Modular forms for even integral quadratic forms

ref: Serre GTM 7.

Introduction:

For a self dual even lattice $\Gamma \subseteq \mathbb{R}^n$, we define

$$r_\Gamma(m) = \#\{ x \in \Gamma \mid x.x = 2m \}.$$ depends on Γ

We have the following results.

Thm 1: The rank n of a self dual even lattice $\Gamma \subseteq \mathbb{R}^n$ must be divided by 8, i.e. $8 \mid n$.

Thm 2: The generating function $\theta_\Gamma(\tau) = \sum_{m=0}^{\infty} r_\Gamma(m) q^m$, $q = e^{2\pi i \tau}$ is a modular form of weight $n/2$.

Cor 1: There exists a cusp form f_Γ of weight $n/2$ such that

$$\theta_\Gamma = E_{2k} + f_\Gamma \quad \text{where } k = \frac{n}{4}.$$

Cor 2: We have

$$r_\Gamma(m) = \frac{4k}{B_k} \sigma_{2k-1}(m) + O(m^k) \quad \text{where } k = \frac{n}{4}.$$

Self dual even lattices

We will consider lattices $\Gamma \subseteq \mathbb{R}^n$, $\Gamma \simeq \mathbb{Z}^n$ s.t.

(i) The dual lattice $\Gamma' = \{ y \in \mathbb{R}^n \mid (x.y) \in \mathbb{Z} \ \forall x \in \Gamma \} = \Gamma$.

(ii) We have $(x \cdot x) \in 2\mathbb{Z} \ \forall x \in \Gamma$.

Examples:

For $n = 8k$. We can take $\Gamma_{n,0} = \{ x \in \mathbb{Z}^n \mid (x.x) \in 2\mathbb{Z} \}$.

and $e = (\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}) \in \mathbb{R}^n$. Consider the lattice Γ_n generated by $\Gamma_{n,0}$ and $\{e\}$ over $\mathbb{Z}$. Then Γ_n is a self dual even lattice.

For $k \geq 2$ the vector $x \in \Gamma_{8k}$ s.t. $x \cdot x = 2$ are the vectors $\pm e_i \pm e_j$ $(i \neq j)$. which implies $r_{\Gamma_{8k}}(1) = 4 \cdot \binom{8k}{2} = 2 \cdot 8k \cdot (8k-1)$.

Theta function:

Given an even self dual lattice Γ as above, we define the theta function associated to Γ:

$$\theta_\Gamma(z) = 1 + \sum_{m=1}^{\infty} r_\Gamma(m) q^m, \quad q = e^{2\pi i z}$$

We have $\theta_\Gamma(z) = \sum_{x \in \Gamma} q^{\frac{(x \cdot x)}{2}} = \sum_{x \in \Gamma} e^{\pi i z (x \cdot x)}$

Thm 1: rank of an even self dual lattice is divided by 8.

pf: Suppose $rk(\Gamma) \not\equiv 0 \pmod 8$. After replacing Γ by $\Gamma \oplus \Gamma$, $\Gamma \oplus \Gamma \oplus \Gamma$ or $\Gamma \oplus \Gamma \oplus \Gamma \oplus \Gamma$ we may assume $rk(\Gamma) \equiv 4$ (8).

Now we prove an identity

$$\theta_\Gamma\left(-\frac{1}{z}\right) = (iz)^{\frac{n}{2}} \theta_\Gamma(z)$$

Since two sides are analytic functions, we only need to show the identity holds for $z = it$, $t > 0$.

We have
$$\theta_\Gamma(it) = \sum_{x \in \Gamma} e^{-\pi t (x \cdot x)} = \sum_{x \in t^{\frac{1}{2}}\Gamma} e^{-\pi (x \cdot x)}$$

$$\theta_\Gamma\left(-\frac{1}{it}\right) = \sum_{x \in \Gamma} e^{-\pi \frac{1}{t}(x \cdot x)} = \sum_{x \in t^{-\frac{1}{2}}\Gamma} e^{-\pi (x \cdot x)}$$

By Poisson summation formula we have

$$\theta_\Gamma(it) = \sum_{x \in t^{\frac{1}{2}}\Gamma} e^{-\pi (x \cdot x)} = \frac{1}{Vol(t^{\frac{1}{2}}\Gamma)} \sum_{y \in t^{-\frac{1}{2}}\Gamma} e^{-\pi (y \cdot y)} = t^{-\frac{n}{2}} \cdot \theta_\Gamma\left(-\frac{1}{it}\right)$$

Which gives us the desired identity

$$\theta_\Gamma\left(-\frac{1}{z}\right) = (iz)^{\frac{n}{2}} \theta_\Gamma(z).$$

i.e. θ_Γ is a weight $\frac{n}{2}$ modular form.

Now we have

$$\theta_\Gamma\left(-\frac{1}{z}\right) = -z^{\frac{n}{2}} \theta_\Gamma(z) \quad \text{Since} \quad 8 \nmid n. \text{ and } 4 \mid n.$$

Let $w = \Theta_\Gamma(z)\, dz^{\frac{n}{4}} = \frac{n}{4} z^{\frac{n}{4}-1}\Theta_\Gamma(z)\, dz$ be a differential form

Then $w \mapsto -w$ under $S: z \mapsto -\frac{1}{z}$

$w \mapsto w$ under $T: z \mapsto z+1$

But $(ST)^3 = \mathrm{id}$ which is absurd.

So we conclude that $8 \mid n$ and Θ_Γ is a modular form of wt $n/2$.

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<u>Thm 2</u>: The generating function $\Theta_\Gamma(\tau) = \sum_{m=0}^{\infty} r_\Gamma(m) q^m$, $q = e^{2\pi i \tau}$

is a modular form of weight $n/2$.

<u>Cor 1</u>: There exists a cusp form f_Γ of weight $n/2$ s.t.

$$\Theta_\Gamma = E_{2k} + f_\Gamma \quad \text{where } k = \frac{n}{4}.$$

This follows from the fact that $\Theta_\Gamma(\infty) = 1 \rightsquigarrow \Theta_\Gamma - E_{2k}$ is a cusp form.

<u>Rmk</u>: Although the error term f_Γ is unknown for arbitrary Γ.

Siegel has proved the following "Weight mean formula".

Let C_n be the set of equivalence classes of self dual even lattice.

Then

$$\sum_{\Gamma \in C_n} \frac{1}{|\mathrm{Aut}\,\Gamma|} f_\Gamma = 0$$

i.e.

$$\sum_{\Gamma \in C_n} \left(\frac{1}{|\mathrm{Aut}(\Gamma)|} \cdot \Theta_\Gamma \right) = \left(\sum_{\Gamma \in C_n} \frac{1}{|\mathrm{Aut}\,\Gamma|} \right) E_k \qquad (*)$$

Examples:

(i) $n=8$.

We know that there is no cusp form of wt 4. So

$$\Theta_{\Gamma} = E_4 = 1 + \sum_{m=1}^{\infty} 240\, \sigma_3(m) q^m$$

Notice that $\Gamma_8^?$ is the only even self-dual lattice of rk 8.

(ii). $n=16$.

For the same reason we have

$$\Theta_{\Gamma} = E_8 = 1 + 480 \sum_{m=1}^{\infty} \sigma_7(m) q^m$$

Note that there are exactly 2 non-isomorphic even self dual lattices of rank 16, namely, $\Gamma_8^? \oplus \Gamma_8^?$ and $\Gamma_{16}^?$. So the above formula gives

$$\Theta_{\Gamma_8 \oplus \Gamma_8} = (\Theta_{\Gamma_8})^2 = E_8 = 1 + 480 \sum_{m=1}^{\infty} \sigma_7(m) q^m$$

This recover the identity:

$$\left(1 + 240 \sum_{m=1}^{\infty} \sigma_3(m) \cdot q^m\right)^2 = 1 + 480 \sum_{m=1}^{\infty} \sigma_7(m) q^m .$$

(iii) $n=24$.

When $n \geq 24$, the space of cusp form is non-empty so we have to determined the error term f_Γ.

When $n=24$. the space of cusp forms is generated by

$$F = (2\pi)^{-12}\Delta = q \cdot \prod_{m=1}^{\infty} (1-q^m)^{24} = \sum_{m=1}^{\infty} \tau(m)\, q^m$$

So the theta function associated with the lattice Γ can be written

$$\theta_\Gamma = E_{12} + C_\Gamma \cdot F \quad \text{with} \quad C_\Gamma \in \mathbb{Q}$$

Now we look at the Eisenstein serie E_{12}.

$$E_{12} = 1 - \frac{24}{B_{12}} \sum_{m=1}^{\infty} \sigma_{11}(m)\, q^m$$

$$= 1 + 24 \cdot \frac{2730}{691} \sum_{m=1}^{\infty} \sigma_{11}(m)\, q^m$$

$$= 1 + \frac{65520}{691} \sum_{m=1}^{\infty} \sigma_{11}(m) \cdot q^m$$

Compare the coefficient we get

$$r_\Gamma(m) = \frac{65520}{691} \sigma_{11}(m) + C_\Gamma \cdot \tau(m) \qquad \forall m.$$

So we only need to put $m=1$ to get the coefficient C_Γ.

i.e. $$C_\Gamma = r_\Gamma(1) - \frac{65520}{691} \quad \left(\neq 0 \text{ since } r_\Gamma(1) \in \mathbb{Z} \right).$$

(a). For $\Gamma = \Gamma_8 \oplus \Gamma_8 \oplus \Gamma_8$.

$$C_\Gamma = 3 \cdot r_{\Gamma_8}(1) - \frac{65520}{691}$$
$$= 3 \cdot 240 - \frac{65520}{691} = \frac{432000}{691}$$

(b) For $\Gamma = \Gamma_{24}$

$$r_\Gamma(1) = \#\{\pm e_i \pm e_k \mid i \neq k\} = 4 \cdot C_2^{24} = 2 \cdot 23 \cdot 24.$$

$$\Rightarrow \quad C_\Gamma = 2 \cdot 23 \cdot 24 - \frac{65520}{691} = \frac{697344}{691}$$

(c) J. LEECH Constructed a lattice $\Gamma \in C_{24}$ s.t. $r_\Gamma(1) = 0$.

Hence
$$C_\Gamma = -\frac{65520}{691}$$

Rmk: There is only 1 lattice in C_{24} satisfies $r_\Gamma(1) = 0$.

More on Even self dual lattice :

Q: For $n=8, 16, 24$, $\#C_n = ?$

In Serre's book [GTM 7], page 54, there is a mass formula.

Define $M_n := \sum_{\Gamma \in C_n} \frac{1}{|Aut(\Gamma)|}$: mass of C_n. Then

$$M_n = \frac{B_{\frac{n}{2}}}{n} \cdot \prod_{j=1}^{\frac{n}{2}-1} \frac{B_{2j}}{4j} \qquad (**)$$

Using this formula we can estimate the number of C_n.

(i) $n=8$: $M_8 = \frac{B_4}{8} \cdot \frac{B_2}{4} \cdot \frac{B_4}{8} \cdot \frac{B_6}{12} = 2^{-14} 3^{-5} 5^{-2} 7^{-1}$

$$Aut(\Gamma_8) = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$$

Hence $\# C_8 = 1$. i.e. Γ_8 is the only element in C_8.

(ii) $n=16$: $M_{16} = \frac{B_8}{16} \cdot \frac{B_2}{4} \cdot \frac{B_4}{8} \cdot \frac{B_6}{12} \cdot \frac{B_8}{16} \cdot \frac{B_{10}}{20} \cdot \frac{B_{12}}{24} \cdot \frac{B_{14}}{28}$

$$= 691 / 2^{30} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13$$

$$Aut(\Gamma_8 \oplus \Gamma_8) = 2^{29} \cdot 3^{10} \cdot 5^4 \cdot 7^2$$

$$Aut(\Gamma_{16}) = 2^{15} \cdot (16!) = 2^{15} \cdot 2^4 \cdot 3 \cdot 5 \cdot 2 \cdot 7 \cdot 13 \cdot 2^2 \cdot 3 \cdot 11 \cdot 2 \cdot 5 \cdot 3^2 \cdot 2^3 \cdot 7 \cdot 2 \cdot 3 \cdot 5 \cdot 2^2 \cdot 3 \cdot 2 = 2^{30} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$$

$$\frac{1}{2^{29} \cdot 3^{10} \cdot 5^4 \cdot 7^2} + \frac{1}{2^{30} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13} = \frac{2 \cdot 11 \cdot 13 + 3^4 \cdot 5}{2^{30} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13} = \frac{691}{2^{30} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13}$$

Which implies $C_{16} = \{\Gamma_{16}, \Gamma_8 \oplus \Gamma_8\}$.

Introduction to the Siegel-Weil formula.

Ref: https://www.math.columbia.edu/~chaoli/ASWSurvey.pdf

The formulas (*) and (**) comes from a deep results from the representation theory.

To understand the formula we need to generalize the modular forms and introduce some definitions.

Def: $\mathbb{H}_n := \{A+Bi \mid A, B \in \mathrm{Sym}_n(\mathbb{R})$: symmetric matrices, B: positive definite$\}$

this is called the Siegel's half space.

Def: $\mathrm{Sym}_n(\mathbb{Z}) := \{ A \in M_n(\frac{1}{2}\mathbb{Z}) \mid A = A^t, \ A_{ii} \in \mathbb{Z} \ \forall i \}$.

$\mathrm{Sym}_n(\mathbb{Z})_{\geq 0}$: the subset of positive semi-definite matrices.

$$Sp_{2n}(\mathbb{Z}) = \left\{ g \in GL_{2n}(\mathbb{Z}) \;\middle|\; g \cdot \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} g^t = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \right\}$$

Def: Let $T \in \mathrm{Sym}_n(\mathbb{Z})_{\geq 0}$, We define q function

$$q^T := e^{2\pi i\, tr(Tz)} : \mathbb{H}_n \to \mathbb{C}.$$

We can then define the Siegel Eisenstein series

$$E_k(z) := \sum_{\substack{C, D \in M_n(\mathbb{Z}) \\ (C,D) \text{ coprime}}} \det(Cz+D)^{-k} \overset{\text{Fourier expansion}}{=} \sum_{T \in \mathrm{Sym}_n(\mathbb{Z})_{\geq 0}} a(T) \cdot e^{2\pi i\, tr(Tz)}$$

here (C, D) coprime $:= \exists A, B \in M_n(\mathbb{Z})$ s.t. $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2n}(\mathbb{Z})$.

Thm (Siegel Weil formula). Let Λ be a quadratic lattice, define

$\mathrm{Gen}(\Lambda)$ = the set of equivalence classes of quadratic lattice Λ' s.t.

$\forall p$: prime $\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \Lambda' \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \simeq \Lambda' \otimes_{\mathbb{Z}} \mathbb{R}$.

For each $n \in \mathbb{N}$, $r_\Lambda(T) := \#\{(l_1, \cdots, l_n) \in \Lambda^n \mid ((x_i, x_j))_{ij} = 2T\}$

$$\Theta_\Lambda(z) := \sum_{T \in \mathrm{Sym}_n(\mathbb{Z})_{\geq 0}} r_\Lambda(T) \cdot q^T$$

Then

$$\sum_{\Lambda' \in \mathrm{Gen}(\Lambda)} \left(\frac{1}{|\mathrm{Aut}(\Lambda')|} \cdot \Theta_{\Lambda'}(z) \right) = \left(\sum_{\Lambda' \in \mathrm{Gen}(\Lambda)} \frac{1}{|\mathrm{Aut}(\Lambda')|} \right) E^*_{\frac{l}{2}}(z)$$

where $l = \dim \Lambda$. $E^*_{\frac{l}{2}}(z)$ is a certain normalization s.t. const $= 1$.

Rmk: This formula is ture for all $n \in \mathbb{N}$, so we can take some special $n \in \mathbb{N}$ to get different results.

Now, our goal is to use this formula appropriate to get the classical results $(*)$ and $(**)$.

First, we use the result in the classification of unimodular lattice over $\mathbb{Z}_p$.

Thm: Let Λ be a unimodular lattice. Then

(i) $\Lambda \otimes \mathbb{Z}_p \simeq \langle 1 \rangle \oplus \langle 1 \rangle \oplus \cdots \oplus \langle d_\Lambda \rangle$

(ii) $\Lambda \otimes \mathbb{Z}_2 \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Kitaoka,
Arithmetic of quadratic forms
Thm 5.2.4 + Thm 5.2.5.

This classification theorem implies that the set of unimodular even lattice of a given rank forms a genus.

Hence, take any unimodular even lattice Λ, $\mathrm{rk}\Lambda = l$

Then $C_l = \mathrm{Gen}(\Lambda)$. let $k = \frac{l}{4}$

So the Siegel-Weil formula becomes.

$$\sum_{\Lambda \in C_l} \frac{1}{|\mathrm{Aut}(\Lambda)|} \Theta_\Lambda(Z) = \left(\sum_{\Lambda \in C_l} \frac{1}{|\mathrm{Aut}(\Lambda)|} \right) E^*_{2k}(Z)$$

When we take $n=1$, we return to the classical case and get formula $(*)$

$$\sum_{\Lambda \in C_l} \frac{1}{|\mathrm{Aut}(\Lambda)|} \Theta_\Lambda(z) = \sum_{\Lambda \in C_l} \frac{1}{|\mathrm{Aut}(\Lambda)|} E_{2k}(z)$$

To get formula $(**)$. take $n=l$, by definition we have

$$\gamma_\Lambda(T) = \#\left\{ (v_1, \cdots, v_l) \in \Lambda^l \,\middle|\, \tfrac{1}{2}(v_i, v_j) = T \right\}.$$

Now we fix any lattice Λ, take an $\mathbb{Z}$-basis $v_1, \cdots, v_l$ of Λ,

Let $T = \frac{1}{2}(v_i, v_j)$. Then for all $g \in \mathrm{Aut}(\Lambda)$, let $w_i = g v_i$.

Then $(w_i, w_j) = (g v_i, g v_j) = (v_i, v_j) = 2T$.

For any $(w_1, \cdots, w_l) \in \Lambda^l$ s.t. $(w_i, w_j) = 2T$.

Just take $g : \Lambda \to \Lambda$ s.t. $v_i = g w_i$.

Then g satisfies $(g v_i, g v_j) = (v_i, v_j) \Rightarrow g \in \mathrm{Aut}(\Lambda)$.

And also, $g v_i = v_i \ \forall i \Rightarrow g = id_\Lambda$.

So we conclude $r_\Lambda(T) = \# \mathrm{Aut}(\Lambda)$.

For other lattice $\Lambda' \in C_\ell$, if $\exists (w_1, \cdots, w_\ell) \in (\Lambda')^\ell$.

Then $(w_i, w_j) = 2T = (v_i, v_j) \ \forall i, j \Rightarrow \Lambda \xrightarrow{\sim} \Lambda'$ (via $v_i \mapsto w_i$)

So $r_{\Lambda'}(T) = 0 \ \forall \Lambda' \in C_\ell$ with $\Lambda' \neq \Lambda$.

Now we look at the Fourier coefficient at T.

$$a_T(LHS) = \sum_{\Lambda \in C_\ell} \frac{1}{|\mathrm{Aut}(\Lambda)|} r_\Lambda(T) = 1.$$

For the RHS, we need to use the results about the Fourier coefficient of the Siegel Eisenstein series $E_{2k}(Z)$.

$$a_T(RHS) = \left(\frac{B_{\frac{\ell}{2}}}{\ell} \cdot \prod_{j=1}^{\frac{n}{2}-1} \frac{B_{2j}}{4j} \right)^{-1}$$

(ref: Euler products and Eisenstein series, Shimura)

which give us the formula (**).