N)') \end{tikzcd} \quad (*)$$

We have

$$\begin{array}{ccccc} D(\bar{S}_1(N)') & \longrightarrow & D(\bar{X}_1(N)^{\text{planar}}) & \longrightarrow & \bar{J}_1(N) \\ \circlearrowright & & \circlearrowright & \uparrow & \circlearrowright \\ \sigma_p + p\langle p\rangle \sigma_p^{-1} & & \sigma_{p*} + \langle p\rangle_* \sigma_p^* & & \sigma_{p*} + \langle p\rangle_* \sigma_p^* \end{array} \quad (**)$$

by Igusa's thm.

$$(j(\bar{E}^{\sigma_p}), x(\bar{Q}^{\sigma_p})) = (j(\bar{E})^{\sigma_p}, x(\bar{Q})^{\sigma_p})$$

$$p(j(\bar{E}^{\sigma_p^{-1}}), x(\bar{Q}^{\sigma_p^{-1}})) = \sigma_p^*(j(\bar{E}), x(\bar{Q}))$$

Hence, we get

$$\begin{tikzcd} T_p \circlearrowright D(S_1(N)'_{\text{good}}) \arrow[r, "☆"] \arrow[d, "(*)"'] \arrow[dr, phantom, "\text{Igusa}"] & J_1(N) \circlearrowleft T_p \arrow[d, "\text{Eichler-Shimura relation.}"] \\ D(\bar{S}_1(N)') \arrow[r, "(**)"'] & \bar{J}_1(N) \end{tikzcd}$$

$$\begin{array}{ccc} \circlearrowright & & \circlearrowright \\ \sigma_p + p\langle p\rangle \sigma_p^{-1} & & \sigma := \sigma_{p*} + \langle p\rangle_* \sigma_p^* \end{array}$$

This diagram commutes: ☆ by

$$\begin{array}{ccc} D(S_1(N)) & \xrightarrow{\sim} & D(Y_1(N)) \\ \circlearrowright & & \circlearrowright \\ T_p & & T_p \end{array}$$

$$\left([E/C_j, \tfrac{1}{N} + C_j] \mapsto \Gamma_1(N)\beta_j. \quad C_j = \langle \tfrac{\tau + j}{p} \rangle, \ C_\infty = \langle \tfrac{1}{p} \rangle. \quad \beta_j = \begin{pmatrix} 1 & j \\ & p \end{pmatrix}, \ \beta_\infty = \begin{pmatrix} m & n \\ N & p \end{pmatrix}\begin{pmatrix} p & \\ & 1 \end{pmatrix} \right)$$

To check E-S relation, consider

$$\begin{tikzcd} T_p \circlearrowright D(S_1(N)'_{\text{good}}) \arrow[r, "g"] \arrow[dr, "f"'] & J_1(N) \circlearrowleft T_p \arrow[d, "h"] \\ & \bar{J}_1(N) \circlearrowleft \sigma, \bar{T}_p \end{tikzcd}$$

surj. by R-R

$$\sigma f = f T_p = h g T_p = h T_p g = \bar{T}_p h g = \bar{T}_p f \Rightarrow \sigma = \bar{T}_p.$$

Thm. (Eichler-Shimura) The diagram

$$\begin{array}{ccc} J_1(N) & \longrightarrow & \bar{J}_1(N) \\ \circlearrowright & & \circlearrowright \\ T_p & & \sigma_{p*} + \langle p\rangle_* \sigma_p^* \end{array}$$

commutes.

In particular,

$$\begin{array}{ccc} J_0(N) & \longrightarrow & \bar{J}_0(N) \\ \circlearrowright & & \circlearrowright \\ T_p & & \sigma_{p*} + \sigma_p^* \end{array}$$

commutes.

Application : E : ell. curve / $\mathbb{Q}$ with conductor N_E

Thm (Modularity thm., ver. a_p) $\exists$ new form $f \in S_2(\Gamma_0(N_E))$ s.t. $a_p(f) = a_p(E)$ $\forall p$.

A weaker version : Given $X_0(N) \overset{\alpha \neq 0}{\longrightarrow} E$. $\exists$ new form $f \in S_2(\Gamma_0(M_f))$, $M_f \mid N$

s.t. $a_p(f) = a_p(E)$ $\forall p \nmid N_E N$

pf: Consider $J_0(N) \xrightarrow{\Psi} \bigoplus_{f,n} A_f'$, which is an isogeny (between abelian var.)

$$A_f' = J_0(M_f) / \underset{\underset{\ker(\mathbb{T}_{\mathbb{Z}} \xrightarrow{\lambda_f} K_f)}{\shortparallel}}{I_f} J_0(M_f) \simeq \underset{\underset{\langle f^\sigma \rangle}{\shortparallel}}{V_f^\vee} / \underset{\underset{H_1(X_0(N), \mathbb{Z})|_{V_f}}{\shortparallel}}{\Lambda_f'}$$

$$\Psi_{f,n}(\varphi) : f^\sigma(\tau) \mapsto \varphi(n f^\sigma(n\tau)) , \quad n \mid \frac{N}{M_f}.$$

$\rightsquigarrow$ dual isogeny

$$\begin{tikzcd}
\bigoplus_{f,n} A_f' \arrow[r, "\Phi"] \arrow[d, "\bigoplus_{f,n}(a_p(f) - a_p(E))"'] \arrow[dr, phantom, "\circlearrowright"] & J_0(N) \arrow[d, "T_p - a_p(E)"] \arrow[dd, bend left, "\beta"] \\
\bigoplus_{f,n} A_f' \arrow[r, "\Phi"] & J_0(N) \arrow[d, "\alpha_*"] \\
& J(E)
\end{tikzcd}$$

Want : $= 0$ for some f

(Here, $(a_p(f)(\varphi))(f^\sigma) := a_p(f)^\sigma \varphi(f^\sigma)$)

The diagram commutes $\Leftrightarrow$

$$\begin{tikzcd}
J_0(N) \arrow[r, "\Psi"] \arrow[loop below, "T_p"] & \bigoplus_{f,n} A_f' \arrow[loop below, "\bigoplus_{f,n} a_p(f)"]
\end{tikzcd}$$

commutes.

($\Psi f = g \Psi \Rightarrow \Psi f \Phi = g \underbrace{\Psi \Phi}_{[\deg \Psi]} = \Psi \Phi g \Rightarrow \mathrm{Im}(f\Phi - \Phi g) \subseteq \ker \Psi$ ← finite set

$\Rightarrow f \Phi = \Phi g$)

For $\varphi \in J_0(N)$,

$(\Psi_{f,n} \circ T_p)(\varphi)(f^\sigma) = (T_p \varphi)(n f^\sigma(n\tau)) = n \varphi(T_p(f^\sigma \cdot n))$

$(a_p(f) \circ \Psi_{f,n})(\varphi)(f^\sigma) = a_p(f) \varphi(n f^\sigma(n\tau)) = n \varphi((T_p f^\sigma) \cdot n)$ $\shortparallel$ by $p \nmid n$ ($\because p \nmid N$)

Claim 1. If $a_p(f) \neq a_p(E)$ for some f. then $\bigoplus_n (a_p(f) - a_p(E))$ is surj.

pf. $\underset{\substack{\neq \\ 0}}{\delta} = \underset{\substack{\cap \\ \overline{\mathbb{Z}}}}{a_p(f)^\sigma} - \underset{\substack{\cap \\ \overline{\mathbb{Z}}}}{a_p(E)} \in \overline{\mathbb{Z}} \Rightarrow \delta^e + \underset{\substack{\uparrow \\ \text{indep of } \sigma}}{a_{e-1}}\delta^{e-1} + \cdots + \underset{\substack{\neq \\ 0}}{a_0} = 0$ (as operators).

$\Rightarrow -a_e = \delta \cdot (\delta^{e-1} + \cdots + a_{e-1})$ is surj. $\Rightarrow \delta$ is surj. ■

Claim 2. $\beta = \alpha_* \circ (T_p - a_p(E)) = 0$

pf. Consider

$$\begin{tikzcd}
J_0(N) \arrow[loop left, "T_p - a_p(E)"] \arrow[r, "\alpha_*"] \arrow[d, "\text{E-S relation}"'] & J(E) \arrow[d, "\pi"] \\
\bar{J}_0(N) \arrow[loop below, "\sigma_{p*} + \sigma_p^* - a_p(E)"] \arrow[r, "\bar{\alpha}_*"] & J(\bar{E}) \arrow[loop below, "\sigma_{p*} + \sigma_p^* - a_p(E) = 0"]
\end{tikzcd}$$

$\pi \circ \beta = 0 \circ \pi \circ \alpha_* = 0$

need $p \nmid N_E \Rightarrow$ good red.

$\pi \circ \underbrace{\alpha_* \circ (T_p - a_p(E))}_{\beta} = 0 \Rightarrow \operatorname{Im}\beta \subseteq \ker \pi \subsetneq J(E)$

$\Rightarrow \beta = 0$ since $J(E) \simeq E$ is a curve. ■

Hence, if for each f, $a_p(f) \neq a_p(E)$ for some $p \nmid N_E N$

$\overset{(a)}{\Rightarrow} \bigoplus_{f,n} (a_p(f) - a_p(E))$ is surj. $\overset{(b)}{\Rightarrow} \operatorname{Im}\Phi \subseteq \operatorname{Im} (T_p - a_p(E)) \circ \Phi$

Φ is an isogeny $\hookrightarrow$ $\|$ $\quad \cap$ (c)

$J_0(N) = \ker \alpha_*$

$\Rightarrow [\deg \alpha] = \alpha^* \alpha_* = 0$ $\quad \rightarrow\!\!\!\!\times$ □.