Ochanine's theorem on spin manifolds

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In this report, we will investigate the integrality of some genera. To do this, we need to view a family of genera as a modular form, which comes naturally if we defined the genera by power series $f(\tau, x)$ that are Jacobi form of weight -1. We will introduce some theory of Jacobi form to investigate elliptic genera of level 2. The modular form expanding at different cusps (0 and ∞) corresponds to twisted $\hat{A}$-genera and signature introduced in subsection 3.1 and 3.2. Using the theory of modular forms and the integrality of twisted $\hat{A}$-genera on spin manifold, we obtain the theorem of Ochanine.

The argument can be applied to more general situation, elliptic genera of level N defined for complex manifolds. We will introduce the Φ-function, which is a theta function, and define a family of complex elliptic genera. This time, involving twisted χ_y-genera and Euler characteristic, we can obtain similar result under appropriate assumptions.

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1 Jacobi form and theta function

1.1 Jacobi form

Definition 1.1. For $\tau \in \mathbb{H}$, let $\Psi(\tau, x)$ be an elliptic function in x (i.e., meromorphic, doubly periodic with respect to $2\pi i(\mathbb{Z}\tau + \mathbb{Z})$). Let $k \in \mathbb{Z}$, $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ congruent, then we call Ψ a Jacobi form on Γ of weight k if

$$\Psi(\gamma(\tau), \frac{x}{c\tau + d})(c\tau + d)^{-k} = \Psi(\tau, x) \quad \text{for} \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$$

Notice that the Weierstrass $\wp$-function

$$\wp(\tau, x) = \frac{1}{x^2} + \sum_{\omega \in 2\pi i(\mathbb{Z}\tau + \mathbb{Z}), \omega \neq 0} \left(\frac{1}{(x - \omega)^2} - \frac{1}{\omega^2}\right)$$

is a Jacobi form on $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ of weight 2.

$\Psi(\tau, x) = x$, a function only in x, can be viewed as a Jacobi form of weight -1.

Theorem 1.2. Let $\Psi(\tau, x)$ be a Jacobi form on Γ of weight k as above. Let $n \in \mathbb{Z}, \alpha, \beta \in \mathbb{R}$, and let $g_n(\tau)$ be the n-th coefficient in the Laurent expansion of $\Psi(\tau, \cdot)$ at $2\pi i(\alpha\tau + \beta)$. Then we have

$$g_n[\gamma]_{k+n} = g_n \quad \text{for all} \quad \gamma \in \Gamma \quad \text{with} \quad (\alpha, \beta)\gamma \equiv (\alpha, \beta) \mod \mathbb{Z}^2$$

Proof. By the Cauchy integral formula,

$$g_n(\tau) = \frac{1}{2\pi i} \oint \frac{\Psi(\tau, x + 2\pi i(\alpha\tau + \beta))}{x^{n+1}}\, dx$$

Thus by definition,

$$\begin{aligned} g_n[\gamma]_{k+n}(\tau) &= g_n(\gamma(\tau))(c\tau + d)^{-(k+n)} \\ &= \frac{1}{2\pi i} \oint \frac{\Psi(\gamma(\tau), (x(c\tau + d) + 2\pi i(\alpha'\tau + \beta'))/(c\tau + d))}{(x(c\tau + d))^{n+1}} (c\tau + d)^{-k}\, d(x(c\tau + d)) \\ &= \frac{1}{2\pi i} \oint \frac{\Psi(\tau, y + 2\pi i(\alpha'\tau + \beta'))}{y^{n+1}}\, dy \end{aligned}$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $(\alpha', \beta') = (\alpha, \beta)\gamma$. Notice that $y = x(c\tau + d)$ preserves the orientation of the closed loop when changing variable.

$\square$

Corollary 1.3. Define $e_1(\tau) = \wp(\tau, \pi i)$, $e_2(\tau) = \wp(\tau, \pi i\tau)$, $e_3(\tau) = \wp(\tau, \pi i(\tau + 1))$, the value of Weierstrass $\wp$-function at the 2-division points $\pi i, \pi i\tau, \pi i(\tau + 1)$. Then

$$e_1 \in M_2(\Gamma_0(2)), \qquad e_2 \in M_2(\Gamma^0(2)), \qquad \text{and} \quad e_3 \in M_2(\left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)\Gamma_0(2)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right))$$

Moreover, for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, $[\gamma]_2$ permutes e_1, e_2, e_3 (This follows directly from the fact that $\wp$ is a Jacobi form of weight 2.). In particular,

$$e_1[\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)]_2 = e_2,\ e_2[\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)]_2 = e_1,\ e_3[\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)]_2 = e_3$$

Proof. $e_1(\tau)$ is the 0-th coefficient of $\wp(\tau, \cdot)$ at $2\pi i(0\tau + \frac{1}{2})$, so by 1.2 it is modular of weight 2 with respect to

$$\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : (0, \frac{1}{2})\gamma \equiv (0, \frac{1}{2}) \mod \mathbb{Z}^2\}$$

which is exactly $\Gamma_0(2)$. Similar for $e_2(\tau)$. For e_3 one should check

$$\{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a + c, b + d \quad \text{odd} \quad\} = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)\Gamma_0(2)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$$

This can be done by, for example, two directions of containment using conjugation.
In the proof of 1.2, we see that since $(0, \frac{1}{2})\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = (\frac{1}{2}, 0)$, we have $e_1[\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)]_2 = e_2$. Similar for e_2, e_3. ☐

Theorem 1.4. We have

$$\wp(\tau, x) = \sum_{n\in\mathbb{Z}} \frac{1}{(q^{n/2}e^{x/2} - q^{-n/2}e^{-x/2})^2} - (-\frac{1}{12} + \sum_{n\in Z, n\neq 0} \frac{1}{(q^{n/2} - q^{-n/2})^2})$$

where $q = e^{2\pi i\tau}$.

Proof. For a fixed $\tau \in \mathbb{H}$, notice that $|q| < 1$, so the first summand converges absolutely and uniformly in x at least on bounded subset, but it is also periodic in x with respect to $2\pi i(\mathbb{Z}\tau + \mathbb{Z})$. It has pole of order 2 exactly at those x so that $e^{2\pi i n\tau + x} = 1$ for some $n \in \mathbb{Z}$, which are the lattice points. Moreover the local behavior is $\frac{1}{x^2}$ Thus it differs by $\wp(\tau, \cdot)$ up to addition of a constant. To determine the constant notice that $\wp(x) = \frac{1}{x^2} + O(x)$, and for $x \to 0$ we have

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \frac{1}{(q^{n/2}e^{x/2} - q^{-n/2}e^{-x/2})^2} &= \frac{1}{(e^{x/2} - e^{-x/2})^2} + \sum_{n\neq 0} \frac{1}{(q^{n/2} - q^{-n/2})^2} + O(x) \\ &= \frac{1}{x^2} - \frac{1}{12} + \sum_{n\neq 0} \frac{1}{(q^{n/2} - q^{-n/2})^2} + O(x) \end{aligned}$$

where we computed $(e^{x/2} - e^{-x/2})^2 = (x + \frac{x^3}{24} + O(x^4))^2 = x^2 + \frac{1}{12}x^4 + O(x^6)$. ☐

Corollary 1.5. For $|q| < \min\{|e^x|, |e^{-x}|\}$,

$$\wp(\tau, x) = \frac{1}{(e^{x/2} - e^{-x/2})^2} + \sum_{n=1}^{\infty}(\sum_{d|n} d(e^{dx} + e^{-dx}))q^n + \frac{1}{12}(1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n)$$

where $\sigma_r(n) = \sum_{d|n} d^r$.

Proof. Apply

$$\frac{1}{(a^{1/2}-a^{-1/2})^2} = \frac{a}{(1-a)^2} = \sum_{k=1}^{\infty} k \cdot a^k$$

for $|a| < 1$ to 1.4, we get

$$\begin{aligned}\wp(\tau, x) &= \frac{1}{(e^{x/2}-e^{-x/2})^2} + \sum_{n\neq 0} \frac{1}{(q^{n/2}e^{x/2}-q^{-n/2}e^{-x/2})^2} + \frac{1}{12} - \sum_{n\neq 0} \frac{1}{(q^{n/2}-q^{-n/2})^2} \\ &= \frac{1}{(e^{x/2}-e^{-x/2})^2} + \sum_{n>0}\sum_{d=1}^{\infty} d(e^{dx}+e^{-dx})q^{dn} + \frac{1}{12} - 2\sum_{n>0}\sum_{d=1}^{\infty} dq^{dn} \\ &= \frac{1}{(e^{x/2}-e^{-x/2})^2} + \sum_{n=1}^{\infty}(\sum_{d|n} d(e^{dx}+e^{-dx}))q^n + \frac{1}{12}(1-24\sum_{n=1}^{\infty}\sigma_1(n)q^n)\end{aligned}$$

□

Corollary 1.6. The functions $e_1(\tau) = \wp(\tau, \pi i), e_2(\tau) = \wp(\tau, \pi i\tau), e_3(\tau) = \wp(\tau, \pi i(\tau+1))$ defined in 1.3 are modular form of weight 2 (with respect to different groups), and we have the Fourier expansion

$$e_1(\tau) = -\frac{1}{6}(1+24\sum_{n=1}^{\infty}\sigma_1^{\mathrm{odd}}(n)q^n) \in M_2(\Gamma_0(2))$$

$$e_2(\tau) = \frac{1}{12}(1+24\cdot\sum_{n=1}^{\infty}\sigma_1^{\mathrm{odd}}(n)q^{n/2}) \in M_2(\Gamma^0(2))$$

$$e_3(\tau) = \frac{1}{12}(1+24\cdot\sum_{n=1}^{\infty}(-1)^n\sigma_1^{\mathrm{odd}}(n)q^{n/2}) \in M_2(\left(\begin{smallmatrix}1&1\\-1&0\end{smallmatrix}\right)\Gamma_0(2)\left(\begin{smallmatrix}0&-1\\1&1\end{smallmatrix}\right))$$

where $\sigma_1^{\mathrm{odd}}(n) = \sum_{d|n, d \text{ odd}} d$. In particular we see that $e_2(2\tau) = \frac{-1}{2}e_1(\tau)$.

Proof.

- Simply put $x = \pi i$ in 1.5 and $e^{d\pi i} + e^{-d\pi i} = 2(-1)^d$ we get the formula for $e_1(\tau)$.
- Note that $e^{d\pi i\tau} - e^{-d\pi i\tau} = q^{d/2} - q^{-d/2}$. So

$$\begin{aligned}e_2(\tau) &= \frac{1}{12} + \sum_{n=1}^{\infty} nq^{n/2} + \sum_{n=1}^{\infty}\sum_{d|n}(dq^{d(1+2n/d)/2} + dq^{d(-1+2n/d)/2}) - 2\sum_{n=1}^{\infty}\sigma_1(n)q^n \\ &= \frac{1}{12} + \sum_{m=1}^{\infty} mq^{m/2} + \sum_{m=1}^{\infty}(\sum_{1\neq m/d \text{ odd}} dq^{m/2} + \sum_{m/d \text{ odd}} dq^{m/2}) - 2\sum_{n=1}^{\infty}\sigma_1(n)q^n \\ &= \frac{1}{12} + \sum_{m=1}^{\infty}(2\sum_{m/d \text{ odd}} d)q^{m/2} - 2\sum_{n=1}^{\infty}\sigma_1(n)q^n\end{aligned}$$

  The coefficient of $q^{m/2}$, m odd, is $2\sum_{m/d \text{ odd}} d = 2\sigma_1(m) = 2\sigma_1^{\mathrm{odd}}(m)$.

The coefficient of $q^{m/2}$, m even, is $2\sum_{m/d \text{ odd}} d - 2\sigma_1(m/2)$. Now write $m/2 = 2^l \cdot a$, with a odd. Then

$$2\sum_{m/d \text{ odd}} d = 2^{l+2}\sum_{a/d \text{ odd}} d = 2^{l+2}\sigma_1^{\text{odd}}(a)$$

Also, we have the formula $\sigma_1(m/2) = (2^{l+1}-1)\sigma_1^{\text{odd}}(a)$, which can be easily obtained by induction on l. To sum up, the coefficient is

$$2^{l+2}\sigma_1^{\text{odd}}(a) - 2(2^{l+1}-1)\sigma_1^{\text{odd}}(a) = 2\sigma_1^{\text{odd}}(a) = 2\sigma_1^{\text{odd}}(m)$$

- Finally,

$$e_3(\tau) = \frac{1}{12} + \sum_{n=1}^{\infty}(-1)^n nq^{n/2} + \sum_{n=1}^{\infty}\sum_{d|n}((-1)^d dq^{d(1+2n/d)/2} + (-1)^d dq^{d(-1+2n/d)/2}) - 2\sum_{n=1}^{\infty}(-1)^n\sigma_1(n)q^n$$

$$= \frac{1}{12} + \sum_{m=1}^{\infty}(-1)^m mq^{m/2} + \sum_{m=1}^{\infty}(\sum_{1\neq m/d \text{ odd}}(-1)^d dq^{m/2} + \sum_{m/d \text{ odd}}(-1)^d dq^{m/2}) - 2\sum_{n=1}^{\infty}(-1)^n\sigma_1(n)q^n$$

$$= \frac{1}{12} + \sum_{m=1}^{\infty}(2\sum_{m/d \text{ odd}}(-1)^d d)q^{m/2} - 2\sum_{n=1}^{\infty}(-1)^n\sigma_1(n)q^n$$

The coefficient of $q^{m/2}$, m odd, is $2\sum_{m/d \text{ odd}}(-1)^d d = -2\sigma_1(m) = 2(-1)^m\sigma_1^{\text{odd}}(m)$.

The coefficient of $q^{m/2}$, m even, is $2\sum_{m/d \text{ odd}}(-1)^d d - 2(-1)^m\sigma_1(m/2)$. Now write $m/2 = 2^l \cdot a$, with a odd. Then

$$2\sum_{m/d \text{ odd}}(-1)^d d = 2^{l+2}\sum_{a/d \text{ odd}} d = 2^{l+2}\sigma_1^{\text{odd}}(a)$$

Also, we have the formula $\sigma_1(m/2) = (2^{l+1}-1)\sigma_1^{\text{odd}}(a)$, which can be easily obtained by induction on l. To sum up, the coefficient is

$$2^{l+2}\sigma_1^{\text{odd}}(a) - 2(2^{l+1}-1)\sigma_1^{\text{odd}}(a) = 2\sigma_1^{\text{odd}}(a) = 2(-1)^m\sigma_1^{\text{odd}}(m)$$

$\square$

Definition 1.7. Define the $\tilde{}$ operator for a function $f(\tau,x)$ by $\tilde{f}(\tau,x) = f(\frac{-1}{2\tau},\frac{x}{2\tau})(2\tau)^{-k}$ for an appropriate weight k for f. If $f(\tau) \in M_k(\Gamma_0(2))$, a function only in τ, we give it weight k, the weight as a modular form, then

$$\tilde{f}(\tau) = f(\frac{-1}{2\tau})(2\tau)^{-k} = f[\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)]_k(2\tau)$$

Recall that $\Gamma_0(2)$ has two cusp, ∞ and 0. Thus the operator on f is simply computing the expansion at 0, and composing with $\tau \mapsto 2\tau$ since 0 has width 2. One can easily verify that $f[\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)]_k(\tau) \in M_k(\Gamma^0(2))$ and $\tilde{f}(\tau) = f[\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)]_k(2\tau) \in M_k(\Gamma_0(2))$, and in fact $\tilde{}$ is a graded algebra automorphism of $M_*(\Gamma_0(2))$.

Corollary 1.8. We can define two modular forms $\delta \in M_2(\Gamma_0(2))$ and $\epsilon \in M_4(\Gamma_0(2))$ and compute their Fourier coefficients:

$$\delta = -\frac{3}{2}e_1 = \frac{1}{4} + 6\sum_{n=1}^{\infty}\sigma_1^{\mathrm{odd}}(n)q^n = \frac{1}{4} + 6q + \cdots$$

$$\begin{aligned}
\epsilon &= (e_1 - e_2)(e_1 - e_3) \\
&= (\frac{-1}{4} - 4\sum_{n=1}^{\infty}\sigma_1^{\mathrm{odd}}(n)q^n - 2\sum_{n=1}^{\infty}\sigma_1^{\mathrm{odd}}(n)q^{n/2})(\frac{-1}{4} - 4\sum_{n=1}^{\infty}\sigma_1^{\mathrm{odd}}(n)q^n - 2\sum_{n=1}^{\infty}(-1)^n\sigma_1^{\mathrm{odd}}(n)q^{n/2}) \\
&= (\frac{-1}{4} - 2q^{1/2} - 6q + \cdots)(\frac{-1}{4} + 2q^{1/2} - 6q + \cdots) \\
&= \frac{1}{16} - q + \cdots
\end{aligned}$$

Notice that for $\gamma \in \Gamma_0(2)$, $[\gamma]_2$ fixes e_1 and either fixes e_2, e_3 or interchanges e_2 with e_3 (1.3), thus indeed $\epsilon \in M_4(\Gamma_0(2))$.

Next, apply the $\tilde{\cdot}$ operator we get $\tilde{\delta} \in M_2(\Gamma_0(2))$ and $\tilde{\epsilon} \in M_2(\Gamma_0(2))$:

$$\tilde{\delta}(\tau) = -\frac{3}{2}e_2(2\tau) = \frac{3}{4}e_1(\tau) = \frac{-1}{8} - 3\sum_{n=1}^{\infty}\sigma_1^{\mathrm{odd}}(n)q^n$$

$$\begin{aligned}
\tilde{\epsilon}(\tau) &= (e_2(2\tau) - e_1(2\tau))(e_2(2\tau) - e_3(2\tau)) \\
&= (\frac{1}{4} + 2\sum_{n=1}^{\infty}\sigma_1^{\mathrm{odd}}(n)q^n + 4\sum_{n=1}^{\infty}\sigma_1^{\mathrm{odd}}(n)q^{2n}) \cdot 2(\sum_{n=1}^{\infty}(1 - (-1)^n)\sigma_1^{\mathrm{odd}}(n)q^n) \\
&= (1 + 8\sum_{n=1}^{\infty}\sigma_1^{\mathrm{odd}}(n)q^n + 16\sum_{n=1}^{\infty}\sigma_1^{\mathrm{odd}}(n)q^{2n}) \cdot (\sum_{n=1}^{\infty}\frac{(1 - (-1)^n)}{2}\sigma_1^{\mathrm{odd}}(n)q^n) \\
&= q + \cdots
\end{aligned}$$

From the above expression, an important observation is that 4δ, 16ϵ, $8\tilde{\delta}$ and $\tilde{\epsilon}$ has an integral q-expansion, which is the key in our proof of theorem of Ochanine.

Theorem 1.9.

$$M_*(\Gamma_0(2)) = \mathbb{C}[\delta, \epsilon]$$

Proof. Notice that $\Gamma_0(2) = \Gamma_1(2)$. $-I \in \Gamma_0(2)$ so there is no odd weight. By the table in 3.9 of [Diam], we have for k even,

$$\dim M_k(\Gamma_0(2)) = \left\lfloor \frac{k}{4} \right\rfloor + 1$$

Now since δ is of weight 2 and ϵ of weight 4, in $M_{2n}(\Gamma_0(2))$ we have $\delta^{2n}, \delta^{2n-2}\epsilon, \ldots, \epsilon^{n/2}$ (or $\delta\epsilon^{(n-1)/2}$ if n odd). Since the number matches $\left\lfloor \frac{2n}{4} \right\rfloor + 1$, we only need to prove they are linearly independent. Suppose there is such relation, dividing by δ^{2n} we know that ϵ/δ^2 satisfies a polynomial of constant coefficient everywhere, hence a constant. But from the Fourier coefficient expanded at ∞ (1.8),

$$\epsilon = \frac{1}{16} - q + \cdots$$

$$\delta^2 = \frac{1}{16} + 3q + \cdots$$

We see that ϵ and δ^2 are linearly independent over $\mathbb{C}$, a contradiction. $\square$

Remark 1.10. In the above proof, we see that if n is odd, every modular form in $M_{2n}(\Gamma_0(2))$ can be divided by δ. In the proof of Ochanine's theorem, we essentially will use this observation from the condition $\dim X \equiv 4\ (8)$.

1.2 Theta function

Definition 1.11. Let $L \subset \mathbb{C}$ be a lattice. A meromorphic function f is called a **theta function** for L if $\dfrac{d^2}{dx^2}\log f(x)$ is elliptic with respect to L. The theta functions $f(x) = e^{ax^2+bx+c}$ for $a, b, c \in \mathbb{C}$ are called **trivial theta functions**.

Definition 1.12. The Weierstrass σ-function for the lattice $2\pi i(\mathbb{Z}\tau + \mathbb{Z})$ is defined by the product

$$\sigma(\tau, x) = x \prod_{\omega \in 2\pi i(\mathbb{Z}\tau+\mathbb{Z}),\omega\neq 0} (1 - \frac{x}{\omega}) \cdot \exp(\frac{x}{\omega} + \frac{1}{2}(\frac{x}{\omega}))$$

It is clear that $\dfrac{d^2}{dx^2}\log(\sigma) = -\wp(\tau, x)$. Hence σ is a theta function. Also observe easily that

$$\sigma(\gamma(\tau), \frac{x}{c\tau+d}) \cdot (c\tau + d) = \sigma(\tau, x)$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, so it is a Jacobi form of weight -1.

Theorem 1.13. We have a product expansion

$$\sigma(\tau, x) = \exp(\frac{1}{2}G_2(\tau) \cdot x^2) \cdot (e^{x/2} - e^{-x/2}) \cdot \prod_{n=1}^{\infty} \frac{(1 - q^n e^x)(1 - q^n e^{-x})}{(1 - q^n)^2}$$

where $G_2(\tau) = \displaystyle\sum_c \sum_d \frac{1}{(2\pi i(c\tau + d))^2}$, and the sum omits $(c, d) = (0, 0)$.

Proof. Apply $\dfrac{d^2}{dx^2}\log$ to right-hand side we get

$$\begin{aligned}
&G_2(\tau) + \frac{d}{dx}(\frac{1}{2}\coth(\frac{x}{2}) + \sum_{n=1}^{\infty}(\frac{-q^n e^x}{1 - q^n e^x} + \frac{q^n e^{-x}}{1 - q^n e^{-x}}))\\
&= G_2(\tau) - \frac{1}{4\sinh(\frac{x}{2})^2} + \frac{d}{dx}(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(e^{-kx} - e^{kx}) \cdot q^{nk})\\
&= \frac{-1}{12} + 2\sum_{n=1}^{\infty}\sigma_1(n)q^n - \frac{1}{(e^{x/2} - e^{-x/2})^2} - \sum_{n=1}^{\infty}\sum_{d|n} d(e^{dx} + e^{-dx})q^n\\
&= -\wp(\tau, x)
\end{aligned}$$

We have used the expansion $G_2(\tau) = \dfrac{-1}{12} + 2\displaystyle\sum_{n=1}^{\infty}\sigma_1(n)q^n$ and 1.5.

We then know that σ differs from right-hand side up to a multiplication of $e^{b(\tau)x+c(\tau)}$. Both sides begins with $x + O(x^2)$ as $x \to 0$, so $c(\tau) = 0$. Also both sides are odd, so $b(\tau) = 0$. Hence they are equal. $\square$

Definition 1.14. Define the Φ-function

$$\Phi(\tau, x) = \sigma(\tau, x) \cdot \exp(-\frac{1}{2} G_2(\tau) \cdot x^2)$$

Then it is also a theta function since $\sigma(\tau, x)$ is. We then from 1.13 obtain a product formula

$$\begin{aligned}\Phi(\tau, x) &= (e^{x/2} - e^{-x/2}) \cdot \prod_{n=1}^{\infty} \frac{(1-q^n e^x)(1-q^n e^{-x})}{(1-q^n)^2} \\ &= 2\sinh(\frac{x}{2}) \cdot \prod_{n=1}^{\infty} \frac{(1-q^n e^x)(1-q^n e^{-x})}{(1-q^n)^2}\end{aligned}$$

We are going to investigate some properties of Φ, which is important for elliptic genera.

Theorem 1.15.

(a)

$$\Phi(\gamma(\tau), \frac{x}{c\tau + d}) \cdot (c\tau + d) = \exp(\frac{cx^2}{4\pi i(c\tau + d)}) \cdot \Phi(\tau, x)$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

(b)

$$\Phi(\tau, x + 2\pi i \cdot (\lambda\tau + \mu)) = q^{-\frac{\lambda^2}{2}} e^{-\lambda x} (-1)^{\lambda+\mu} \cdot \Phi(\tau, x)$$

for $\lambda, \mu \in \mathbb{Z}$.

Proof.

(a) Since $\sigma(\tau, x)$ is a Jacobi form, the extra factor comes from the term $\exp(-\frac{1}{2} G_2(\tau) \cdot x^2)$ and the formula $G_2(\tau)[\gamma]_2(\tau) = G_2(\tau) - \frac{c}{2\pi i(c\tau + d)}$.

(b) We only need to prove for $\lambda > 0$, since changing left hand side to right hand side we get the formula for $(-\lambda, -\mu)$.

$$\begin{aligned}\Phi(\tau, x + 2\pi i \cdot (\lambda\tau + \mu)) &= (q^{\lambda/2}(-1)^{\mu} e^{x/2} - q^{-\lambda/2}(-1)^{\mu} e^{-x/2}) \prod_{n=1}^{\infty} \frac{(1-q^{n+\lambda} e^x)(1-q^{n-\lambda} e^{-x})}{(1-q^n)^2} \\ &= (-1)^{\mu} \cdot (-1) q^{-\lambda/2} e^{-x/2} (1 - q^{\lambda} e^x) \cdot \prod_{n=\lambda+1}^{\infty} (1 - q^n e^x) \\ &\cdot \prod_{n=1-\lambda}^{0} (-q^n e^{-x})(1 - q^{-n} e^x) \cdot \prod_{n=1}^{\infty} (1 - q^n e^{-x}) \cdot \prod_{n=1}^{\infty} (1-q^n)^{-2} \\ &= (-1)^{\mu} \cdot (-1)^{\lambda+1} q^{-\lambda/2 - \lambda(\lambda-1)/2} e^{-\lambda x} e^{-x/2} (1 - e^x) \prod_{n=1}^{\infty} \frac{(1-q^n e^x)(1-q^n e^{-x})}{(1-q^n)^2} \\ &= q^{-\frac{\lambda^2}{2}} e^{-\lambda x} (-1)^{\lambda+\mu} \cdot \Phi(\tau, x)\end{aligned}$$

□

Theorem 1.16. For a lattice L and $E = \mathbb{C}/L$, we define Θ_L to be theta functions for L and $\tilde{\Theta}_L$ the trivial ones. Then, the assignment $f \mapsto \mathrm{div}(f)$ induces an exact sequence:

$$1 \to \tilde{\Theta}_L \to \Theta_L \to \mathrm{Div}(E) \to 1$$

Proof. If $f \in \Theta_L$ has neither zeros nor poles, then $\frac{d}{dx}\log(f) = \frac{f'}{f}$ is globally defined, thus $\frac{d^2}{dx^2}\log(f)$ is a constant, say $2a$, we get $f = e^{ax^2+bx+c}$ for some $a, b, c \in \mathbb{C}$.
It is clear that the Weierstrass σ-function has divisor (0) and is a theta function, so considering the products of $\sigma(x - x_i)^{n_i}$ ensures the surjectivity. $\square$

2 Elliptic genera

Recall the fact that on the elliptic curve $E = \mathbb{C}/L$, a divisor $D = \sum n_P \cdot (P)$ is principal if and only if $\sum n_P = 0$ and $\sum n_P \cdot P \equiv 0\ (L)$.

2.1 Elliptic genera of level 2

Theorem 2.1.

(a) For $\tau \in \mathbb{H}$, let $\varphi_1(\tau, x)$ be the elliptic function in x for the lattice $2\pi i(\mathbb{Z}\cdot 2\tau + \mathbb{Z})$ with the divisor $(\pi i) + (\pi i(1 + 2\tau)) - (\pi i \cdot 2\tau) - (0)$ and the normalization $\varphi_1(\tau, x) = \frac{1}{x} + O(1)$ as $x \to 0$. Then we have

$$\varphi_1(\gamma(\tau), \frac{x}{c\tau + d})(c\tau + d)^{-1} = \varphi_1(\tau, x) \quad \text{for all} \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$$

$$\varphi_1(\tau, x)^2 = \wp(\tau, x) - e_1(\tau)$$

$$\varphi_1(\tau, x) = \frac{1}{2}\frac{e^{x/2} + e^{-x/2}}{e^{x/2} - e^{-x/2}} \cdot \prod_{n=1}^{\infty} \frac{(1 + q^n e^x)(1 + q^n e^{-x})/(1 + q^n)^2}{(1 - q^n e^x)(1 - q^n e^{-x})/(1 - q^n)^2}$$

(b) For another 2-division point, let $\varphi_2(\tau, x)$ be the elliptic function in x for the lattice $2\pi i(\mathbb{Z}\tau + 2\mathbb{Z})$ with the divisor $(\pi i\tau) + (\pi i(2 + \tau)) - (2\pi i) - (0)$ and the normalization $\varphi_2(\tau, x) = \frac{1}{x} + O(1)$ as $x \to 0$. Then

$$\varphi_2(\gamma(\tau), \frac{x}{c\tau + d})(c\tau + d)^{-1} = \varphi_2(\tau, x) \quad \text{for all} \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma^0(2)$$

$$\varphi_2(\tau, x)^2 = \wp(\tau, x) - e_2(\tau)$$

$$\varphi_2(\tau, x) = \frac{1}{e^{x/2} - e^{-x/2}} \frac{\prod_{n=2m+1}(1 - q^{n/2}e^x)(1 - q^{n/2}e^{-x})/(1 - q^{n/2})^2}{\prod_{n=2m+2}(1 - q^{n/2}e^x)(1 - q^{n/2}e^{-x})/(1 - q^{n/2})^2}$$

where the product is over $m \geq 0$.

(c)

$$\varphi_1(\frac{-1}{\tau}, \frac{x}{\tau})\tau^{-1} = \varphi_2(\tau, x)$$

Thus $\tilde{\varphi}_1(\tau, x) = \varphi_1(\frac{-1}{2\tau}, \frac{x}{2\tau})(2\tau)^{-1} = \varphi_2(2\tau, x)$, here we give φ_1, φ_2 weight 1 with respect to the $\tilde{}$ operator (1.7).

Proof.

(a)

- For Jacobi form, for a fixed $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$ define

$$\psi_1(\tau, x) = \varphi_1(\gamma(\tau), \frac{x}{c\tau + d}) \cdot (c\tau + d)^{-1} = \frac{1}{x} + O(1)$$

  For $\lambda, \mu \in \mathbb{Z}$ compute

$$\psi_1(\tau, x + 2\pi i(2\lambda\tau + \mu)) = \varphi_1(\gamma(\tau), \frac{x + 2\pi i(2\lambda\tau + \mu)}{c\tau + d}) \cdot (c\tau + d)^{-1}$$

$$= \varphi_1(\gamma(\tau), \frac{x}{c\tau + d} + 2\pi i(\lambda' \cdot 2\gamma(\tau) + \mu')) \cdot (c\tau + d)^{-1}$$

  where (λ', μ') satisfies $\left(\begin{smallmatrix} \lambda' & \mu' \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2a & 2b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2\lambda & \mu \\ c & d \end{smallmatrix}\right)$, that is, $(\lambda' \ \mu')\left(\begin{smallmatrix} a & 2b \\ c/2 & d \end{smallmatrix}\right) = (\lambda \ \mu)$. So $\lambda', \mu' \in \mathbb{Z}$ and ψ is also doubly periodic. Also using this on the pair $(0\ 0), (\frac{1}{2}\ 0), (0\ \frac{1}{2}), (\frac{1}{2}\ \frac{1}{2})$, one can check that ψ_1 has the same zeros and poles as φ_1, thus they are equal.

- Second, we prove $\varphi_1(\tau, x + 2\pi i\tau) = -\varphi_1(\tau, x)$. Since $\varphi_1(\tau, x + 2\pi i\tau) + \varphi_1(\tau, x)$ is elliptic with respect to $2\pi i(\mathbb{Z}\tau + \mathbb{Z})$ and has at most one pole of order one on this lattice, we know that it should be a constant. Take $x = \pi i$ we find that the terms vanish and the constant is 0. So now $\varphi_1^2(\tau, x)$ is elliptic with respect to $2\pi i(\mathbb{Z}\tau + \mathbb{Z})$ and has the same poles of order two as $\wp(\tau, x)$, so $\varphi_1^2(\tau, x) = \wp(\tau, x) + C_1(\tau)$. Again set $x = \pi i$ we see $C_1(\tau) = -e_1(\tau)$.

- Call the right hand side $p_1(x)$. It is meromorphic since $|q| < 1$. Denominator gives poles at $x \in 2\pi i\mathbb{Z}, 2\pi in\tau \pm x \in 2\pi i\mathbb{Z}$, $n \in \mathbb{N}$, which are $2\pi i(\mathbb{Z}\tau + \mathbb{Z})$. Numerator gives zeros at $x \in \pi i + 2\pi i\mathbb{Z}, 2\pi in\tau \pm x \in \pi i + 2\pi i\mathbb{Z}$, $n \in \mathbb{N}$, which are $\pi i + 2\pi i(\mathbb{Z}\tau + \mathbb{Z})$. So the divisors of φ_1 and p_1 coincides. It is clear that $p_1(x + 2\pi i) = p_1(x)$, and

$$p_1(x + 2\pi i\tau) = \frac{1}{2}\frac{qe^x + 1}{qe^x - 1} \cdot \prod_{n=1}^{\infty} \frac{(1 + q^{n+1}e^x)(1 + q^{n-1}e^{-x})/(1 + q^n)^2}{(1 - q^{n+1}e^x)(1 - q^{n-1}e^{-x})/(1 - q^n)^2}$$

$$= -\frac{1}{2}\frac{1 + e^{-x}}{1 - e^{-x}} \cdot \prod_{n=1}^{\infty} \frac{(1 + q^n e^x)(1 + q^n e^{-x})/(1 + q^n)^2}{(1 - q^n e^x)(1 - q^n e^{-x})/(1 - q^n)^2} = -p_1(x)$$

  Finally we see that as $x \to 0$ the product cancel out and the behavior is $\frac{1}{x} + O(1)$. Thus $p_1 = \varphi_1$.

(b)

- First two equations are similar to (a).

- Call the right hand side $p_2(x)$. It is meromorphic since $|q| < 1$. Denominator gives poles at $x \in 2\pi i\mathbb{Z}, \pi in\tau \pm x \in 2\pi i\mathbb{Z}$, $n \in 2\mathbb{N}$, which are $2\pi i(\mathbb{Z}\tau + \mathbb{Z})$. Numerator gives zeros at $\pi in\tau \pm x \in \pi i + 2\pi i\mathbb{Z}$, $n \in 2\mathbb{N} - 1$, which are $\pi i\tau + 2\pi i(\mathbb{Z}\tau + \mathbb{Z})$. So the divisors of φ_2 and p_2 coincides. It is clear that $p_2(x + 2\pi i) = -p_2(x)$, and

$$p_2(x + 2\pi i\tau) = \frac{1}{q^{1/2}e^{x/2} - q^{-1/2}e^{-x/2}} \frac{\prod_{n=2m+1}(1 - q^{(n+2)/2}e^x)(1 - q^{(n-2)/2}e^{-x})/(1 - q^{n/2})^2}{\prod_{n=2m+2}(1 - q^{(n+2)/2}e^x)(1 - q^{(n-2)/2}e^{-x})/(1 - q^{n/2})^2}$$

$$= \frac{1}{q^{1/2}e^{x/2} - q^{-1/2}e^{-x/2}} \frac{(1 - qe^x)(1 - q^{-1/2}e^{-x})}{(1 - q^{1/2}e^x)(1 - e^{-x})} \frac{\prod_{n=2m+1}(1 - q^{n/2}e^x)(1 - q^{n/2}e^{-x})/(1 - q^{n/2})^2}{\prod_{n=2m+2}(1 - q^{n/2}e^x)(1 - q^{n/2}e^{-x})/(1 - q^{n/2})^2}$$

$$= \frac{1}{e^{x/2}-e^{-x/2}} \frac{\prod_{n=2m+1}(1-q^{n/2}e^{x})(1-q^{n/2}e^{-x})/(1-q^{n/2})^2}{\prod_{n=2m+2}(1-q^{n/2}e^{x})(1-q^{n/2}e^{-x})/(1-q^{n/2})^2} = p_2(x)$$

Finally we see that as $x \to 0$ the product cancel out and the behavior is $\frac{1}{x} + O(1)$. Thus $p_2 = \varphi_2$.

(c) We have (1.3)

$$\varphi_1(\frac{-1}{\tau}, \frac{x}{\tau})^2\tau^{-2} = \wp(\frac{-1}{\tau}, \frac{x}{\tau})\tau^{-2} - e_1[\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)]_2(\tau) = \wp(\tau, x) - e_2(\tau) = \varphi_2(\tau, x)^2$$

So $\varphi_1(\frac{-1}{\tau}, \frac{x}{\tau})\tau^{-1} = \pm\varphi_2(\tau, x)$, but $\varphi_1(\frac{-1}{\tau}, \frac{x}{\tau})\tau^{-1} = \frac{1}{x} + O(1)$ as $x \to 0$, so they are equal.

□

Definition 2.2. For each $\tau \in \mathbb{H}$, take $f(x) = f(\tau, x) = (\wp(\tau, x) - e_1(\tau))^{-1/2}$ with the choice of normalization $(\wp(\tau, x) - e_1(\tau))^{1/2} = \frac{1}{x} + O(1)$. Then f is an odd function and $Q(x) = \frac{x}{f(x)}$ defines a genus for compact oriented differentiable manifold of dimension $4k$:

$$\varphi(X) = (\prod_{i=1}^{2k} \frac{x_i}{f(x_i)})[X]$$

where $p(TX) = (1 + x_1^2) \cdots (1 + x_{2k}^2)$. Notice that $\varphi(X) = \varphi(X)(\tau)$ can be written as a function of τ. Notice that the choice of normalization says that $(\wp(\tau, x) - e_1(\tau))^{1/2} = \varphi_1(\tau, x)$, the function in 2.1. Thus we can also write the genus as

$$\varphi(X)(\tau) = (\prod_{i=1}^{2k} x_i\varphi_1(\tau, x_i))[X]$$

Remark 2.3. In [Hirz] 1.7, we say that a genus is an elliptic genus (of level 2) if it is defined by a power series $Q(x) = \frac{x}{f(x)}$ with f satisfying a differential equation

$$f'^2 = 1 - 2\delta \cdot f^2 + \epsilon \cdot f^4$$

for some constant δ, ϵ.

In fact, the genera defined in 2.2 are elliptic genera, and for each $\tau \in \mathbb{H}$, the corresponding $f(\tau, x)$ satisfies the differential equation with $\delta(\tau), \epsilon(\tau)$ defined in 1.8.

Corollary 2.4. For a compact oriented differentiable manifold X of dimension $4k$, the elliptic genus $\varphi(X)(\tau)$ is a modular form of weight $2k$ on $\Gamma_0(2)$.

Proof. Notice that the elliptic genus is defined by the even power series $x\varphi_1(\tau, x)$. (We can see that this is even by action of $-I$ to obtain $\varphi_1(\tau, -x) = -\varphi_1(\tau, x)$.) That is, formal factorize $p(X) = p(TX) = \prod_{i=1}^{2k}(1+x_i^2)$ and we can write

$$\varphi(X)(\tau) = \{\prod_{i=1}^{2k} x_i\varphi_1(\tau, x_i)\}_{4k}[X]$$

where $\{\cdot\}_{4k}$ takes the weight $4k$ part of polynomials in x_i, view each x_i as weight 2 indeterminates. Thus, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$,

$$\varphi(X)[\gamma]_{2k}(\tau) = \{\prod_{i=1}^{2k} x_i\varphi_1(\gamma(\tau), x_i)\}_{4k}(c\tau+d)^{-2k}[X] = \{\prod_{i=1}^{2k} \frac{x_i}{c\tau+d}\varphi_1(\gamma(\tau), \frac{x_i}{c\tau+d})\}_{4k}[X]$$

$$= \{\prod_{i=1}^{2k} x_i\varphi_1(\tau, x_i)\}_{4k}[X] = \varphi(X)(\tau)$$

We have used that $x\varphi_1(\tau, x)$ is a Jacobi form of weight 0 proved in 2.1. It is holomorphic on $\mathbb{H}$ and at ∞ since from the expansion of $\varphi_1(\tau, x_i)$, we see that $\varphi(X)(\tau)$ is a power series in q converging for $|q| < 1$.
$\Gamma_0(2) = \Gamma_1(2)$ has another cusps at 0, we consider

$$\varphi(X)[\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)]_{2k}(\tau) = \{\prod_{i=1}^{2k} \frac{x_i}{\tau}\varphi_1(\frac{-1}{\tau}, \frac{x_i}{\tau})\}_{4k}[X] = \{\prod_{i=1}^{2k} x_i\varphi_2(\tau, x_i)\}_{4k}[X]$$

which is the elliptic genus defined by $x\varphi_2(\tau, x)$, thus has a power series expansion in $q^{1/2}$. We conclude that $\varphi(X)(\tau)$ is a modular form of weight $2k$ on $\Gamma_0(2)$. □

2.2 Elliptic genera of level N for complex manifolds

Similar to the construction of the real cobordism ring, one can define the complex cobordism ring. This time, the objects collected are the stably almost complex manifolds, which means that we can give $TM \oplus \mathbb{R}^k$ a structure of complex vector bundle for some trivial bundle $\mathbb{R}^k$. Then we can again define the cobordant relation and form a cobordism ring. For details and more general definition of cobordism, see [Ston]. Complex genera are again defined simply as ring homomorphisms.
Recall that given an odd power series $f(x) = x + \cdots$, we can define a genus on a compact oriented differentiable (real) manifold X of dimension $4k$ by

$$\varphi(X) = (\prod_{i=1}^{2k} \frac{x_i}{f(x_i)})[X]$$

where $p(TX) = (1 + x_1^2)\cdots(1 + x_{2k}^2)$.
But if the manifold is almost complex of dimension $2d$, we can use Chern class $c(X)$ instead of Pontryagin class. Then our power series $f(x) = x + \cdots$ needs not be odd and we can define a complex genus

$$\varphi(X) = (\prod_{i=1}^{d} \frac{x_i}{f(x_i)})[X]$$

where $c(X) = (1 + x_1)\cdots(1 + x_d)$.
We are now going to define a particular family of genera called complex elliptic genera.

Definition 2.5. Recall (1.14) the Φ-function

$$\Phi(\tau, x) = 2\sinh(\frac{x}{2})\prod_{n=1}^{\infty} \frac{(1 - q^n e^x)(1 - q^n e^{-x})}{(1 - q^n)^2}$$

If the power series $f(x)$ is given by

$$f(x) = e^{kx} \frac{\Phi(\tau, x)\Phi(\tau, -\omega)}{\Phi(\tau, x - \omega)}$$

for some k, τ, ω, we call the resulting complex genus a **complex elliptic genus**.

In elliptic genera of level 2, we use the power series $f(x) = \dfrac{1}{\varphi_1(\tau, x)}$. Also, in Theorem 2.1 we proved that

$$f(x) = 2\frac{e^{x/2} - e^{-x/2}}{e^{x/2} + e^{-x/2}} \cdot \prod_{n=1}^{\infty} \frac{(1 - q^n e^x)(1 - q^n e^{-x})/(1 - q^n)^2}{(1 + q^n e^x)(1 + q^n e^{-x})/(1 + q^n)^2} = \frac{\Phi(\tau, x)\Phi(\tau, -\pi i)}{\Phi(\tau, x - \pi i)}$$

Thus they are also elliptic genera in this sense with $k = 0, \omega = \pi i$.

There is a complex analogue of 2.1, called elliptic genera of level N that can be defined on complex manifolds.

Theorem 2.6. Let $N \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{Z}$, not both divisible by N, that is, $2\pi i \dfrac{\alpha\tau + \beta}{N}$ is a nonzero N-division point on the elliptic curve $\mathbb{C}/2\pi i(\mathbb{Z}\tau + \mathbb{Z})$. Then there is a unique function $f_{\alpha,\beta}(\tau, x)$ with the following properties:

1) $x \mapsto f_{\alpha,\beta}(\tau, x)$ is a theta function on the lattice $2\pi i(\mathbb{Z}\tau + \mathbb{Z})$.

2) The divisor of $f_{\alpha,\beta}(\tau, x)$ is $(0) - (2\pi i \dfrac{(\alpha\tau + \beta)}{N})$.

3) $f_{\alpha,\beta}(\tau, x) = x + O(x^2)$ for $x \to 0$ and for all $\lambda, \mu \in \mathbb{Z}$ there exists an N-th root of unity ν so that

$$f_{\alpha,\beta}(\tau, x + 2\pi i(\lambda\tau + \mu)) = \nu \cdot f_{\alpha,\beta}(\tau, x)$$

i.e. $x \mapsto f^N_{\alpha,\beta}(\tau, x)$ is doubly periodic with respect to $2\pi i(\mathbb{Z}\tau + \mathbb{Z})$.

Proof. From 1.16, we see that there exists a $f_{\alpha,\beta}(\tau, x)$ satisfying $f_{\alpha,\beta}(\tau, x)$, and is unique up to multiplication with a trivial theta function. Now $f^N_{\alpha,\beta}$ is a theta function with divisor $N \cdot ((0) - (2\pi i \dfrac{\alpha\tau + \beta}{N}))$. But $N \cdot (0 - 2\pi i \dfrac{\alpha\tau + \beta}{N}) \in 2\pi i(\mathbb{Z}\tau + \mathbb{Z})$, so there is an elliptic function g with the same divisor. $f^N_{\alpha,\beta}/g$ is then a theta function without zeros and poles, by 1.16 this should be trivial θ. So by multiplying $f_{\alpha,\beta}$ by $\theta^{1/N}$ and constant, we may assume $f^N_{\alpha,\beta}$ is elliptic with respect to $2\pi i(\mathbb{Z}\tau + \mathbb{Z})$ and satisfies $f_{\alpha,\beta}(\tau, x) = x + O(x^2)$ as $x \to 0$. This $f^N_{\alpha,\beta}$ is uniquely determined and $f_{\alpha,\beta}$ is uniquely determined by the normalization condition. □

Theorem 2.7. Let $\omega = 2\pi i \dfrac{\alpha\tau + \beta}{N}$, and $f_{\alpha,\beta}(\tau, x)$ as before.

(a) $f_{\alpha,\beta}(\tau, x) = e^{\alpha x/N} \dfrac{\Phi(\tau, x)\Phi(\tau, -\omega)}{\Phi(\tau, x - \omega)}$.

(b) $f_{\alpha,\beta}(\tau, x + 2\pi i(\lambda\tau + \mu)) = e^{2\pi i(\alpha\mu - \beta\lambda)/N} \cdot f_{\alpha,\beta}(\tau, x)$ for all $\lambda, \mu \in \mathbb{Z}$.

(c) $f_{\alpha,\beta}(\gamma(\tau), \dfrac{x}{c\tau + d})(c\tau + d) = f_{(\alpha,\beta)\gamma}(\tau, x)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. In particular, $f_{0,\beta}(\tau, x)$ is a Jacobi form on $\Gamma_1(N)$ of weight -1.

Proof. To prove (a), we need to show that right-hand side satisfies the conditions in 2.6. From the properties of Φ we immediately see 1) and 2). For 3) we prove (b) for right-hand side and we will conclude (a),(b) at once. By Theorem 1.15 we see the change $x \mapsto x + 2\pi i(\lambda\tau + \mu)$ produces the extra factor

$$e^{2\pi i(\alpha\lambda\tau+\alpha\mu)/N} \cdot \frac{e^{-\lambda^2/2}e^{-\lambda x}(-1)^{\lambda+\mu}}{e^{-\lambda^2/2}e^{-\lambda(x-\omega)}(-1)^{\lambda+\mu}} = e^{2\pi i(\alpha\lambda\tau+\alpha\mu)/N} \cdot e^{-2\pi i(\lambda\alpha\tau+\lambda\beta)/N} = e^{2\pi i(\alpha\mu-\beta\lambda)/N}$$

Finally, for (c) we again claim that left-hand side satisfies conditions 1),2),3) for $(\alpha', \beta') = (\alpha, \beta)\gamma$. For 3), write $(\lambda', \mu')\gamma = (\lambda, \mu)$,

$$\begin{aligned} f_{\alpha,\beta}(\gamma(\tau), \frac{x + 2\pi i(\lambda\tau + \mu)}{c\tau + d}) &= f_{\alpha,\beta}(\gamma(\tau), \frac{x + 2\pi i(\lambda'(a\tau + b) + \mu'(c\tau + d))}{c\tau + d}) \\ &= f_{\alpha,\beta}(\gamma(\tau), \frac{x}{c\tau + d} + 2\pi i(\lambda'\gamma(\tau) + \mu')) \\ &= e^{2\pi i(\alpha\mu'-\beta\lambda')} \cdot f_{\alpha,\beta}(\gamma(\tau), \frac{x}{c\tau + d}) \end{aligned} \tag{1}$$

This also implies 1), and clearly the function is also normalized as $x + O(x^2)$ for $x \to 0$. Simple zero at 0, and simple pole at $\frac{x}{c\tau + d} = 2\pi i\frac{\alpha\gamma(\tau + \beta)}{N}$, $x = 2\pi i\frac{\alpha(a\tau + b) + \beta(c\tau + d)}{N} = 2\pi i\frac{\alpha'\tau + \beta'}{N}$. This checks 2) and finished the proof. □

We call the genera defined by these $f_{\alpha,\beta}$ **complex elliptic genera of level** N.

Corollary 2.8. The complex elliptic genera of level N defined by $f_{\alpha,\beta}$ are modular forms of weight d on Γ, where Γ consists of those $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $(\alpha, \beta)\gamma \equiv (\alpha, \beta)$ (N). For $f_{0,\beta}$ this is $\Gamma_1(N)$.

Proof. Exactly same argument as in level 2. Remember that 2.7 tells us that $f_{\alpha,\beta}(\tau, x)$ is a Jacobi form on Γ of weight -1. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & g \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$,

$$\varphi(X)[\gamma]_d(\tau) = \{\prod_{i=1}^{d} \frac{x_i}{f_{\alpha,\beta}(\gamma(\tau), x_i)}\}_{2d}(c\tau + d)^{-d}[X] = \{\prod_{i=1}^{d} \frac{x_i/(c\tau + d)}{f_{\alpha,\beta}(\gamma(\tau), x_i/(c\tau + d))}\}_{2d}[X]$$

$$= \{\prod_{i=1}^{d} \frac{x_i}{f_{(\alpha,\beta)\gamma}(\tau, x_i)}\}_{2d}[X] = \varphi(X)(\tau)$$

So if in particular $\gamma \in \Gamma$, this shows that $\varphi(X)(\tau)$ is invariant under $[\gamma]_d$ for $\gamma \in \Gamma$. In general $[\gamma]_d$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ transforms it to other elliptic genera of level N defined by $f_{(\alpha,\beta)\gamma}$. Thus it remains to show that all elliptic genera of level N defined by $f_{\alpha,\beta}$ is holomophic on $\mathbb{H}$ and at ∞. We may assume $0 \le \alpha, \beta < N$. Recall that we have the product expansion ($\omega = 2\pi i\frac{(\alpha\tau + \beta)}{N}$)

$$\begin{aligned} f_{\alpha,\beta}(\tau, x) &= e^{\alpha x/N} \cdot \frac{\Phi(\tau, x)\Phi(\tau, -\omega)}{\Phi(\tau, x - \omega)} \\ &= e^{\alpha x/N} \frac{(e^{x/2} - e^{-x/2})(1 - e^{\omega})}{(e^{x/2} - e^{-x/2+\omega})} \cdot \prod_{n=1}^{\infty} \frac{(1 - q^n e^x)(1 - q^n e^{-x})(1 - q^n e^{\omega})(1 - q^n e^{-\omega})}{(1 - q^n e^{x-\omega})(1 - q^n e^{-x+\omega})(1 - q^n)^2} \end{aligned}$$

From our assumption $0 \le \alpha < N$ and all $n \ge 1$ we see that the elliptic genera has an expansion in $q^{1/N}$, converging for $|q| < 1$. □

3 More genera and twisted genera

3.1 $\hat{A}$ genus

Definition 3.1. Let X be a differentiable manifold of dimension $2k$ and W a complex vector bundle over X. Then the twisted $\hat{A}$-genus $\hat{A}(X,W)$ is defined as

$$\hat{A}(X,W) = (\prod_{i=1}^{k} \frac{x_i/2}{\sinh(x_i/2)} \cdot \mathrm{ch}(W))[X]$$

where $c(T_{\mathbb{C}}X) = (1+x_1)\cdots(1+x_k)(1-x_1)\cdots(1-x_k)$, $p(TX) = (1+x_1^2)\cdots(1+x_k^2)$, and ch is the Chern character.

Theorem 3.2. For a spin manifold X, $\hat{A}(X,W) \in \mathbb{Z}$.

Proof. Since X is spin, it has a global spinor bundle $S \to X$. Now for any complex vector bundle W, give it a trivial Clifford module structure and we obtain a Clifford module $E = S \otimes W$. Take a Clifford connection ∇^E on E, consider the associated Dirac operator D, defined by the composition

$$\Gamma(X,E) \xrightarrow{\nabla^E} \Gamma(X, T^*X \otimes E) \xrightarrow{c} \Gamma(X,E)$$

where c is the bundle isomorphism from $\bigwedge T^*X$ to the Clifford bundle $C(X)$. In local coordinates, $D = \sum_i c(dx^i)\nabla^E_{\partial_i}$. We see that $D : \Gamma(X,E^{\pm}) \to \Gamma(X,E^{\mp})$, so that we can write

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

where $D^{\pm}$ are the restrictions of D to $\Gamma(X,E^{\pm})$. The index of D is defined to be

$$\mathrm{ind}(D) = \dim(\mathrm{Ker}\, D^+) - \dim(\mathrm{Ker}\, D^-)$$

which is obviously an integer. The Atiyah-Singer index theorem says that

$$\mathrm{ind}(D) = \int_X \hat{\mathfrak{A}}(X).\,\mathrm{ch}(W)$$

where the $\hat{\mathfrak{A}}(X)$ means the class $\prod_{i=1}^{k} \frac{x_i/2}{\sinh(x_i/2)}$ here, so the right hand side is exact the definition of $\hat{A}(X,W)$ in our article.

$\square$

Definition 3.3. Now let X be a compact oriented differentiable manifold of dimension $4k$. Define $\tilde{\varphi}(X)(\tau)$ as viewing the components $x_i\varphi_1(\tau, x_i)$ as weight 0 with respect to the $\tilde{}$ operator (1.7), that is,

$$\tilde{\varphi}(X)(\tau) = \{\prod_{i=1}^{2k} \frac{x_i}{2\tau}\varphi_1(\frac{-1}{2\tau}, \frac{x_i}{2\tau})\}_{4k}[X]$$

On the other hand, view $\varphi(X)(\tau)$ as a modular form of weight $2k$ with respect to $\Gamma_0(2)$ (2.4), so that we have the expression

$$\widetilde{\varphi(X)}(\tau) = \varphi(X)(\frac{-1}{2\tau})(2\tau)^{-2k} = \varphi(X)[\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)]_{2k}(2\tau)$$

In fact, $\tilde{\varphi}(X)(\tau) = \widetilde{\varphi(X)}(\tau)$, since

$$\tilde{\varphi}(X)(\tau) = \{\prod_{i=1}^{2k} \frac{x_i}{2\tau}\varphi_1(\frac{-1}{2\tau}, \frac{x_i}{2\tau})\}_{4k}[X] = \{\prod_{i=1}^{2k} x_i\varphi_1(\frac{-1}{2\tau}, x_i)\}_{4k}[X] \cdot (2\tau)^{-2k} = \varphi(X)(\frac{-1}{2\tau})(2\tau)^{-2k} = \widetilde{\varphi(X)}(\tau)$$

Since $\tilde{\cdot}$ induces a graded automorphism of $M_*(\Gamma_0(2))$, we conclude that $\tilde{\varphi}(X)(\tau)$ is also a modular form of weight $2k$.

Definition 3.4. Let $r = \operatorname{rank} E$. We define formally the expressions

$$\bigwedge\nolimits_t E = \sum_{k=0}^{\infty} (\bigwedge\nolimits^k E) \cdot t^k, \ S_t E = \sum_{k=0}^{\infty} (S^k E) \cdot t^k$$

Then in view of the formulas

$$\sum_{k=0}^{r} \mathrm{ch}(\bigwedge\nolimits^k E^*) \cdot t^k = \prod_{i=1}^{r}(1 + te^{-x_i}), \ \sum_{k=0}^{\infty} \mathrm{ch}(S^k E^*) \cdot t^k = \prod_{i=1}^{r}(1 + te^{-x_i} + t^2 e^{-2x_i} + \cdots)$$

We will denote these expressions by

$$\mathrm{ch}(\bigwedge\nolimits_t E^*) = \prod_{i=1}^{r}(1 + te^{-x_i}), \ \mathrm{ch}(S_t E^*) = \prod_{i=1}^{r} \frac{1}{1 - te^{-x_i}}$$

$r = \operatorname{rank} E$. Thus we can also put these expression into the twisted genera.

Theorem 3.5.

$$\tilde{\varphi}(X)(\tau) = \hat{A}(X, \bigotimes_{n=2m+1} \bigwedge\nolimits_{-q^n} T_{\mathbb{C}} \otimes \bigotimes_{n=2m+2} S_{q^n} T_{\mathbb{C}}) \cdot (\frac{\prod_{n=2m+2}(1-q^n)^2}{\prod_{n=2m+1}(1-q^n)^2})^{2k}$$

Proof.

$$\tilde{\varphi}(X)(\tau) = \{\prod_{i=1}^{2k} \frac{x_i}{2\tau}\varphi_1(\frac{-1}{2\tau}, \frac{x_i}{2\tau})\}_{4k}[X] = \{\prod_{i=1}^{2k} x_i\varphi_2(2\tau, x_i)\}_{4k}[X]$$

$$= \{\prod_{i=1}^{2k} \frac{x_i}{e^{x_i/2} - e^{-x_i/2}} \frac{\prod_{n=2m+1}(1-q^n e^{x_i})(1-q^n e^{-x_i})/(1-q^n)^2}{\prod_{n=2m+2}(1-q^n e^{x_i})(1-q^n e^{-x_i})/(1-q^n)^2}\}_{4k}[X]$$

$$= \{\prod_{i=1}^{2k} \frac{x_i/2}{\sinh(x_i/2)} \frac{\prod_{n=2m+1}(1-q^n e^{x_i})(1-q^n e^{-x_i})}{\prod_{n=2m+2}(1-q^n e^{x_i})(1-q^n e^{-x_i})}\}_{4k}[X] \cdot (\frac{\prod_{n=2m+2}(1-q^n)^2}{\prod_{n=2m+1}(1-q^n)^2})^{2k}$$

$$= \hat{A}(X, \bigotimes_{n=2m+1} \bigwedge\nolimits_{-q^n} T_{\mathbb{C}} \otimes \bigotimes_{n=2m+2} S_{q^n} T_{\mathbb{C}}) \cdot (\frac{\prod_{n=2m+2}(1-q^n)^2}{\prod_{n=2m+1}(1-q^n)^2})^{2k}$$

$\square$

3.2 Equivariant signature of the loop space

Definition 3.6. let X be a compact oriented differentiable manifold of dimension $4k$. The equivariant signature of the loop space is (defined as)

$$\operatorname{sign}(q, \mathscr{L} X) = \prod_{i=1}^{2k}(x_i \frac{1+e^{-x_i}}{1-e^{-x_i}} \cdot \prod_{n=1}^{\infty}(\frac{(1+q^n e^{-x_i})(1+q^n e^{x_i})}{(1-q^n e^{-x_i})(1-q^n e^{x_i})}))[X]$$

Corollary 3.7.

$$\operatorname{sign}(q, \mathscr{L} X) = 2^{2k} \cdot \varphi(X) \cdot (\prod_{n=1}^{\infty} \frac{(1+q^n)^2}{(1-q^n)^2})^{2k}$$

Proof. By definition and the product expansion of $\varphi_1(\tau, x)$ (2.1),

$$\operatorname{sign}(q, \mathscr{L} X) = 2^{2k} \cdot \prod_{i=1}^{2k}(x_i \cdot \frac{1}{2} \frac{e^{x_i/2}+e^{-x_i/2}}{e^{x_i/2}-e^{-x_i/2}} \cdot \prod_{n=1}^{\infty}(\frac{(1+q^n e^{-x_i})(1+q^n e^{x_i})/(1+q^n)^2}{(1-q^n e^{-x_i})(1-q^n e^{x_i})/(1-q^n)^2}))[X] \cdot (\prod_{n=1}^{\infty} \frac{(1+q^n)^2}{(1-q^n)^2})^{2k}$$

$$= 2^{2k} \cdot \prod_{i=1}^{2k} x_i \varphi_1(\tau, x_i)[X] \cdot (\prod_{n=1}^{\infty} \frac{(1+q^n)^2}{(1-q^n)^2})^{2k}$$

$$= 2^{2k} \cdot \varphi(X) \cdot (\prod_{n=1}^{\infty} \frac{(1+q^n)^2}{(1-q^n)^2})^{2k}$$

□

Theorem 3.8 (Hirzebruch signature theorem)**.** For a compact oriented differentiable manifold X of dimension $4k$,

$$L(X) = \operatorname{sign}(X)$$

where the L-genus is defined by the even power series $\dfrac{x}{\tanh x}$.

Proof. (*Sketch*) The signature is a cobordism invariant, which is compatible with the ring structure, so it is a genus. Moreover, $\operatorname{sign}(\mathbb{C}\mathbb{P}^{2n}) = 1 = L(\mathbb{C}\mathbb{P}^{2n})$, and $\mathbb{C}\mathbb{P}^{2n}$ generates $\Omega \otimes \mathbb{Q}$, we have sign $= L$. □

Corollary 3.9. The constant term of $\operatorname{sign}(q, \mathscr{L} X)$ is $\operatorname{sign}(X)$.

Proof. The constant term is

$$\{\prod_{i=1}^{2k} x_i \frac{1+e^{-x_i}}{1-e^{-x_i}}\}_{4k}[X] = 2^{-2k}\{\prod_{i=1}^{2k}(2x_i)\frac{1+e^{-2x_i}}{1-e^{-2x_i}}\}_{4k}[X] = \{\prod_{i=1}^{2k} \frac{x_i}{\tanh x_i}\}_{4k}[X] = L(X) = \operatorname{sign}(X)$$

□

3.3 χ_y-genus of the loop space

Definition 3.10. Let X be a compact complex manifold of complex dimension d, then the Hodge numbers are defined as $h^{p,q} = h^q(X, \Omega^p)$ and the p-th holomorphic Euler number is defined as

$$\chi^p = \sum_{q=0}^{d} (-1)^q \cdot h^{p,q}$$

χ^p is the index of the Dolbeault complex. Further define

$$\chi_y(X) = \sum_{p=0}^{d} \chi^p(X) \cdot y^p$$

Then the yields

Theorem 3.11.

$$\chi_y(X) = \prod_{i=1}^{d} (x_i \frac{1 + y \cdot e^{-x_i}}{1 - e^{-x_i}})[X]$$

Proof. This is a corollary of the Atiyah-Singer index theorem on elliptic complexes. Apply it on the Dolbeault complex:

$$\begin{aligned}
\chi^p = \sum_{q=0}^{d} (-1)^q \cdot h^{p,q} &= ((\sum_{q=0}^{d} (-1)^q \operatorname{ch}(\bigwedge^p T^* \otimes \bigwedge^q T)) \prod_{i=1}^{d} (\frac{x_i}{1 - e^{-x_i}} \cdot \frac{1}{1 - e^{x_i}}))[X] \\
&= (\operatorname{ch}(\bigwedge^p T^*)(\sum_{q=0}^{d} (-1)^q \operatorname{ch}(\bigwedge^q T)) \prod_{i=1}^{d} (\frac{x_i}{1 - e^{-x_i}} \cdot \frac{1}{1 - e^{x_i}}))[X] \\
&= (\operatorname{ch}(\bigwedge^p T^*) \prod_{i=1}^{d} (\frac{x_i}{1 - e^{-x_i}}))[X]
\end{aligned}$$

So now

$$\chi_y(X) = ((\sum_{p=0}^{d} \operatorname{ch}(\bigwedge^p T^*) \cdot y^p) \prod_{i=1}^{d} (\frac{x_i}{1 - e^{-x_i}}))[X] = \prod_{i=1}^{d} (x_i \frac{1 + y \cdot e^{-x_i}}{1 - e^{-x_i}})[X]$$

□

This is not actually a genus since the power series does not begin with 1, so we must normalize it to obtain a genus

$$\frac{\chi_y(X)}{(1+y)^d} = \prod_{i=1}^{d} (x_i \frac{1 + y e^{-x_i}}{1 - e^{-x_i}} \cdot \frac{1}{1+y})[X]$$

Also, we can define the twisted χ_y-genus for a complex vector bundle E over X by replacing $h^{p,q} = h^q(X, \Omega^p)$ by $h^q(X, \Omega^p \otimes E)$, similarly we get

$$\chi_y(X, E) = (\prod_{i=1}^{d} x_i \frac{1 + y e^{-x_i}}{1 - e^{-x_i}} \cdot \operatorname{ch}(E))[X]$$

In the last section, we will turn some expression into these twisted genera and deduce integrality from that of $h^q(X, \Omega^p \otimes E)$.

Definition 3.12. Assume $2N|d$. Let $-y = e^{\alpha}$ be an N-th root of unity. We define the χ_y-genus of the loop space

$$\chi_y(q, \mathscr{L}X) = (\prod_{i=1}^{d} x_i \cdot \frac{1+ye^{-x_i}}{1-e^{-x_i}}(\prod_{n=1}^{\infty} \frac{1+yq^n e^{-x_i}}{1-q^n e^{-x_i}} \cdot \frac{1+y^{-1}q^n e^{x_i}}{1-q^n e^{x_i}}))[X]$$

In view of 3.4,

$$\chi_y(q, \mathscr{L}X) = \chi_y(X, \bigotimes_{n=1}^{\infty} \bigwedge\nolimits_{yq^n} T^* \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T^* \otimes \bigotimes_{n=1}^{\infty} \bigwedge\nolimits_{y^{-1}q^n} T \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T)$$

Theorem 3.13. Assume $2N|d$. Let $\alpha = 2\pi i k/N$ with $0 < k < N$ and $\gcd(k, N) = 1$, $-y = e^{\alpha}$, then

$$\chi_y(q, \mathscr{L}X) = \varphi_{N,\alpha}(X) \cdot \Phi(-\alpha)^d \cdot (-y)^{d/2}$$

Also, $\Phi(\tau, -\alpha)^{-d}$ is a modular form of weight d on $\Gamma_1(N)$. Hence, $\chi_y(q, \mathscr{L}X)$ is a **modular function** on $\Gamma_1(N)$, that is, an automorphic form of weight 0.

Proof. Simply recall $-y = e^{\alpha}$,

$$\Phi(x) = (e^{x/2} - e^{-x/2}) \cdot \prod_{n=1}^{\infty} \frac{(1-q^n e^x)(1-q^n e^{-x})}{(1-q^n)^2}$$

and $\varphi_{N,\alpha}(X)$ is defined by $f(x) = \dfrac{\Phi(x)\Phi(x-\alpha)}{\Phi(-\alpha)}$. Now we get

$$\begin{aligned}
\chi_y(q, \mathscr{L}X) &= (\prod_{i=1}^{d} x_i \cdot \frac{1-e^{-(x_i-\alpha)}}{1-e^{-x_i}}(\prod_{n=1}^{\infty} \frac{1-q^n e^{-(x_i-\alpha)}}{1-q^n e^{-x_i}} \cdot \frac{1-q^n e^{x_i-\alpha}}{1-q^n e^{x_i}}))[X] \\
&= (\prod_{i=1}^{d} x_i \cdot \frac{1-e^{-(x_i-\alpha)}}{1-e^{-x_i}} \cdot \frac{e^{x_i/2}-e^{-x_i/2}}{e^{(x_i-\alpha)/2}-e^{-(x_i-\alpha)/2}} \cdot \frac{\Phi(x_i-\alpha)}{\Phi(x_i)})[X] \\
&= (\prod_{i=1}^{d} x_i e^{\alpha/2} \frac{\Phi(x_i-\alpha)}{\Phi(x_i)})[X] \\
&= (\prod_{i=1}^{d} \frac{x_i}{f(x_i)})[X] \cdot \Phi(-\alpha)^d \cdot e^{d\alpha/2} \\
&= \varphi_{N,\alpha}(X) \cdot \Phi(-\alpha)^d \cdot (-y)^{d/2}
\end{aligned}$$

Next, using 1.15 we compute for $\gamma = \left(\begin{smallmatrix} a & b \\ c & g \end{smallmatrix}\right) \in \Gamma_1(N)$,

$$\begin{aligned}
\Phi(\gamma(\tau), -\alpha) \cdot (c\tau+g) &= \Phi(\gamma(\tau), \frac{-\alpha(c\tau+g)}{c\tau+g}) \cdot (c\tau+g) \\
&= \exp(\frac{c\alpha^2(c\tau+g)^2}{4\pi i(c\tau+g)}) \cdot \Phi(\tau, -\alpha(c\tau+g)) \\
&= \exp(2\pi i\tau\frac{k^2c^2}{2N^2}) \cdot \exp(2\pi i\frac{k^2cg}{2N^2}) \cdot \Phi(\tau, \frac{-2\pi ik}{N} + 2\pi i \cdot (\frac{-kc}{N}\tau + \frac{-k(g-1)}{N})) \\
&= q^{k^2c^2/2N^2} \cdot \exp(2\pi i\frac{k^2cg}{2N^2}) \cdot q^{-k^2c^2/2N^2} \exp(-2\pi i\frac{k^2c}{N^2}) \cdot (-1)^{\frac{-kc}{N} - \frac{k(g-1)}{N}} \cdot \Phi(\tau, \frac{-2\pi ik}{N}) \\
&= \exp(2\pi i\frac{k^2c(g-2)}{2N^2}) \cdot (-1)^{-k(c+g-1)/N} \cdot \Phi(\tau, -\alpha)
\end{aligned}$$

Notice that $N|c$ and $N|(g-1)$. Now if we consider the power $\Phi(\tau,-\alpha)^{2N}$, we will see the difference factor becomes 1.

Next we show that $\Phi(\tau,-\alpha)^{-2N}$ is holomorphic at all cusps. For general $\gamma = \left(\begin{smallmatrix} a & b \\ c & g \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we see during the above computation that

$$\Phi(\gamma(\tau),-\alpha)^{-1} \cdot (c\tau+g)^{-1} = C_1 q^{-k^2c^2/2N^2} \cdot \Phi(\tau, 2\pi i\tau(\frac{-kc}{N}) + C_2)$$

where C_1, C_2 are constants independent of τ that are not important for now. We may assume $c \geq 0$, if not, replacing γ by $-\gamma$. Plug in the product form of Φ and try to cancel the negative powers of q, it suffices to show that

$$-\frac{k^2c^2}{2N^2} + \frac{kc}{2N} + (\frac{kc}{N} - 1) + \cdots + (\frac{kc}{N} - l) \geq 0$$

where $l = \left\lfloor \frac{kc}{N} \right\rfloor$. Summing and completing the square, the above quantity is

$$-\frac{1}{2}(l + \frac{1}{2} - \frac{kc}{N})^2 + \frac{1}{8}$$

But by the definition of l,

$$-\frac{1}{2} < l + \frac{1}{2} - \frac{kc}{N} \leq \frac{1}{2}$$

we thus conclude that the quantity is nonnegative and $\Phi(\tau,\alpha)^{-2N}$ has an expansion in $q^{1/N}$, i.e. it is holomorphic at every cusp of $\Gamma_1(N)$ and on $\mathbb{H}$.

Summing up, we get $\Phi(\tau,-\alpha)^{-2N}$ is a modular form on $\Gamma_1(N)$ of weight $2N$. Since $2N|d$, $\Phi(\tau,-\alpha)^{-d}$ is then a modular form on $\Gamma_1(N)$ of weight d. Quotient and we know $\chi_y(q, \mathscr{L}X)$ is a modular function.

□

4 A divisibility theorem for elliptic genera

4.1 Ochanine's theorem on spin manifolds

Theorem 4.1. Let W be the complex extension of a real vector bundle over a compact, oriented, differentiable spin manifold X with $\dim X \equiv 4\ (8)$. Then $\hat{A}(X,W) \in 2\,\mathbb{Z}$.

Proof. (*Sketch*) In the proof of 3.2, we consider the Dirac operator $D : \Gamma(X, E^{\pm}) \to \Gamma(X, E^{\mp})$ with

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

where $E = S \otimes W$. The Atiyah-Singer index theorem says that

$$\hat{A}(X,W) = \mathrm{ind}(D) = \dim(\mathrm{Ker}\, D^+) - \dim(\mathrm{Ker}\, D^-)$$

But when $\dim M \equiv 4\ (8)$ and that W comes from a real bundle, in fact there is a quaternionic structure on $\mathrm{Ker}\, D^+$ and $\mathrm{Ker}\, D^-$, so as complex vector spaces they have even dimensions. Thus $\hat{A}(X,W) \in 2\,\mathbb{Z}$.

For the original proof, see [AtHi].

□

Theorem 4.2 (Ochanine)**.** Let X be a compact, oriented, differentiable spin manifold with $\dim X \equiv 4\ (8)$. Then

$$\mathrm{sign}(X) \equiv 0\ (16)$$

Proof. First recall that the elliptic genus (2.2)

$$\varphi(X)(\tau) = \{\prod_{i=1}^{2k} x_i \varphi_1(\tau, x_i)\}_{4k}[X]$$

Recall that $\tilde{\cdot} : M_*(\Gamma_0(2)) \to M_*(\Gamma_0(2))$ is a well-defined automorphism of graded algebra (1.7). From 1.9, $M_*(\Gamma_0(2)) = \mathbb{C}[\delta, \epsilon]$, so we know that $M_*(\Gamma_0(2)) = \mathbb{C}[\tilde{\delta}, \tilde{\epsilon}]$ as well. Also, recall that $\varphi(X)(\tau), \tilde{\varphi}(X)(\tau) \in M_{2k}(\Gamma_0(2))$ (2.4, 3.3). We now can write $\tilde{\varphi}(X)$ as a polynomial in $8\tilde{\delta}$ and $\tilde{\epsilon}$:

$$\tilde{\varphi}(X) = \sum_{2a+4b=2k} c_{a,b} \cdot (8\tilde{\delta})^a \cdot \tilde{\epsilon}^b = P(8\tilde{\delta}, \tilde{\epsilon})$$

where $c_{a,b} \in \mathbb{C}$ for now. But $\tilde{\varphi}_1(\tau, x) = \varphi_1(\frac{-1}{2\tau}, \frac{x}{2\tau})(2\tau)^{-1} = \varphi_2(2\tau, x)$. Recall that $\tilde{\varphi}(X)(\tau)$ has the expansion (3.5)

$$\tilde{\varphi}(X)(\tau) = \hat{A}(X, \bigotimes_{n=2m+1} \bigwedge\nolimits_{-q^n} T_{\mathbb{C}} \otimes \bigotimes_{n=2m+2} S_{q^n} T_{\mathbb{C}}) \cdot (\frac{\prod_{n=2m+2}(1-q^n)^2}{\prod_{n=2m+1}(1-q^n)^2})^{2k}$$

The product factor has an integral q-expansion, while the q-coefficient of the other factor is some twisted $\hat{A}$-genera $\hat{A}(X, W)$.

Now if X is spin, we know that $\hat{A}(X, W) \in \mathbb{Z}$ (3.2). Thus $\tilde{\varphi}(X)(\tau)$ has an integral q-expansion. Recall that we have computed the Fourier coefficients (1.8)

$$8\tilde{\delta} = -1 + \cdots,\ \tilde{\epsilon} = q + \cdots$$

These facts implies that the coefficients $c_{a,b}$ must also be integral by induction. Hence $\tilde{\varphi} \in \mathbb{Z}[8\tilde{\delta}, \tilde{\epsilon}]$. Apply the inverse of $\tilde{\cdot}$ we get $\varphi \in \mathbb{Z}[8\delta, \epsilon]$.

Finally, by 3.7,

$$\mathrm{sign}(q, \mathscr{L} X) = 2^{2k}\varphi(X) \cdot (\prod_{n=1}^{\infty} \frac{(1+q^n)^2}{(1-q^n)^2})^{2k}$$

The product factor again has an integral q-expansion, and the other factor

$$2^{2k}\varphi(X) = 2^{2k}P(8\delta, \epsilon) = P(32\delta, 16\epsilon)$$

where we again have computed the Fourier coefficients

$$32\delta = 8(1 + 24q + \cdots),\ 16\epsilon = 1 + 16q + \cdots$$

If our manifold has dimension $4k$ with k odd, then the polynomial $P(32\delta, 16\epsilon)$ is divisible by 32δ, otherwise the weight could not be $2k$. (Note that ϵ has weight 4.) Hence all q-coefficients of $\mathrm{sign}(q, \mathscr{L} X)$ are divisible by 8:

$$\mathrm{sign}(q, \mathscr{L} X) \equiv 0 \ (8)$$

Moreover, by 4.1, in above argument the genera $\hat{A}(X, W) \in 2\,\mathbb{Z}$ since each W comes from real operations of $T_{\mathbb{C}} = TX \otimes_{\mathbb{R}} \mathbb{C}$, thus a complex extension of a real vector bundle. We conclude that the coefficients $c_{a,b}$ of P must be even, and that

$$\mathrm{sign}(q, \mathscr{L} X) \equiv 0 \ (16)$$

In particular this holds for the constant term, the signature of X (3.9).

□

4.2 Complex analogue

Let X be a compact complex manifold of dimension d.

Theorem 4.3. Let $\alpha = 2\pi ik/N$ with $0 < k < N$ and $\gcd(k, N) = 1$. $-y = e^{\alpha}$, then

$$\varphi_{N,\alpha}(\tau) = \frac{(q\text{-expansion with coefficients in } \mathbb{Z}[y])}{(1+y)^d}$$

Proof. Recall 3.13 that

$$\chi_y(q, \mathscr{L} X) = \varphi_{N,\alpha}(X) \cdot \Phi(\tau, -\alpha)^d \cdot (-y)^{d/2}$$

But on the other hand, from 3.12,

$$\chi_y(q, \mathscr{L} X) = \chi_y(X, \bigotimes_{n=1}^{\infty} \bigwedge\nolimits_{yq^n} T^* \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T^* \otimes \bigotimes_{n=1}^{\infty} \bigwedge\nolimits_{y^{-1}q^n} T \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T)$$

Notice that the coefficients of q^i are the sum of twisted χ_y-genera multiplying with powers of y, thus in $\mathbb{Z}[y]$. Next we directly expand $\Phi(\tau, -\alpha)$:

$$\begin{aligned}\Phi(\tau, -\alpha)^d &= ((e^{-\alpha/2} - e^{\alpha/2}) \cdot (1 + \text{integral } q\text{-expansion}))^d \\ &= e^{-d\alpha/2}(1 - e^{\alpha})^d \cdot (1 + \text{integral } q\text{-expansion}) \\ &= (-y)^{-d/2}(1+y)^d \cdot (1 + \text{integral } q\text{-expansion})\end{aligned}$$

Combining all of these we get the desired result.

□

Since the q-expansion of $\chi_y(q, \mathscr{L} X)$ starts with $\chi_y(X)$, the value of $\varphi_{N,\alpha}(X)$ at ∞ is $\dfrac{\chi_y(X)}{(1+y)^d}$.
If $N = 2$ and $y = 1$, $\mathbb{Z}[y] = \mathbb{Z}$ and the numerator is an integral q-expansion.
We consider the expansion at some other cusps. To do this we can also change the N-division point to $\alpha = 2\pi ik\tau/N$.

Theorem 4.4. Assume $c_1(X) \equiv 0\ (N)$, which is equivalent to that $K = L^N$ for some line bundle L. For $\alpha = 2\pi ik\tau/N$ with $0 < k < N$ and $\gcd(k, N) = 1$, the q-expansion of $\tilde{\varphi}_{N,\alpha}(X)(\tau) = \varphi_{N,\alpha}(N\tau)$ has integral coefficients.

Proof. Recall (2.7) that for this N-division point, the power series we consider takes the form

$$f(x) = e^{(k/N)x} \cdot \frac{\Phi(\tau, x)\Phi(\tau, -\alpha)}{\Phi(\tau, x - \alpha)}$$

Insert $\alpha = 2\pi i\tau\dfrac{k}{N}$, we get

$$\varphi_{N,\alpha}(X) = (\prod_{i=1}^{d}(\frac{x_i}{1 - e^{-x_i}} \cdot \frac{1 - q^{k/N}e^{-x_i}}{1 - q^{k/N}} \cdot e^{-(k/N)x_i} \cdot \prod_{n=1}^{\infty} \frac{(1 - q^{n+k/N}e^{-x_i})(1 - q^{n-k/N}e^{x_i})(1 - q^n)^2}{(1 - q^n e^{-x_i})(1 - q^n e^{x_i})(1 - q^{n+k/N})(1 - q^{n-k/N})}))[X]$$

$\varphi_{N,\alpha}(X)(\tau)$ is a modular form on $\Gamma^1(N)$. We replace q by q^N and obtain a modular form on $\Gamma_1(N)$. Now notice that $\prod_{i=1}^{d} \frac{x_i}{1 - e^{-x_i}}$ is the Todd class of X and $\prod_{i=1}^{d} e^{-(k/N)x_i} = e^{(k/N)c_1(K)} = \mathrm{ch}(K^{k/N})$. By Riemann-Roch theorem we get

$$\tilde{\varphi}_{N,\alpha}(X) = \prod_{n=1}^{\infty} \frac{(1 - q^{nN})^2}{(1 - q^{nN+k})(1 - q^{nN-k})}$$
$$\cdot \chi(X, K^{k/N} \otimes \bigotimes_{n \equiv k(N), n \geq N+k} \bigwedge\nolimits_{q^n} T^* \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T^* \otimes \bigotimes_{n \equiv -k(N), n \geq N-k} \bigwedge\nolimits_{q^n} T \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T)$$

Thus the integrality of coefficients of q^i comes from that of $\chi(X, K^{k/N} \otimes W)$, since $c_1(X) \equiv 0\ (N)$.

□

5 References

[Hirz] F. Hirzebruch, Th. Berger, R. Jung: Manifolds and Modular Forms

[Diam] Fred Diamond , Jerry Shurman: A First Course in Modular Forms

[Ston] Stong, Robert E. (1968), Notes on cobordism theory.

[AtHi] M. F. Atiyah, F. Hirzebruch, Riemann-Roch theorems for differentiable manifolds.