EICHLER-SHIMURA ISOMORPHISM

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ABSTRACT. This is the final paper report on Eichler-Shimura isomorphism. Basically, this is the reorganization of the course note [Wie18].

1. Introduction

Let $\Gamma \subseteq SL_2(\mathbb{Z})$ be a torsion free congruence subgroup. Consider the (open) modular curve $Y_\Gamma = \Gamma\backslash\mathbb{H}$ and the compactified one X_Γ. The Hodge decomposition provides a decomposition

$$H^{1,0}(X_\Gamma) \oplus H^{0,1}(X_\Gamma) \cong H^1_{dR}(X_\Gamma).$$

Recall [DS05, §3.3] that $f(z) \mapsto f(z)dz$ defines an isomorphism

$$S_2(\Gamma) \xrightarrow{\sim} H^0(X, \Omega_{X_\Gamma}) \cong H^{1,0}(X_\Gamma).$$

Similarly, $\overline{f(z)} \mapsto \overline{f(z)}d\bar{z}$ defines an isomorphism $\overline{S_2(\Gamma)} \to H^{0,1}(X_\Gamma)$, where $\overline{S_2(\Gamma)} := S_2(\Gamma) \otimes_{\mathbb{C},\sigma} \mathbb{C}$ and σ is the complex conjugation. In particular, we obtain an isomorphism

$$S_2(\Gamma) \oplus \overline{S_2(\Gamma)} \xrightarrow{\sim} H^1_{dR}(X_\Gamma) \cong H^1_{sing}(X_\Gamma, \mathbb{C}).$$

Let U be the intersection of Y_Γ with a sufficiently small neighborhood of cusps in X_Γ. Then there is an exact sequence

$$0 \longrightarrow H^1_{sing}(X_\Gamma, \mathbb{C}) \longrightarrow H^1_{sing}(Y_\Gamma, \mathbb{C}) \longrightarrow H^1_{sing}(U, \mathbb{C}).$$

This follows from, for example, the excision and the long exact sequence for relative cohomology groups. By deRham theorem, $H^1(Y_\Gamma, \mathbb{C}) \cong H^1_{dR}(Y_\Gamma)$, but this time $H^0(Y_\Gamma, \Omega_{Y_\Gamma}) \to H^1_{dR}(Y_\Gamma)$ is not injective due to the non-compactness of Y_Γ. Nevertheless, consider the composition

$$M_2(\Gamma) \longrightarrow H^0(Y_\Gamma, \Omega_{Y_\Gamma}) \longrightarrow H^1_{dR}(Y_\Gamma) \cong H^1_{sing}(Y_\Gamma, \mathbb{C})$$

Then we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & S_2(\Gamma) \arrow[r] \arrow[d] & M_2(\Gamma) \arrow[r] \arrow[d] & M_2(\Gamma)/S_2(\Gamma) \arrow[d] \\
0 \arrow[r] & H^1_{sing}(X_\Gamma, \mathbb{C}) \arrow[r] & H^1_{sing}(Y_\Gamma, \mathbb{C}) \arrow[r] & H^1_{sing}(U, \mathbb{C}).
\end{tikzcd}$$

The map $M_2(\Gamma) \to H^1_{sing}(U, \mathbb{C}) \cong \mathbb{C}^{\oplus\{\text{cusps}\}}$ can be described by residues, so its kernel is exactly $S_2(\Gamma)$. This shows the rightmost arrow is injective. We already see the first vertical arrow is injective, so this implies so is the middle. Comparing the dimension yields an isomorphism

$$M_2(\Gamma) \oplus \overline{S_2(\Gamma)} \xrightarrow{\sim} H^1_{sing}(Y_\Gamma, \mathbb{C}).$$

The image of $H^1_{\mathrm{sing}}(X_\Gamma, \mathbb{C})$ in $H^1_{\mathrm{sing}}(Y_\Gamma, \mathbb{C})$ coincides of that of $H^1_{\mathrm{sing},c}(Y_\Gamma, \mathbb{C})$ in $H^1_{\mathrm{sing}}(Y_\Gamma, \mathbb{C})$; we denote it by $H^1_{\mathrm{inn}}(Y_\Gamma, \mathbb{C})$, the inner cohomology of Y_Γ. Hence

$$\begin{tikzcd} M_2(\Gamma) \oplus \overline{S_2(\Gamma)} \arrow[r, "\sim"] & H^1_{\mathrm{sing}}(Y_\Gamma, \mathbb{C}) \\ S_2(\Gamma) \oplus \overline{S_2(\Gamma)} \arrow[r, "\sim"] \arrow[u, "\subseteq"'] & H^1_{\mathrm{inn}}(Y_\Gamma, \mathbb{C}) \arrow[u, "\subseteq"'] \end{tikzcd}$$

Since Γ is torsion free, the fundamental group of Y_Γ is isomorphic to Γ and $\pi : \mathbb{H} \to Y_\Gamma$ is the universal cover. Then $H^1_{\mathrm{dR}}(Y_\Gamma) \cong H^1_{\mathrm{sing}}(Y_\Gamma, \mathbb{C}) \cong \mathrm{Hom}_{\mathbf{Grp}}(\Gamma, \mathbb{C}) = H^1(\Gamma, \mathbb{C})$; explicitly, a form ω is sent to the homomorphism $\gamma \mapsto \displaystyle\int_{z_0}^{\gamma z_0} \pi^*\omega$, where z_0 is a fixed base point of $\mathbb{H}$. In other words, we obtain an isomorphism

$$\begin{aligned} M_2(\Gamma) \oplus \overline{S_2(\Gamma)} &\longrightarrow H^1(\Gamma, \mathbb{C}) \\ (f, \overline{g}) &\longmapsto \gamma \mapsto \int_{z_0}^{\gamma z_0} f(z)dz + \int_{z_0}^{\gamma z_0} \overline{g(z)}d\overline{z} \end{aligned}$$

This is the easiest case of the so-called **Eichler-Shimura isomorphism**. In the following this is going to be generalized to higher weight k. The constant sheaf $\mathbb{C}$ played in the cohomology will be replaced by a certain local system over Y_Γ. Our exposition will use group cohomology exclusively. For an approach using sheaf cohomology of local systems, see [DI95, §12.2] and [Li20, §9].

2. Review on group cohomology

2.1. **Inhomogeneous complexes.** Fix a unital ring R and a group G. Let M be an $R[G]$-module. Define $C^0_{\mathrm{inhom}}(G, M) = M$, and for $p \geqslant 1$, set $C^p_{\mathrm{inhom}}(G, M) := \mathrm{Hom}_{\mathbf{Set}}(G^p, M)$. These have natural R-module structures induced from that on M. Define the coboundary map

$$\partial^p : C^p_{\mathrm{inhom}}(G, M) \longrightarrow C^{p+1}_{\mathrm{inhom}}(G, M)$$

by the formula:

$$\begin{aligned} &\partial u(g_1, \dots, g_{p+1}) \\ &\quad = g_1.u(g_2, \dots, g_{p+1}) + \sum_{i=1}^{p} (-1)^i u(g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+1}, \dots, g_{p+1}) + (-1)^{p+1} u(g_1, \dots, g_p). \end{aligned}$$

It is straightforward to verify that $\partial^{p+1} \circ \partial^p = 0$ for $p \geqslant 0$, so $(C^\bullet_{\mathrm{inhom}}(G, M), \partial^\bullet)$ forms a complex, called the **inhomogeneous complex**. As usual, put

$$\begin{aligned} Z^p(G, M) &= \ker(\partial^p : C^p_{\mathrm{inhom}}(G, M) \to C^{p+1}_{\mathrm{inhom}}(G, M)) \\ B^p(G, M) &= \begin{cases} 0 & \text{, if } p = 0 \\ \mathrm{Im}(\partial^{p-1} : C^{p-1}_{\mathrm{inhom}}(G, M) \to C^p_{\mathrm{inhom}}(G, M)) & \text{, if } p \geqslant 1 \end{cases} \\ H^p(G, M) &= Z^p(G, M)/B^p(G, M). \end{aligned}$$

The group $H^p(G, M)$ is the p-th cohomology group of G with coefficients in M.

Let us look at the cohomology groups in low dimension. For $x \in M = C^0_{\mathrm{inhom}}(G, M)$, by definition

$$\partial x(g) = gx - x.$$

Therefore $H^0(G, M) = Z^0(G, M) = M^G := \{x \in G \mid gx = x \text{ for all } g \in G\}$. For $u \in C^1_{\mathrm{inhom}}(G, M)$,

$$\partial u(g_1 g_2) = g_1.u(g_2) - u(g_1 g_2) + u(g_1).$$

This shows that

$$Z^1(G,M) = \{u : G \to M \mid u(xy) = x.u(y) + u(x)\}.$$

In particular, for $u \in Z^1(G,M)$, we have $u(1) = 0$ and $u(x^{-1}) = -x^{-1}u(x)$.

2.2. **Derived functors.** Consider a complex

$$\cdots \xrightarrow{d} R[G^3] \xrightarrow{d} R[G^2] \xrightarrow{d} R[G^1] \xrightarrow{\varepsilon} R \longrightarrow 0$$

defined as follows. For $g \in G$, set $\varepsilon(g) = 1 \in R$ and extend R-linearly to a map $\varepsilon : R[G] \to R$. For $p \geqslant 2$ and $(g_1, \dots, g_p) \in G^p$,

$$d(g_1, \dots, g_p) := \sum_{j=0}^{p} (-1)^j (g_0, \dots, g_{j-1}, g_{j+1}, \dots, g_p).$$

One checks quickly that $d \circ d = 0$. Fix $s \in G$ and define $h : R[G^p] \to R[G^{p+1}]$ by $h(g_1, \dots, g_p) = (s, g_1, \dots, g_p)$. One checks $d \circ h + h \circ d = 1$, so the above complex is in fact exact. This is a free solution of the trivial $R[G]$-module R, called the **bar resolution**.

For each $R[G]$-module M, applying $\mathrm{Hom}_R(\cdot, M)$ to the bar resolution yields the **standard resolution** of M:

$$0 \longrightarrow M \longrightarrow \mathrm{Hom}_R(R[G], M) \longrightarrow \mathrm{Hom}_R(R[G^2], M) \longrightarrow \cdots .$$

We put $X^p(G,M) = \mathrm{Hom}_R(R[G^{p+1}], M)$ $(p \geqslant 0)$ and let G act on $X^p(G,M)$ by conjugation: $(g.f)(x) = g.f(g^{-1}x)$ for $f \in X^p(G,M)$, $g \in G$. Define the **homogeneous complex**

$$C^p_{\mathrm{hom}}(G,M) = X^p(G,M)^G = \mathrm{Hom}_{R[G]}(R[G^{p+1}], M) \qquad (p \geqslant 0).$$

We then have a sequence

$$0 \longrightarrow M^G \longrightarrow C^0_{\mathrm{hom}}(G,M) \longrightarrow C^1_{\mathrm{hom}}(G,M) \longrightarrow \cdots$$

An element $f \in C^p_{\mathrm{hom}}(G,M)$ satisfies $f(sg_1, \dots, sg_{p+1}) = sf(g_1, \dots, g_{p+1})$, so its values are completely determined by its evaluation on elements of the form $(1, g_1, g_1g_2, \dots, g_1 \cdots g_p)$. We then obtain an isomorphism

$$\begin{aligned} C^p_{\mathrm{hom}}(G,M) &\xrightarrow{\sim} C^p_{\mathrm{inhom}}(G,M) \\ f &\longmapsto u_f \end{aligned}$$

by the formula

$$u_f(g_1, \dots, g_p) = f(1, g_1, \dots, g_1 \cdots g_p).$$

By transferring the induced coboundary map on $C^p_{\mathrm{hom}}(G,M)$ to $C^p_{\mathrm{inhom}}(G,M)$, we see it coincides with the previously defined ∂. In particular, this shows

$$H^\bullet(G,M) \cong H^\bullet(C^\bullet_{\mathrm{hom}}(G,M)) = H^\bullet(X^\bullet(G,M)^G).$$

Note that since the bar resolution is a free resolution, it follows that $H^\bullet(G,M) \cong \mathrm{Ext}^\bullet_{R[G]}(R,M)$ as well. Since $\mathrm{Hom}_{R[G]}(R,M) \cong M^G$ functorially, we further see that $H^\bullet(G,\cdot) \cong R^\bullet(\cdot)^G$, the right derived functor of $(\cdot)^G$.

Lemma 2.1. The $R[G]$-module $X^p(G,M)$ $(p \geqslant 0)$ is $(\cdot)^G$-acyclic.

Proof. Note that $X^0(G, X^{p-1}(G,M)) \cong X^p(G,M)$ $(p \geqslant 1)$ as $R[G]$-modules. Explicitly, the maps

$$\begin{aligned} \mathrm{Hom}_R(R[G], \mathrm{Hom}_R(R[G^p], M)) &\longrightarrow \mathrm{Hom}_R(R[G^{p+1}], M) \\ T &\longmapsto f_T : (g, g_1, \dots, g_p) \mapsto T(g)(g_1, \dots, g_p) \\ T_f : g \mapsto [(g_1, \dots, g_p) \mapsto f(g, g_1, \dots, g_p)] &\longleftarrow f \end{aligned}$$

are R[G]-isomorphism. For $x \in G$,

$$\begin{aligned}(x.f_T)(g, g_1, \dots, g_p) &= x.f_T(x^{-1}g, x^{-1}g_1, \dots, x^{-1}g_p) = x.T(x^{-1}g)(x^{-1}g_1, \dots, x^{-1}g_p) \\ &= (x.T(x^{-1}g))(g_1, \dots, g_p) = ((x.T)(g))(g_1, \dots, g_p) = f_{x.T}(g, g_1, \dots, g_p)\end{aligned}$$

so $T \mapsto f_T$ is G-equivariant. It is clear that the maps are inverse to each other. Hence to show the lemma it suffices to show the R[G]-module $\mathrm{Hom}_R(R[G], M)$ is $(\cdot)^G$-acyclic.

Observe that for any R[G]-module N, we have an R-isomorphism

$$\begin{aligned}\mathrm{Hom}_R(N, M) &\longrightarrow \mathrm{Hom}_{R[G]}(N, \mathrm{Hom}_R(R[G], M)) \\ \varphi &\longmapsto n \mapsto [g \mapsto g.\varphi(g^{-1}n)] \\ n \mapsto \phi(n)(1_G) &\longleftarrow\!\shortmid\ \phi,\end{aligned}$$

In particular, $X^\bullet(G, M) \cong C^\bullet_{\mathrm{hom}}(G, \mathrm{Hom}_R(R[G], M))$ as R-modules. Furthermore, it is an isomorphism of complexes. Hence

$$R^p(\cdot)^G(\mathrm{Hom}_R(R[G], M)) \cong H^p(G, \mathrm{Hom}_R(R[G], M)) \cong H^p(C^\bullet_{\mathrm{hom}}(G, \mathrm{Hom}_R(R[G], M))) \cong H^p(X^\bullet(G, M)) = 0$$

for $p \geqslant 1$. This finishes the proof. □

Corollary 2.1.1. Let M be an arbitrary R[G]-module.

(i) The standard resolution $X^\bullet(G, M)$ is a $(\cdot)^G$-acyclic resolution of M.
(ii) The module $\mathrm{Hom}_R(R[G], M)$ is acyclic.

2.3. **Cohomology of cyclic groups.** Let R be a unital ring, let $G = \langle\sigma\rangle$ be a finite cyclic group with order n. Let $\mathrm{Tr} = 1 + \sigma + \cdots + \sigma^{n-1} \in \mathbb{Z}[G]$; then $(1-\sigma)\mathrm{Tr} = \mathrm{Tr}(1-\sigma) = 0$ in $\mathbb{Z}[G]$. The complex

$$\cdots \xrightarrow{\mathrm{Tr}} R[G] \xrightarrow{1-\sigma} R[G] \xrightarrow{\mathrm{Tr}} R[G] \xrightarrow{1-\sigma} R[G] \xrightarrow{\varepsilon} R \longrightarrow 0.$$

is then a free resolution of R as a R[G]-module. This shows $H^p(G, M) = H^{p+2}(G, M)$ for all $p \geqslant 1$.

Assume $G = \langle\sigma\rangle$ is an infinite cyclic group. There is an isomorphism

$$\begin{aligned}Z^1(G, M) &\longrightarrow M \\ u &\longmapsto u(\sigma).\end{aligned}$$

Also, for $m \geqslant 1$, we have $1 - \sigma^m = (1-\sigma)(1 + \cdots + \sigma^{m-1})$, implying that the image of $B^1(G, M)$ under the above isomorphism is $(1-\sigma)M$ for all G-modules M. This shows

$$H^1(G, M) \cong M/(1-\sigma)M.$$

For the higher, consider the short exact sequence

$$0 \longrightarrow R[G] \xrightarrow{1-\sigma} R[G] \xrightarrow{\varepsilon} R \longrightarrow 0.$$

This is a free resolution of R as R[G]-modules, so $H^p(G, M) = 0$ for $p \geqslant 2$. This also recovers the above interpretation of H^1 abstractly.

2.4. **Functoriality.** Let $\varphi : H \to G$ be a group homomorphism. Let M be an $R[G]$-module, which can be viewed as an $R[H]$-module via φ, and we denote it by M^{φ}. We have an inclusion $M^G \subseteq (M^{\varphi})^H$. By universality of derived functors, this extends uniquely to maps between higher cohomology modules[1]

$$(\varphi^*)^p : H^p(G, M) \longrightarrow H^p(H, M^{\varphi}).$$

If $\varphi : H \to G$ is an inclusion, write $M^{\varphi} = \mathrm{Res}^G_H M = M$. In this case, we call $(\varphi^*)^p$ the **restriction**, and write it as

$$\mathrm{res}^p_{G|H} : H^p(G, M) \longrightarrow H^p(H, M).$$

Let $\phi : M \to N$ be an $R[G]$-homomorphism. It restricts to a map $\phi : M^G \to N^G$, so by universality it induces uniquely to maps between higher cohomology modules

$$\phi^p_* : H^p(G, M) \longrightarrow H^p(G, N)$$

Let $H \leqslant G$, and for $g \in G$ put $H^g := g^{-1}Hg \leqslant G$. Let M be an $R[G]$-module. The conjugation $x \mapsto gxg^{-1}$ restricts to a group isomorphism $H^g \xrightarrow{\sim} H$ which makes M the $R[H^g]$ module M^g by $g^{-1}hg.m := hm$. This induces a map $H^\bullet(H, M) \to H^\bullet(H^g, M^g)$. The map $M^g \to M$ given by $m \mapsto g^{-1}m$ is H^g-equivariant, where H^g acts on M since $H^g \leqslant G$. This then induces a map $H^\bullet(H^g, M^g) \to H^\bullet(H^g, M)$. They are in fact both isomorphisms, and their composition gives

$$\mathrm{conj}_g : H^\bullet(H, M) \xrightarrow{\sim} H^\bullet(H^g, M).$$

This is called the **conjugation by** g.

Lemma 2.2.

(i) The conjugation $\mathrm{conj}_g : H^\bullet(G, M) \to H^\bullet(G, M)$ is the identity map for all $g \in G$

(ii) The image $\mathrm{res}_{G|H} : H^\bullet(G, M) \to H^\bullet(H, M)$ lies in $H^\bullet(H, M)^G$.

(iii) The composition

$$H^\bullet(H, M) \xrightarrow{\mathrm{conj}_g} H^\bullet(H^g, M) \xrightarrow{\mathrm{conj}_h} H^\bullet((H^g)^h, M)$$

is the map $\mathrm{conj}_{gh} : H^\bullet(H, M) \to H^\bullet(H^{gh}, M)$.

Proof. For (i) and (iii), by the universality it suffices to check the degree 0 map. This is clear. (ii) is a direct computation. □

By (iii), we see G acts on the cohomology $H^\bullet(H, M)$. If H is normal, by (i) we have $H^\bullet(H, M)^G = H^\bullet(H, M)^{G/H}$. Hence by (ii) the restriction can be viewed as a map

$$\mathrm{res}^p_{G|H} : H^p(G, M) \longrightarrow H^p(H, M)^{G/H}$$

in the case $H \trianglelefteq G$.

Suppose $H \trianglelefteq G$ is a normal subgroup. Then M^H is naturally an $R[G/H]$-module. If we denote by $\pi : G \to G/H$ the quotient map, then we have the **inflation**

$$\mathrm{inf}^p_{G|H} : H^p(G/H, M^H) \xrightarrow{(\pi_*)^p} H^p(G, M^H) \longrightarrow H^p(G, M)$$

[1]Maps obtained in this way automatically commute with the connecting homomorphisms of cohomology groups. The same remark holds for all maps obtained in this way. Concisely speaking, these are called the morphisms of δ-functors.

Suppose $H \leqslant G$ has finite index. Let $\{g_i\}_{i=1}^n$ be a system of representatives of $H\backslash G$. We then have the norm $N_{G|H} = \sum_{i=1}^n g_i \in R[G]$, and hence a map $M^H \to M^G$ defined by $m \mapsto N_{G|H} m$. By universality, this extends uniquely to maps between cohomology

$$\mathrm{cores}_{G|H}^p : H^p(H, M) \longrightarrow H^p(G, M)$$

called the **corestriction**. Here is a subtlety. We must show the functor $M \mapsto H^p(H, M)$ is a universal δ-functor on the category of G-modules, and we prove it is erasable[2]. We have a bijection of sets $H \times H\backslash G \to G$ given by $(h, g_i) \mapsto hg_i$. Then

$$\begin{aligned} \mathrm{Hom}_R(R[G], M) \cong \mathrm{Hom}_{\mathbf{Set}}(G, M) \cong \mathrm{Hom}_{\mathbf{Set}}(H \times H\backslash G, M) &\cong \mathrm{Hom}_{\mathbf{Set}}(H, \mathrm{Hom}_{\mathbf{Set}}(H\backslash G, M)) \\ &\cong \mathrm{Hom}_R(R[H], \mathrm{Hom}_{\mathbf{Set}}(G/H, M)) \end{aligned}$$

as $R[H]$-modules (H acts on $H \times H\backslash G$ from the left), and such module is acyclic by Corollary 2.1.1.(ii). Also, the map $m \mapsto N_{G|H} m$ on M^H is independent of the choice of representatives, so we can write it as $m \mapsto \sum_{g \in H\backslash G} gm$.

Lemma 2.3. Let R be a unital ring and M an $R[G]$-module.

(i) Let $H \leqslant G$ have finite index. Then $\mathrm{cores}_{G|H} \circ \mathrm{res}_{G|H} = [G : H]$.
(ii) Suppose $\#G < \infty$. If $\#G \in R^\times$, then $H^p(G, M) = 0$ for $p \geqslant 0$.

Proof. Let $\{g_i\}$ be as above. Then for $m \in M^G$, clearly $N_{G|H} m = [G : H]m$. This shows $\mathrm{cores}_{G|H}^0 \circ \mathrm{res}_{G|H}^0 = [G : H]$, the multiplication by $[G : H]$. It then follows from universality that (i) holds for the higher. (ii) follows from taking $H = \{1\}$. □

2.5. **Hochschild-Serre spectral sequence.** Let $H \trianglelefteq G$ and let M be an $R[G]$-module. Consider the first quadrant double complex $X^p(G/H, X^q(G, M)^H)^{G/H}$. The sign convention is that we twist the vertical differentials by $(-1)^p$. Then the Hochschild-Serre spectral sequence is the resulting spectral sequence

$$E_2^{p,q} \Rightarrow H^{p+q}(\mathrm{Tot}^\bullet(X^\bullet(G/H, X^\bullet(G, M)^H)^{G/H}).$$

We compute $E_2^{p,q}$ and the limit term. By definition,

$$E_2^{p,q} = H^p(H^q(X^\bullet(G/H, X^\bullet(G, M)^H)^{G/H})).$$

Lemma 2.4. The functor $M \mapsto X^p(G, M)^G$ is exact.

Proof. This follows from Corollary 2.1.1.(i). □

In particular, applying this lemma to the group G/H, we see

$$H^q(X^p(G/H, X^\bullet(G, M)^H)^{G/H}) = X^p(H^q(X^\bullet(G, M)^{G/H}))^{G/H} = X^p(H^q(H, M))^{G/H}$$

and hence $E_2^{p,q} = H^p(G/H, H^q(H, M))$. The limit term is computed via the transpose complex. By Corollary 2.1.1.(i) again, we have

$$H^p(X^\bullet(X^q(G, M)^H)^{G/H}) = \begin{cases} X^q(G, M)^G & , \text{if } p = 0 \\ 0 & , \text{if } p \geqslant 1. \end{cases}$$

This shows

$$H^{p+q}(\mathrm{Tot}^\bullet(X^\bullet(G/H, X^\bullet(G, M)^H)^{G/H}) = H^{p+q}(G, M).$$

Hence the Hochschild-Serre spectral sequence now takes the form

$$E_2^{p,q} = H^p(G/H, H^q(H, M)) \Rightarrow H^{p+q}(G, M)$$

[2] Effaceable in Hartshorne.

2.6. **Inflation-restriction exact sequence.**

Lemma 2.5. Consider the Hochschild-Serre spectral sequence.

(1) The edge map $H^p(G/H, M^H) = E_2^{p,0} \to H^p(G, M)$ coincides with the inflation $\inf^p_{G|H}$.
(2) The edge map $H^q(G, M) \to E_2^{0,q} = H^q(H, M)^{G/H}$ coincides with the restriction $\mathrm{res}^q_{G|H}$.

Proof. See [Mac95, Chapter XI 9. 10.] and [NSW13, Chapter II §4]. □

Consider its associated five term exact sequence

$$\begin{tikzcd} 0 \arrow[r] & E_2^{1,0} \arrow[r] \arrow[d, equal] & H^1(G,M) \arrow[r] & E_2^{0,1} \arrow[r] \arrow[d, equal] & E_2^{2,0} \arrow[r] \arrow[d, equal] & H^2(G,M) \\ & H^1(G/H, M^H) & & H^1(H,M)^{G/H} & H^2(G/H, M^H) & \end{tikzcd}$$

It follows from the previous lemma that the second and the last arrows are inflations, and the third arrow is the restriction. In other words, we have the so-called **inflation-restriction exact sequence**

$$0 \longrightarrow H^1(G/H, M^H) \xrightarrow{\inf^1_{G|H}} H^1(G,M) \xrightarrow{\mathrm{res}^1_{G|H}} H^1(H,M)^{G/H} \longrightarrow H^2(G/H, M^H) \xrightarrow{\inf^2_{G|H}} H^2(G,M).$$

The fourth arrow is called the **transgression**, and it is given by the differential of the E_2 page of the Hochschild-Serre spectral sequence. For a later use, we mention some application.

Lemma 2.6. Let G be a group, $H \leqslant G$ a subgroup of finite index and $K \leqslant G$ a finite normal subgroup. Let R be a unital ring such that $\#K \in R^\times$, and let V be an $R[H]$-module. Suppose either $K \cap H = \{1_G\}$, or $K \subseteq H$ and K acts on V trivially. Then the inflation induces an isomorphism

$$\inf_{G|K} : H^1(G/K, \mathrm{Ind}_{H/K}^{G/K} V) \longrightarrow H^1(G, \mathrm{Ind}_H^G V) \ .$$

Here H/K means $H/K \cap H \subseteq G/K$, and the induced module is defined in the next subsection.

Proof. The assumption implies V is naturally an $R[H/K]$-module. If $K \cap H = \{1_G\}$, $H \cong H/K$ and $(\mathrm{Ind}_H^G V)^K \cong \mathrm{Ind}_{H/K}^{G/K} V$. If $K \subseteq H$ and K acts on V trivially, then $(\mathrm{Ind}_H^G V)^K \cong \mathrm{Ind}_{H/K}^{G/K} V$. The displayed isomorphism now follows from the inflation-restriction exact sequence and Lemma 2.3.(ii).

□

Similarly, the same proof gives

Lemma 2.7. Let G be a group, $H \leqslant G$ and let $K \leqslant G$ be a finite normal subgroup. Let R be a unital ring with $\#K \in R^\times$, and let M be an $R[G]$-module. Suppose either $K \subseteq H$ and K acts on M trivially, or $K \cap H = \{1_G\}$. Then the inflation induces an isomorphism

$$\inf_{H|H \cap K} : H^1(H/K, M) \longrightarrow H^1(H, M).$$

Here H/K means $H/K \cap H \subseteq G/K$.

2.7. **Shapiro's lemma.** Let $H \leqslant G$. For any $R[H]$-module M, define the **induced module**

$$\mathrm{Ind}_H^G M := \mathrm{Hom}_{R[H]}(R[G], M),$$

For $f \in \mathrm{Ind}_H^G M$ and $x \in G$, define $x.f \in \mathrm{Ind}_H^G$ by $(x.f)(g) = f(gx)$. Then $\mathrm{Ind}_H^G M$ is an $R[G]$-module, and clearly, $M \mapsto \mathrm{Ind}_H^G M$ defines a functor.

Lemma 2.8.

(i) The functor $M \mapsto \mathrm{Ind}_H^G M$ is exact.
(ii) For any $R[H]$-module M, we have $\mathrm{Hom}_R(R[G], M) \cong \mathrm{Ind}_H^G \mathrm{Hom}_R(R[H], M)$ as G-modules.

Proof. Note that we have functorial isomorphisms

$$\mathrm{Hom}_{R[H]}(R[G], M) \cong \{f : G \to M \mid f(hg) = h.f(g)\} \cong \mathrm{Hom}_{\mathbf{Set}}(H\backslash G, M)$$

of G-modules, where G acts on the last two set by right translation. The functor $M \mapsto \mathrm{Hom}_{\mathbf{Set}}(H\backslash G, M)$ is clearly exact. This shows (i). For (ii), using the bijection $H \times H/G \to G$, we have

$$\begin{aligned}\{f : G \to \mathrm{Hom}_{\mathbf{Set}}(H, M) \mid f(hg) = hf(g)\} \cong \mathrm{Hom}_{\mathbf{Set}}(H\backslash G, \mathrm{Hom}_{\mathbf{Set}}(H, M)) &\cong \mathrm{Hom}_{\mathbf{Set}}(H \times H\backslash G, M)\\ &\cong \mathrm{Hom}_{\mathbf{Set}}(G, M)\end{aligned}$$

as G-modules, where G acts on every set above by right translation. Consider the bijection

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Set}}(G, M) &\longrightarrow \mathrm{Hom}_{\mathbf{Set}}(G, M)\\ f &\longmapsto F_f : g \mapsto g.f(g^{-1})\end{aligned}.$$

Here G acts on the domain by conjugation, while acts on the codomain by right translation. For $x, g \in G$ and $f : G \to M$, compute

$$F_{x.f}(g) = g.(x.f)(g^{-1}) = gx.f(x^{-1}g^{-1}) = F_f(gx) = (x.F_f)(g).$$

Hence the two different G-actions on $\mathrm{Hom}_R(R[G], M)$ are isomorphic. This proves (ii). □

The lemma shows that the functors $M \mapsto H^p(G, \mathrm{Ind}_H^G M)$ $(p \geqslant 0)$ are erasable, so $H^\bullet(G, \mathrm{Ind}_H^G(\cdot))$ is a universal δ-functor. Consider the functorial bijection

$$\begin{aligned}H^0(G, \mathrm{Ind}_H^G M) &\longrightarrow H^0(H, M)\\ f &\longmapsto f(1).\end{aligned}$$

By the universality, we obtain the so-called **Shapiro's lemma**.

Lemma 2.9. The canonical bijection $(\mathrm{Ind}_H^G M)^G \cong M^H$ extends uniquely to an isomorphism

$$\mathrm{sh}^\bullet_{G|H} : H^\bullet(G, \mathrm{Ind}_H^G M) \longrightarrow H^\bullet(H, M).$$

2.8. **Mackey's theory.** Let $H \leqslant G$. For an R[G]-module M, let $\mathrm{Res}_H^G M$ denote the module M viewed as an R[H]-module. Let $K \leqslant G$ be another subgroup. Then if V is an R[H]-module, there is an R[K]-isomorphism

$$\begin{aligned}\mathrm{Res}_K^G \mathrm{Ind}_H^G V &\xrightarrow{\sim} \prod_{g\in H\backslash G/K} \mathrm{Ind}_{K\cap g^{-1}Hg}^K(\mathrm{Res}^H_{H\cap gKg^{-1}} V)_g\\ \phi : R[G] \to V &\longrightarrow (k \mapsto \phi(k))_{g\in H\backslash G/K}\end{aligned}$$

where for $x \in (\mathrm{Res}^H_{H\cap gKg^{-1}} V)_g$ and $h \in H$ with $g^{-1}hg \in K$, $g^{-1}hg.x := h.x$. Indeed, the last space is

$$\begin{aligned}\mathrm{Ind}_{K\cap g^{-1}Hg}^K(\mathrm{Res}^H_{H\cap gKg^{-1}} V)_g &= \mathrm{Hom}_{R[K\cap g^{-1}Hg]}(R[K], (\mathrm{Res}^H_{H\cap gKg^{-1}} V)_g)\\ &\cong \mathrm{Hom}_{R[gKg^{-1}\cap H]}(R[gKg^{-1}], \mathrm{Res}^H_{H\cap gKg^{-1}} V)\end{aligned}$$

where the last isomorphism is given by conjugation. The bijection

$$\begin{aligned}(gKg^{-1} \cap H)\backslash gKg^{-1} &\longrightarrow H\backslash HgK\\ gkg^{-1} &\longrightarrow Hgk\end{aligned}$$

implies that we have an isomorphism

$$\begin{aligned} \mathrm{Res}^G_K \mathrm{Ind}^G_H V &\xrightarrow{\sim} \prod_{g\in H\backslash G/K} \mathrm{Hom}_{R[gKg^{-1}\cap H]}(R[gKg^{-1}], \mathrm{Res}^H_{H\cap gKg^{-1}} V) \\ \phi : R[G]\to M &\longmapsto (gkg^{-1}\mapsto \phi(gkg^{-1}))_g \end{aligned}$$

This proves the claimed isomorphism. Combined with Shapiro's lemma, we have

Lemma 2.10. Let $H, K \leqslant G$ and let M be an $R[H]$-module. Then we have an isomorphism

$$H^\bullet(K, \mathrm{Ind}^G_H V) \cong \prod_{g\in H\backslash G/K} H^\bullet(gKg^{-1}\cap H, V).$$

For a later use, let V be an $R[H]$-module, and consider the diagram

$$\begin{tikzcd} H^\bullet(H,V) \arrow[r, "\mathrm{res}"] & \prod_{g\in H\backslash G/K} H^\bullet(gKg^{-1}\cap H, V) \\ H^\bullet(G, \mathrm{Ind}^G_H V) \arrow[u, "\mathrm{sh}_{G|H}"] \arrow[r, "\mathrm{res}"] & H^\bullet(K, \mathrm{Ind}^G_H V). \arrow[u, "\mathrm{Mackey}"'] \end{tikzcd}$$

Lemma 2.11. The above diagram commutes.

Proof. At degree 0, the diagram clearly commutes. Since all maps involved are morphisms of δ-functor, it follows from the universality of $V\mapsto H^\bullet(G, \mathrm{Ind}^G_H V)$ that the diagram commutes for all degrees. ☐

2.9. **Cup products.** Let M, N be $R[G]$-modules. For $p, q \geqslant 0$, define the pairing

$$\begin{aligned} C^p_{\mathrm{hom}}(G,M)\times C^q_{\mathrm{hom}}(G,N) &\xrightarrow{\cup} C^{p+q}_{\mathrm{hom}}(G, M\otimes_R N) \\ (u,v) &\longmapsto (u\cup v)(g_1,\dots,g_{p+q+1}) := u(g_1,\dots,g_{p+1})\otimes v(g_{p+1},\dots,g_{p+q+1}). \end{aligned}$$

It is straightforward to check that

$$\partial(u\cup v) = (\partial u)\cup v + (-1)^p u\cup(\partial v),$$

so it induces maps on cohomology groups

$$H^p(G,M)\times H^q(G,N) \xrightarrow{\cup} H^{p+q}(G, M\otimes_R N)$$

This is called the **cup product**. On the inhomogeneous complex, the pairing becomes

$$\begin{aligned} C^p_{\mathrm{inhom}}(G,M)\times C^q_{\mathrm{inhom}}(G,N) &\xrightarrow{\cup} C^{p+q}_{\mathrm{inhom}}(G, M\otimes_R N) \\ (u,v) &\longmapsto (u\cup v)(g_1,\dots,g_{p+q}) := u(g_1,\dots,g_p)\otimes g_1\cdots g_p.v(g_{p+1},\dots,g_{p+q}), \end{aligned}$$

Lemma 2.12.

(i) The cup product on H^0 is the natural map $M^G\times N^G\to (M\otimes_R N)^G$.
(ii) The cup product is functorial in M and N in an obvious sense.
(iii) The cup product is super-commutative and associative in an obvious sense.

Suppose L is another $R[G]$-module, and we have an $R[G]$-homomorphism $\varphi : M\otimes_R N\to L$. Composing with the cup product, we obtain another map

$$\begin{aligned} H^p(G,M)\times H^q(G,N) &\longrightarrow H^{p+q}(G,L) \\ (u,v) &\longmapsto \varphi^{p+q}_*(u\cup v) \end{aligned}$$

which is sometimes also called the cup product.

2.10. **Mayer-Vietoris sequence.** We copy the discussion from [Bie76]. Let G be a group and let M be an $R[G]$-module. A derivation[3] on G is a map $d : G \to M$ satisfying $d(xy) = d(x) + xd(y)$ for $x, y \in G$. Such a map extends uniquely to an R-homomorphism $d : R[G] \to M$ satisfying $d(xy) = \varepsilon(y)d(x) + xd(y)$ for all $x, y \in R[G]$. If we denote by $\mathrm{Der}(G, M)$ the R-module of all derivations on G, then there is a functorial isomorphism

$$\begin{aligned} \mathrm{Der}(G, M) &\longrightarrow \mathrm{Hom}_{R[G]}(I_G, M) \\ d &\longmapsto [g - 1 \mapsto d(g)] \end{aligned}$$

where $I_G = \bigoplus_{g \in G} R(g - 1) \leqslant R[G]$.

Let X be a set and let F be the free group based on X.

Lemma 2.13. The $R[F]$-submodule I_F is $R[F]$-free with a basis $\{x - 1 \mid x \in X\}$.

Proof. Suppose $\sum_{x \in X} r_x(x - 1) = 0$, where $r_x \in R[G]$ and $r_x \neq 0$ for finitely many x. Then $\sum_{x \in X} r_x x = \sum_{x \in X} r_x$. Suppose $r_y \neq 0$ for some $y \in X$; pick $y \in X$ such that it contains an element of maximal length among those elements in F appearing in $\{r_x \mid r_x \neq 0\}$. Then the left hand side of the identity contains a strictly longer element, which is absurd. $\square$

It follows that there is an isomorphism

$$\begin{aligned} \mathrm{Der}(G, M) &\longrightarrow M^X \\ d &\longrightarrow (d(x))_{x \in X}. \end{aligned}$$

For each $x \in X$, define the derivation $\partial_x : F \to M$ by the formula $\partial_x(y) = \delta_{xy}$, $y \in F$. If $d : F \to M$ is an arbitrary derivation, define $d' : F \to M$ by the formula

$$d'(w) = \sum_{x \in X} (\partial_x w) d(x), \qquad w \in F.$$

This is again a derivation, as for $u, v \in F$, we have

$$d'(uv) = \sum_{x \in X} (\partial_x uv) d(x) = \sum_{x \in X} (\partial_x u + u \partial_x v) d(x) = d'(u) + ud'(v).$$

Since $d'(x) = d(x)$ for $x \in X$, we see $d = d'$ identically. Applying this identity to the derivation $d : F \to R[F]$ given by $g \mapsto g - 1$, we obtain

$$w - \varepsilon(w) = \sum_{x \in X} (\partial_x w)(x - 1), \qquad w \in R[F].$$

Let G be a group with generators $S \subseteq G$. Let F be the free group on S and let $\pi : F \to G$ be the unique map induced from $S \subseteq G$. Then for any $w \in R[G]$, we have

$$w - 1 = \sum_{s \in S} \pi(\partial_s W)(s - 1)$$

where $W \in R[F]$ is any element satisfies $\pi(W) = w$. We now can prove the following

Lemma 2.14. Let G_1, G_2 be two groups and $G = G_1 * G_2$ be their free product. Then

$$\begin{gathered} 0 \longrightarrow R[G] \xrightarrow{\alpha} R[G/G_1] \oplus R[G/G_2] \xrightarrow{\varepsilon} R \longrightarrow 0 \\ x \longmapsto (xG_1, -xG_2) \\ (xG_1, 0),\ (0, yG_1) \longmapsto 1 \end{gathered}$$

[3]This is the same as a 1-cocycle with coefficients in M.

is a short exact sequence. (See [Bro82, p. II.7] for a topological proof.)

Proof. ε is clearly surjective. Let X_i be a set of generators of G_i, F the free group on $X_1 \cup X_2$ and $\pi : F \to G$ the projection. Then for $w \in R[G]$, we have

$$w - \varepsilon(w) = \sum_{x \in X_1 \cup X_2} \pi(\partial_x W)(x-1).$$

Hence $wG_i - \varepsilon(wG_i) = \sum_{x \notin X_i} \pi(\partial_x W)(x-1)G_i$. Suppose $w, w' \in R[G]$ satisfy $\varepsilon(wG_1) = -\varepsilon(w'G_2)$. Then

$$\alpha\left(\sum_{x\in X_2} \pi(\partial_x W)(x-1) - \sum_{x \in X_1} \pi(\partial_x W')(x-1) + \varepsilon(wG_1)1\right) = (wG_1, -w'G_2)$$

where W, W' are lifts of w, w' in $R[F]$. This proves the exactness at the middle. For the injectivity of α, let $0 \neq w = \sum_g a_g g \in R[G]$ with $wG_1 = 0 = wG_2$. Clearly $w \notin R1$; denote by g' the element with $a_{g'} \neq 0$ with maximal length $l(g')$. Then $l(g') > 0$. Assume, by symmetry, that g' ends with a nontrivial element in G_1. Since $\sum_g a_g g G_2 = 0$, it follows that $g'G_2 = hG_2$ for some $h \neq g'$. But g' ends with G_1, we obtain $g'y = h$ for some $y \in G_2$, a contradiction to maximality. ☐

Applying the functor $\mathrm{Hom}_R(\cdot, M)$ to the short exact sequence in the lemma, we obtain the short exact sequence

$$0 \longrightarrow M \longrightarrow \mathrm{Hom}_R(R[G/G_1], M) \oplus \mathrm{Hom}_R(R[G/G_2], M) \longrightarrow \mathrm{Hom}_R(R[G], M) \longrightarrow 0\ .$$

Recall that there is an $R[G]$-isomorphism

$$\begin{aligned} \mathrm{Ind}_{G_i}^G M &\longrightarrow \mathrm{Hom}_R(R[G/G_i], M) \\ f &\longmapsto [gG_i \mapsto g.f(g^{-1})]. \end{aligned}$$

Hence the above sequence becomes

$$0 \longrightarrow M \longrightarrow \mathrm{Ind}_{G_1}^G M \oplus \mathrm{Ind}_{G_2}^G M \longrightarrow \mathrm{Hom}_R(R[G], M) \longrightarrow 0.$$

Taking cohomology, in view of Shapiro's lemma and Corollary 2.1.1, we obtain

Corollary 2.14.1. Let G_1, G_2 be groups and $G = G_1 * G_2$. If M is an $R[G]$-module, then we have an exact sequence

$$0 \longrightarrow M^G \longrightarrow M^{G_1} \oplus M^{G_2} \longrightarrow M \longrightarrow H^1(G, M) \longrightarrow H^1(G_1, M) \oplus H^1(G_2, M) \longrightarrow 0$$

and

$$H^p(G, M) \cong H^p(G_1, M) \oplus H^p(G_2, M) \qquad (p \geqslant 2).$$

The maps $H^p(G, M) \to H^p(G_1, M) \oplus H^p(G_2, M)$ are given by restrictions. We call these sequences the **Mayer-Vietoris sequences**.

3. Cohomology of $\mathrm{PSL}_2(\mathbb{Z})$

3.1. **Free product.** Recall that $\mathrm{PSL}_2(\mathbb{R})$ is the automorphism group of the Poincare half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$, and $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{PSL}_2(\mathbb{R})$ is the discrete subgroup generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Consider the element $ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, which has order 3 in $\mathrm{PSL}_2(\mathbb{Z})$. Then $\mathrm{PSL}_2(\mathbb{Z})$ is generated by S and ST. In fact,

Theorem 3.1. $\mathrm{PSL}_2(\mathbb{Z}) = \langle S\rangle * \langle ST\rangle \cong \langle x, y \mid x^2 = y^3 = 1\rangle$.

Proof. The following proof is taken from [JS97, §6.8]. Let $\pi : \langle x, y \mid x^2 = y^3 = 1\rangle \to \mathrm{PSL}_2(\mathbb{Z})$ be the map defined by sending (x, y) to (S, ST). Let W be a nonempty reduced word in x, y. In other words, $W = \gamma_1 \cdots \gamma_r$ with $\gamma_i \in \{x, y, y^{-1}\}$, and $\gamma_i, \gamma_{i+1} \notin \{x\}, \{y, y^{-1}\}$. To show the theorem, we must show π is an isomorphism, and it comes down to showing that if W is a nonempty reduced word, then $\pi(W) \neq 1$ in $\mathrm{PSL}_2(\mathbb{Z})$.

For this we use some geometry. Consider the following regions

$$A = \{z \in \mathbb{H} \mid \operatorname{Re} z < 0\}, \qquad B = \{z \in \mathbb{H} \mid |z+1| > |z|, |z+1| > z\}, \qquad C = A \cap B$$

We have $S(A) = \{z \in \mathbb{H} \mid \operatorname{Re} z > 0\}$, so $S(A) \cap A = \varnothing$ particularly. Also, if $z \in B$,

$$|STz + 1| = \left|\frac{z}{z+1}\right| < 1$$

so $ST(B) \cap B = \varnothing$. This implies $(ST)^{-1}B \cap B = \varnothing$ as well. Hence

$$S(A) \subseteq \mathbb{H}\backslash A \subseteq B, \qquad ST(B) \subseteq \mathbb{H}\backslash B \subseteq A$$

Let W be a nonempty reduced word. The above relation implies that $\pi(W)(C)$ is either contained in $\mathbb{H}\backslash A$ or $\mathbb{H}\backslash B$, and hence contained in $\mathbb{H}\backslash C$. This implies $\pi(W) \neq 1$ since $\mathrm{PSL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ faithfully. $\square$

For later computational convenience, we do a change of variables. Instead of $\mathrm{PSL}_2(\mathbb{Z}) = \langle S\rangle * \langle ST\rangle$, we will use

$$\mathrm{PSL}_2(\mathbb{Z}) = \langle S\rangle * \langle TS\rangle.$$

This holds because TS and ST are conjugate (by S). Now we can apply Mayer-Vietoris sequence to compute the group cohomology of $\mathrm{PSL}_2(\mathbb{Z})$.

Corollary 3.1.1. Let M be an $R[\mathrm{PSL}_2(\mathbb{Z})]$-module. Then we have a long exact sequence

$$0 \longrightarrow M^{\mathrm{PSL}_2(\mathbb{Z})} \longrightarrow M^S \oplus M^{TS} \longrightarrow M \longrightarrow H^1(\mathrm{PSL}_2(\mathbb{Z}), M) \longrightarrow H^1(\langle S\rangle, M) \oplus H^1(\langle TS\rangle, M) \longrightarrow 0$$

and isomorphisms

$$H^p(\mathrm{PSL}_2(\mathbb{Z}), M) \cong H^p(\langle S\rangle, M) \oplus H^p(\langle TS\rangle, M) \qquad (p \geqslant 2).$$

Moreover, the connecting homomorphism $M \to H^1(\mathrm{PSL}_2(\mathbb{Z}), M)$ is given by $m \mapsto f_m$, where $f_m(S) = (1 - S)m$, $f_m(TS) = 0$.

Corollary 3.1.2. Let $\Gamma \leqslant \mathrm{PSL}_2(\mathbb{Z})$ be a subgroup of finite index and R a unital ring in which all periods of elliptic points are invertible. Let V be an $R[\Gamma]$-module. Then

$$H^1(\Gamma, V) = \frac{M}{M^S \oplus M^{TS}}$$

with $M = \mathrm{Ind}_\Gamma^{\mathrm{PSL}_2(\mathbb{Z})} V$ and $H^p(\Gamma, V) = 0$ for all $p \geqslant 2$.

Proof. By Mackey's formula, we have

$$H^p(\mathrm{PSL}_2(\mathbb{Z})_x, \mathrm{Ind}_\Gamma^{\mathrm{PSL}_2(\mathbb{Z})} V) \cong \prod_{g \in \Gamma\backslash \mathrm{PSL}_2(\mathbb{Z})x} H^p(\Gamma_x, V).$$

By Lemma 2.3, our assumptions imply the last groups are zero. The Corollary 3.1.1 now reads

$$H^1(\Gamma, V) \cong H^1(\mathrm{PSL}_2(\mathbb{Z}), M) = \frac{M}{M^S \oplus M^{TS}}.$$

The last assertion follows from another isomorphisms. $\square$

3.2. **Parabolic group cohomology.** Let $\Gamma \leqslant \mathrm{PSL}_2(\mathbb{Z})$ be a subgroup of finite index and let V an be $R[\Gamma]$-module. We define the **parabolic cohomology** $H^1_{\mathrm{par}}(\Gamma, V)$ by the following exact sequence

$$0 \longrightarrow H^1_{\mathrm{par}}(\Gamma,V) \longrightarrow H^1(\Gamma,V) \xrightarrow{\mathrm{res}} \prod_{g\in \mathrm{PSL}_2(\mathbb{Z})} H^1(\Gamma \cap \langle gTg^{-1}\rangle, V).$$

The last group may be replaced by the product of $H^1(\Gamma_c, V)$ with $c \in \mathbb{P}^1(\mathbb{Q})$. Still, if $c = \gamma c'$ for some $c, c' \in \mathbb{P}^1(\mathbb{Q})$ and $\gamma \in \Gamma$, then $\Gamma_c = \gamma\Gamma_{c'}\gamma^{-1}$ and the conjugation gives an isomorphism $H^1(\Gamma_c, V) \cong H^1(\Gamma_{c'}, V)$. Hence in the exact sequence, it causes no confusion to write

$$0 \longrightarrow H^1_{\mathrm{par}}(\Gamma,V) \longrightarrow H^1(\Gamma,V) \xrightarrow{\mathrm{res}} \prod_{g\in \Gamma\backslash \mathrm{PSL}_2(\mathbb{Z})/\langle T\rangle} H^1(\Gamma \cap \langle gTg^{-1}\rangle, V).$$

Put $G = \mathrm{PSL}_2(\mathbb{Z})$. By Lemma 2.11, we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H^1_{\mathrm{par}}(\Gamma,V) \arrow[r] & H^1(\Gamma,V) \arrow[r, "\mathrm{res}"] & \prod\limits_{g\in \Gamma\backslash \mathrm{PSL}_2(\mathbb{Z})/\langle T\rangle} H^1(\Gamma \cap \langle gTg^{-1}\rangle, V) \\
0 \arrow[r] & H^1_{\mathrm{par}}(G,\mathrm{Ind}_\Gamma^G V) \arrow[r] \arrow[u, "\mathrm{sh}_{G|\Gamma}"] & H^1(G,\mathrm{Ind}_\Gamma^G V) \arrow[r, "\mathrm{res}"] \arrow[u, "\mathrm{sh}_{G|\Gamma}"] & H^1(\langle T\rangle,\mathrm{Ind}_\Gamma^G V) \arrow[u, "\mathrm{mackey}"]
\end{tikzcd}$$

Since $\langle T\rangle \cong \mathbb{Z}$, the map $u \mapsto u(T)$ defines an $H^1(\langle T\rangle, \mathrm{Ind}_\Gamma^G V) \cong \dfrac{\mathrm{Ind}_\Gamma^G V}{(1-T)\,\mathrm{Ind}_\Gamma^G V}$. Put

$$V_\Gamma = \frac{V}{\mathrm{span}_R\{(1-\gamma)v \mid \gamma\in\Gamma,\ v\in V\}},$$

and define $\Phi : \mathrm{Ind}_\Gamma^G V \to V_\Gamma$ by $\Phi(f) = \sum\limits_{g\in\Gamma\backslash G} f(g)$. Since $(1-T)\,\mathrm{Ind}_\Gamma^G V$ lies in the kernel of Φ, it induces

$$\Phi : H^1(\langle T\rangle, \mathrm{Ind}_\Gamma^G V) \longrightarrow V_\Gamma.$$

By Corollary 3.1.1, we can form diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H^1_{\mathrm{par}}(G,\mathrm{Ind}_\Gamma^G V) \arrow[r] \arrow[d, equal] & H^1(G,\mathrm{Ind}_\Gamma^G V) \arrow[r, "\mathrm{res}"] & H^1(\langle T\rangle,\mathrm{Ind}_\Gamma^G V) \arrow[d, "u\mapsto u(T)"] & & \\
0 \arrow[r] & H^1_{\mathrm{par}}(G,\mathrm{Ind}_\Gamma^G V) \arrow[r] & \dfrac{M}{M^S\oplus M^{TS}} \arrow[r, "m\mapsto(S-1)m"] \arrow[u, "m\mapsto f_m"] & \dfrac{M}{(1-T)M} \arrow[r] & M_G \arrow[r] & 0
\end{tikzcd}$$

with $M = \mathrm{Ind}_\Gamma^G V$ and $M_G = \dfrac{M}{\mathrm{span}_R\{(1-g)m \mid g\in G,\ m\in M\}}$. In fact, the second square is commutative:

$$f_m(T) = f_m(TSS) = f_m(TS) + TSf_m(S) = TS(1-S)m \equiv S(1-S)m = (S-1)m.$$

Also, the map Φ just introduced defines a map $\Phi : M_G \to V_\Gamma$. In fact, it is an isomorphism. To see this, define $\Psi : V \to M$ by $\Psi(v)(g) = \mathbf{1}_\Gamma(g)v$. For $\gamma\in\Gamma$, compute $\Psi(v)(g) - \Psi(\gamma v)(g) = gv - g\gamma v = (1-\gamma)\Psi(v)(g)$, so that Ψ defines a map $\Psi : V_\Gamma \to M_G$. It is easy to see that $\Phi\circ\Psi$ is the identity. We claim Ψ is surjective, so Ψ is a bijection, proving Φ is a bijection as well. Indeed, for $f \in M_G$, we have $f = \sum\limits_{g\in\Gamma\backslash G} g^{-1}\Psi(f(g))$, as

$$\sum_{g\in\Gamma\backslash G} \left(g^{-1}\Psi(f(g))\right)(x) = \sum_{g\in\Gamma\backslash G} \Psi(f(g))(xg^{-1}) = xg_x^{-1}f(g_x) = x(xg_x^{-1}g_x) = f(x)$$

where $g_x \in G$ with $\Gamma x = \Gamma g_x$. But then

$$f = \sum_{g\in\Gamma\backslash G} g^{-1}\Psi(f(g)) = \sum_{g\in\Gamma\backslash G} \Psi(f(g)) - \sum_{g\in\Gamma\backslash G} (1-g^{-1})\Psi(f(g)) \in \mathrm{Im}\,\Psi \text{ in } M_G$$

showing the surjectivity. Now we conclude that

Lemma 3.2. The sequence

$$0 \longrightarrow H^1_{\mathrm{par}}(G, \mathrm{Ind}_\Gamma^G V) \longrightarrow H^1(G, \mathrm{Ind}_\Gamma^G V) \xrightarrow{\mathrm{res}} H^1(\langle T\rangle, \mathrm{Ind}_\Gamma^G V) \xrightarrow{\Phi} V_\Gamma \longrightarrow 0$$

is exact.

Remark. I don't know a direct algebraic proof of this exact sequence without the explicit formula for $m \mapsto f_m$. Yet, see [Shi71, Proposition 8.2] and [Hid06, Proposition 6.1.1] for a proof using simplicial cohomology.

3.3. **Dimension formulae.** Let $\Gamma \leqslant SL_2(\mathbb{Z})$ be a congruence subgroup, $k \geqslant 2$ and $V = \mathbb{C}[X,Y]_{k-2}$. If $-I \in \Gamma$, we assume k is even. The goal of this subsection is the compute

$$\dim_{\mathbb{C}} H^1(\Gamma, V), \qquad \dim_{\mathbb{C}} H^1(\Gamma, V) - \dim_{\mathbb{C}} H^1_{\mathrm{par}}(\Gamma, V).$$

Note that by Lemma 2.7, we can freely replace Γ by its image in $PSL_2(\mathbb{Z})$ without altering the cohomology groups. In the following we simply write dim for dimension over $\mathbb{C}$.

Put $M = \mathrm{Ind}_\Gamma^{PSL_2(\mathbb{Z})} V$. On applying $\dim_R$ to the isomorphism in Corollary 3.1.2, we obtain

$$\begin{aligned}\dim H^1(\Gamma, V) &= \dim M - \dim M^S - \dim_R M^{TS} + \dim M^{PSL_2(\mathbb{Z})} \\ &= \dim M - \dim M^S - \dim M^{TS} + \dim V^\Gamma,\end{aligned}$$

where $M = \mathrm{Ind}_\Gamma^{PSL_2(\mathbb{Z})} V$. By Mackey's formula, we have

$$\begin{aligned}\dim M^S &= \sum_{g \in \Gamma\backslash PSL_2(\mathbb{Z})/\langle S\rangle} \dim V^{\langle gSg^{-1}\rangle \cap \Gamma} = \sum_{x \in \Gamma\backslash PSL_2(\mathbb{Z}) i} \dim V^{\Gamma_x} \\ \dim M^{TS} &= \sum_{g \in \Gamma\backslash PSL_2(\mathbb{Z})/\langle TS\rangle} \dim V^{\langle gTSg^{-1}\rangle \cap \Gamma} = \sum_{x \in \Gamma\backslash PSL_2(\mathbb{Z}) \mu_6} \dim V^{\Gamma_x}\end{aligned}$$

By Lemma 3.2, we have

$$\begin{aligned}\dim H^1_{\mathrm{par}}(\Gamma, V) &= \dim H^1(\Gamma, V) + \dim_R V_\Gamma - \dim H^1(\langle T\rangle, \mathrm{Ind}_\Gamma^G V) \\ &= \dim H^1(\Gamma, V) + \dim V_\Gamma - \sum_{g \in \Gamma\backslash PSL_2(\mathbb{Z})/\langle T\rangle} H^1(\Gamma \cap \langle gTg^{-1}\rangle, V)\end{aligned}$$

We now specialize to the representation $V = R[X,Y]_{k-2}$.

Lemma 3.3. Let $n, N \in \mathbb{Z}_{\geqslant 1}$. Suppose R is a unital ring with $n!N \in R^\times$.

(i) $R[X,Y]_n^{\mathbf{n}(N)} = RX^n$ and $R[X,Y]_n^{\mathbf{n}(N)^t} = RY^n$

(ii) $(R[X,Y]_n)_{\mathbf{n}(N)} = \dfrac{R[X,Y]_n}{RX^n \oplus \cdots \oplus RXY^{n-1}}$ and $(R[X,Y]_n)_{\mathbf{n}(N)^t} = \dfrac{R[X,Y]_n}{RY^n \oplus \cdots \oplus RX^{n-1}Y}$.

(iii) $R[X,Y]_n^{\Gamma(N)} = 0 = (R[X,Y]_n)_{\Gamma(N)}$ if R is a field of characteristic 0.

Proof. We have $(\mathbf{n}(N)-1).X^iY^{n-i} = X^i(NX+Y)^{n-i} - X^iY^{n-i} = \sum_{k=1}^{n-i} N^k \binom{n-i}{k} X^{i+k}Y^{n-i-k}$. With this formula it is direct to see (i) and (ii) hold. (iii) follows as $\Gamma(N)$ contains both $\mathbf{n}(N)$ and $\mathbf{n}(N)^t$. □

From the lemma we then have

$$(\mathbb{C}[X,Y]_{k-2})_\Gamma = 0 = \mathbb{C}[X,Y]_{k-2}^\Gamma$$

so

$$\begin{aligned}\dim H^1(\Gamma, R[X,Y]_{k-2}) &= \dim M - \dim M^S - \dim M^{TS} + \delta_{2,k} \\ \dim H^1_{\mathrm{par}}(\Gamma, \mathbb{C}[X,Y]_{k-2}) &= \dim H^1(\Gamma, \mathbb{C}[X,Y]_{k-2}) + \delta_{2,k} - \sum_{g \in \Gamma\backslash PSL_2(\mathbb{Z})/\langle T\rangle} H^1(\Gamma \cap \langle gTg^{-1}\rangle, \mathbb{C}[X,Y]_{k-2})\end{aligned}$$

Lemma 3.4. Let $\Gamma = \Gamma(N)$, $N \geqslant 3$.

(i) $\dim H^1(\Gamma, \mathbb{C}[X,Y]_{k-2}) = (k-1)\dfrac{[SL_2(\mathbb{Z}) : \Gamma(N)]}{12} + \delta_{2,k} = \dim M_k(\Gamma(N)) + \dim S_k(\Gamma(N))$.

(ii) $\dim M_k(\Gamma) - \dim S_k(\Gamma) = \dim H^1(\Gamma, \mathbb{C}[X,Y]_{k-2}) - \dim H^1_{par}(\Gamma, \mathbb{C}[X,Y]_{k-2})$.

Proof. Γ is torsion free by our assumption, so

$$\dim M^S = (k-1)\#(\Gamma\backslash PSL_2(\mathbb{Z})i) = \frac{(k-1)[PSL_2(\mathbb{Z}) : \Gamma(N)]}{2}$$

$$\dim M^{TS} = (k-1)\#(\Gamma\backslash PSL_2(\mathbb{Z})\mu_6) = \frac{(k-1)[PSL_2(\mathbb{Z}) : \Gamma(N)]}{3}.$$

Also, $\dim M = (k-1)[PSL_2(\mathbb{Z}) : \Gamma(N)]$. We conclude that

$$\dim H^1(\Gamma(N), \mathbb{C}[X,Y]_{k-2}) = \frac{(k-1)[SL_2(\mathbb{Z}) : \Gamma(N)]}{12} + \delta_{2,k}.$$

By [DS05, §3.9], the last equality holds. This proves (i).

From [DS05, §3.5-6], we see LHS equals $\varepsilon_\infty - \delta_{2,k}$. Recall that all cusps of $\Gamma(N)$ are regular, so $\Gamma \cap \langle gTg^{-1}\rangle = \langle gT^r g^{-1}\rangle$ (for some r). Hence by Lemma 3.3 we have

$$\dim H^1(\Gamma \cap \langle gTg^{-1}\rangle, \mathbb{C}[X,Y]_{k-2}) = \dim(\mathbb{C}[X,Y]_{k-2})_{gT^r g^{-1}} = 1.$$

The proof of (ii) is then completed in view of the formula

$$\dim H^1(\Gamma, \mathbb{C}[X,Y]_{k-2}) - \dim H^1_{par}(\Gamma, \mathbb{C}[X,Y]_{k-2}) = \sum_{g \in \Gamma\backslash PSL_2(\mathbb{Z})/\langle T\rangle} H^1(\Gamma \cap \langle gTg^{-1}\rangle, \mathbb{C}[X,Y]_{k-2}) - \delta_{2,k}.$$

□

4. Eichler-Shimura isomorphism

For a map $f : \mathbb{H} \to \mathbb{C}$, $k \in \mathbb{Z}_{\geqslant 0}$ and $g \in GL_2(\mathbb{R})^+$, denote $f|_k g : \mathbb{H} \to \mathbb{C}$ given by

$$f|_k g(\tau) = (\det g)^{k-1} j(g,\tau)^{-k} f(g.\tau), \qquad \tau \in \mathbb{H}.$$

This is of course the same notation as $f[g]_k$ used in the course.

4.1. **Eichler-Shimura map.** Let $\Gamma \leqslant SL_2(\mathbb{R})$ be a congruence subgroup. For $z_0, z_1 \in \mathbb{H}$ and $f \in M_k(\Gamma)$ with $k \geqslant 2$, consider the integral

$$I_f(gz_0, hz_0) = \int_{gz_0}^{hz_0} f(z)(Xz+Y)^{k-2}dz \in \mathbb{C}[X,Y]_{k-2}$$

$$J_{\overline{f}}(gz_1, hz_1) = \int_{gz_1}^{hz_1} \overline{f(z)}(X\overline{z}+Y)^{k-2}d\overline{z} \in \mathbb{C}[X,Y]_{k-2}.$$

where $g, h \in GL_2(\mathbb{R})^+$. Since f is holomorphic, the integrals are independent of the choice of paths in $\mathbb{H}$.

Lemma 4.1. For $g, h \in GL_2(\mathbb{R})^+$, we have

$$I_f(z_0, ghz_0) = I_f(z_0, gz_0) + I_f(gz_0, ghz_0)$$

$$I_f(gz_0, ghz_0) = \det(g)^{2-k} g.(I_{f|_k g}(z_0, hz_0))$$

where we let $GL_2(\mathbb{R})$ acts on $\mathbb{C}[X,Y]_{k-2}$ by right translation.

Proof. The first is clear, and for the second

$$I_f(gz_0, ghz_0) = \int_{gz_0}^{ghz_0} f(z)(Xz+Y)^{k-2}dz = \int_{z_0}^{hz_0} f(g.z)(Xg.z+Y)^{k-2}\frac{d(g.z)}{dz}dz$$

$$= \int_{z_0}^{hz_0} f(g.z)g.(Xz+Y)^{k-2} j(g,z)^{-k} \det g\, dz$$

which is exactly what we want.

□

Since we are assuming $f \in M_k(\Gamma)$, so for $g \in \Gamma$, $h \in SL_2(\mathbb{Z})$ we have

$$I_f(z_0, ghz_0) = I_f(z_0, gz_0) + g.I_f(z_0, hz_0).$$

This shows $\Gamma \ni g \mapsto I_f(z_0, gz_0)$ defines an element in $Z^1(\Gamma, \mathbb{C}[X,Y]_{k-2})$. If z_1 is another point on $\mathbb{H}$, take $\gamma \in SL_2(\mathbb{Z})$ with $\gamma z_1 = z_0$ and compute

$$\begin{aligned} I_f(z_1, gz_1) &= I_f(\gamma z_0, g\gamma z_0) = I_f(z_0, g\gamma z_0) - I_f(z_0, \gamma z_0) \\ &= I_f(z_0, gz_0) + g.I_f(z_0, \gamma z_0) - I_f(z_0, \gamma z_0) \\ &= I_f(z_0, gz_0) + (g-1)I_f(z_0, \gamma z_0). \end{aligned}$$

Hence different choices of z_0 differ the integral by an element in $B^1(\Gamma, \mathbb{C}[X,Y]_{k-2})$, so we obtain a well-defined $\mathbb{C}$-linear map

$$\begin{aligned} M_k(\Gamma) &\longrightarrow H^1(\Gamma, \mathbb{C}[X,Y]_{k-2}) \\ f &\longmapsto [g \mapsto I_f(z_0, gz_0)] \end{aligned}$$

Similarly, we have a conjugate-linear map

$$\begin{aligned} M_k(\Gamma) &\longrightarrow H^1(\Gamma, \mathbb{C}[X,Y]_{k-2}) \\ f &\longmapsto [g \mapsto J_{\overline{f}}(z_1, gz_1)] \end{aligned}.$$

They together define the so-called **Eichler-Shimura map**

$$\begin{aligned} ES_\Gamma : M_k(\Gamma) \oplus \overline{S_k(\Gamma)} &\longrightarrow H^1(\Gamma, \mathbb{C}[X,Y]_{k-2}) \\ (f, \overline{g}) &\longmapsto [\gamma \mapsto I_f(z_0, \gamma z_0) + J_{\overline{g}}(z_1, \gamma z_1)]. \end{aligned}$$

Lemma 4.2. The induced map

$$M_k(\Gamma) \oplus \overline{S_k(\Gamma)} \xrightarrow{sh^{-1}_{SL_2(\mathbb{Z})|\Gamma} \circ ES_\Gamma} H^1(SL_2(\mathbb{Z}), \mathrm{Ind}_\Gamma^{SL_2(\mathbb{Z})} \mathbb{C}[X,Y]_{k-2}).$$

is given by

$$\left(sh^{-1}_{SL_2(\mathbb{Z})|\Gamma} \circ ES_\Gamma\right)(f, \overline{g})(\gamma)(\gamma') = [I_f(\gamma' z_0, \gamma'\gamma z_0) + J_{\overline{g}}(\gamma' z_1, \gamma'\gamma z_1)],$$

where $z_0, z_1 \in \mathbb{H}$ are any fixed points.

Proof. First, we show $\gamma' \mapsto I_f(\gamma' z_0, \gamma'\gamma z_0) + I_{\overline{g}}(\gamma' z_1, \gamma'\gamma z_1)$ lies in the induced module. This follows from the first identity in Lemma 4.1. Next, we show $\gamma \mapsto [\gamma' \mapsto I_f(\gamma' z_0, \gamma'\gamma z_0) + I_{\overline{g}}(\gamma' z_1, \gamma'\gamma z_1)]$ is a 1-cocycle. Put $\phi(\gamma) : \gamma' \mapsto I_f(\gamma' z_0, \gamma'\gamma z_0)$. By Lemma 4.1 again, for $\gamma', \gamma_1, \gamma_2 \in SL_2(\mathbb{Z})$ we have

$$\begin{aligned} \phi(\gamma_1\gamma_2)(\gamma') = I_f(\gamma' z_0, \gamma'\gamma_1\gamma_2 z_0) &= I_f(\gamma' z_0, \gamma'\gamma_1 z_0) + I_f(\gamma'\gamma_1 z_0, \gamma'\gamma_1\gamma_2 z_0) \\ &= \phi(\gamma_1)(\gamma') + \phi(\gamma_2)(\gamma'\gamma_1) = (\phi(\gamma_1) + \gamma_1.\phi(\gamma_2))(\gamma'). \end{aligned}$$

Finally, from the definition of $sh_{SL_2(\mathbb{Z})|\Gamma}$, we see $sh^{-1}_{SL_2(\mathbb{Z})|\Gamma} \circ ES_\Gamma$ is given by the displayed formula. □

For a path $\gamma : [0,1] \to \mathbb{H}$ and a function $f : \mathbb{H} \to \mathbb{C}$, one has

$$\overline{\int_\gamma f(z)dz} = \overline{\int_0^1 f(\gamma(t))\gamma'(t)dt} = \int_0^1 \overline{f}(\gamma(t))(\overline{\gamma})'(t)dt = \int_\gamma \overline{f(z)}d\overline{z}.$$

This implies $\overline{I_f(gz_0, hz_0)} = J_{\overline{f}}(gz_0, hz_0)$, where we let complex conjugation act on $\mathbb{C}[X,Y]_{k-2}$ by acting coefficients. Define

$$\begin{aligned} \mathrm{rES}_\Gamma : S_k(\Gamma) &\longrightarrow H^1(\Gamma, \mathbb{R}[X,Y]_{k-2}) \\ f &\longrightarrow [\gamma \mapsto \mathrm{Re}\,(I_f(z_0, \gamma z_0))]. \end{aligned}$$

Since $\mathrm{Re}(I_f(z_0, \gamma z_0)) = \dfrac{I_f(z_0, \gamma z_0) + J_{\overline{f}}(z_0, \gamma z_0)}{2}$, we see rES_Γ is twice the composition

$$\begin{aligned} S_k(\Gamma) &\longrightarrow S_k(\Gamma) \oplus \overline{S_k(\Gamma)} \xrightarrow{\mathrm{ES}_\Gamma} H^1(\Gamma, \mathbb{C}[X,Y]_{k-2}) \\ f &\longmapsto (f, \overline{f}). \end{aligned}$$

where we use the identification described in the following lemma, which follows from the flatness of $\mathbb{R} \to \mathbb{C}$.

Lemma 4.3. Let G be a group and M be an $\mathbb{R}[G]$-module. The natural map $H^p(G, M) \to H^p(G, M_\mathbb{C})$ is injective, and induces an isomorphism $H^p(G, M)_\mathbb{C} \cong H^p(G, M_\mathbb{C})$.

Proof. It is clear that the inclusion $C^p_{\mathrm{inhom}}(G, M) \to C^p_{\mathrm{inhom}}(G, M_\mathbb{C})$ induces an isomorphism $C^p_{\mathrm{inhom}}(G, M)_\mathbb{C} \cong C^p_{\mathrm{inhom}}(G, M_\mathbb{C})$. It is also clear that $Z^p(G, M)_\mathbb{C} \cong Z^p(G, M_\mathbb{C})$. For injectivity, it suffices to show $Z^p(G, M) \cap B^p(G, M_\mathbb{C}) = B^p(G, M_\mathbb{C})$, since it will imply $B^p(G, M)_\mathbb{C} \cong B^p(G, M)$. For $f \in C^{p-1}(G, M_\mathbb{C})$, if $\partial f \in C^p(G, M)$, then $\partial f = \overline{\partial f} = \partial \overline{f}$. Then $\dfrac{f + \overline{f}}{2} \in C^{p-1}(G, M)$ is mapped to $\dfrac{\partial f + \partial \overline{f}}{2} = \partial f$. The last statement now follows from the exactness of $(\cdot) \otimes_\mathbb{R} \mathbb{C}$. ☐

Lemma 4.4. ES_Γ is twice the complexification of rES_Γ. Precisely, the composition

$$S_k(\Gamma) \otimes \overline{S_k(\Gamma)} \xrightarrow{\sim} S_k(\Gamma)_\mathbb{C} \xrightarrow{2 \cdot \mathrm{rES}_\Gamma \otimes \mathrm{id}_\mathbb{C}} H^1(\Gamma, \mathbb{R}[X,Y]_{k-2})_\mathbb{C} \xrightarrow{\sim} H^1(\Gamma, \mathbb{C}[X,Y]_{k-2})$$

coincides with $\mathrm{ES}_\Gamma : S_k(\Gamma) \oplus \overline{S_k(\Gamma)} \to H^1(\Gamma, \mathbb{C}[X,Y]_{k-2})$. Similarly, the map in Lemma 4.2 is twice the complexification of $\mathrm{sh}^{-1}_{\mathrm{SL}_2(\mathbb{Z})|\Gamma} \circ \mathrm{rES}_\Gamma$.

Proof. This follows from the above discussion. ☐

4.2. **Land in parabolic cohomology.**

Theorem 4.5. The kernel of the composition

$$M_k(\Gamma) \oplus \overline{S_k(\Gamma)} \xrightarrow{\mathrm{ES}_\Gamma} H^1(\Gamma, \mathbb{C}[X,Y]_{k-2}) \xrightarrow{\prod_c \mathrm{res}^\Gamma_{\Gamma_c}} \prod_{c \in \Gamma \backslash \mathbb{P}^1(\mathbb{Q})} H^1(\Gamma_c, \mathbb{C}[X,Y]_{k-2})$$

is exactly $S_k(\Gamma) \oplus \overline{S_k(\Gamma)}$.

Proof. The anti-holomorphic part is addressed in the same way as the holomorphic part, so we only consider $M_k(\Gamma)$. Let $f \in M_k(\Gamma)$, and fix a point $z_0 \in \mathbb{H}$. For a cusp $c \in \mathbb{P}^1(\mathbb{Q})$, let $c = \gamma\infty$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$; using the injectivity of res, we can assume $\Gamma_c = \langle \gamma \mathbf{n}(x) \gamma^{-1} \rangle$ for some $x \in \mathbb{N}$. Write

$$f|_k\gamma(\tau) = a_0 + \sum_{n=1}^{\infty} a_n q^n =: a_0 + g(\tau).$$

Then $f(\tau) = a_0|_k\gamma^{-1}(\tau) + g|_k\gamma^{-1}(\tau) = \dfrac{a_0}{j(\gamma^{-1}, \tau)^k} + g|_k\gamma^{-1}(\tau)$, and

$$\begin{aligned} I_f(z_0, hz_0) &= \int_{z_0}^{hz_0} f(z)(Xz+Y)^{k-2} dz \\ &= a_0 \int_{z_0}^{hz_0} \frac{(Xz+Y)^{k-2}}{j(\gamma^{-1}, z)^k} dz + \int_{z_0}^{hz_0} g|_k\gamma^{-1}(z)(Xz+Y)^{k-2} dz. \end{aligned}$$

Since g is of exponential decay near ∞, the integral

$$I_{g|_k\gamma^{-1}}(z_0,\gamma\tau)=\gamma.I_g(\gamma^{-1}z_0,\tau)=\gamma.\int_{\gamma^{-1}z_0}^{\tau}g(z)(Xz+Y)^{k-2}dz$$

converges as $\tau\to\infty$ (within a bounded vertical strip), so the notation $I_{g|_k\gamma^{-1}}(z_0,\gamma\infty)$ makes sense. Consider a general element $h=\gamma\mathbf{n}(N)\gamma^{-1}\in\Gamma_c$. Then

$$\begin{aligned}
I_{g|_k\gamma^{-1}}(z_0,hz_0)&=I_{g|_k\gamma^{-1}}(z_0,\gamma\infty)+I_{g|_k\gamma^{-1}}(\gamma\infty,hz_0)\\
(\text{as }\mathbf{n}(N)\text{ fixes }\infty)&=I_{g|_k\gamma^{-1}}(z_0,\gamma\infty)+I_{g|_k\gamma^{-1}}(h\gamma\infty,hz_0)\\
&=I_{g|_k\gamma^{-1}}(z_0,\gamma\infty)+h.I_{g|_k\gamma^{-1}h}(\gamma\infty,z_0)\\
&=(1-h)I_{g|_k\gamma^{-1}}(z_0,\gamma\infty)
\end{aligned}$$

where the last equality holds as $g|_k\gamma^{-1}h=g|_k\mathbf{n}(N)\gamma^{-1}=g|_k\gamma^{-1}$. This shows the image of f in $H^1(\Gamma_c,\mathbb{C}[X,Y]_{k-2})$ is represented by the 1-cocycle

$$\Gamma_c\ni h\mapsto a_0\int_{z_0}^{hz_0}\frac{(Xz+Y)^{k-2}}{j(\gamma^{-1},z)^k}dz.$$

It remains to shows the 1-cocycle represents zero if and only if $a_0=0$. We need an explicit description of $H^1(\Gamma_c,\mathbb{C}[X,Y]_{k-2})$. From (2.3) we have

$$H^1(\Gamma_c,\mathbb{C}[X,Y]_{k-2})\cong\frac{\mathbb{C}[X,Y]_{k-2}}{(1-\gamma\mathbf{n}(x)\gamma^{-1})\mathbb{C}[X,Y]_{k-2}}\cong\frac{\mathbb{C}[X,Y]_{k-2}}{(1-\mathbf{n}(x))\mathbb{C}[X,Y]_{k-2}}$$

with the isomorphisms given by $u\mapsto u(\gamma\mathbf{n}(x)\gamma^{-1})$ and $f\mapsto\gamma^{-1}.f$. But Lemma 3.3.(ii) implies $f\mapsto f(0,1)$ defines an isomorphism of the last space with $\mathbb{C}$. Finally, write $\gamma^{-1}=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ compute (with $h=\gamma\mathbf{n}(x)\gamma^{-1}$)

$$\begin{aligned}
\left(\gamma^{-1}.\int_{z_0}^{hz_0}\frac{(Xz+Y)^{k-2}}{j(\gamma^{-1},z)^k}dz\right)(0,1)&=\left(\int_{z_0}^{hz_0}\frac{(Xz+Y)^{k-2}}{j(\gamma^{-1},z)^k}dz\right)((0,1)\gamma^{-1})\\
&=\int_{z_0}^{hz_0}\frac{1}{(cz+d)^2}dz\\
&=\int_{z_0}^{hz_0}d(\gamma^{-1}z)=\gamma^{-1}(hz_0-z_0)=\mathbf{n}(x)\gamma^{-1}z_0-\gamma^{-1}z_0=x\neq0.
\end{aligned}$$

This finishes the proof. □

4.3. **Eichler-Shimura isomorphism.** From Theorem 4.5, we obtain a commutative diagram

$$\begin{tikzcd}
M_k(\Gamma)\oplus\overline{S_k(\Gamma)} \arrow[r] & H^1(\Gamma,\mathbb{C}[X,Y]_{k-2}) \arrow[r, "\prod_c \mathrm{res}^{\Gamma}_{\Gamma_c}"] & \prod_{c\in\Gamma\backslash\mathbb{P}^1(\mathbb{Q})} H^1(\Gamma_c,\mathbb{C}[X,Y]_{k-2}) \\
S_k(\Gamma)\oplus\overline{S_k(\Gamma)} \arrow[u] \arrow[r] & H^1_{\mathrm{par}}(\Gamma,\mathbb{C}[X,Y]_{k-2}) \arrow[u] &
\end{tikzcd}$$

We can now state the main theorem of this article.

Theorem 4.6. Let $\Gamma\leqslant SL_2(\mathbb{Z})$ be a congruence subgroup. Let $k\in\mathbb{Z}_{\geqslant2}$ and assume it is even if $-I\in\Gamma$. Then the Eichler-Shimura map

$$ES_\Gamma: M_k(\Gamma)\oplus\overline{S_k(\Gamma)}\longrightarrow H^1(\Gamma,\mathbb{C}[X,Y]_{k-2})$$

is an isomorphism, and the image of $S_k(\Gamma)\oplus\overline{S_k(\Gamma)}$ is isomorphic to $H^1_{\mathrm{par}}(\Gamma,\mathbb{C}[X,Y]_{k-2})$.

The idea of the proof goes as follows. To show ES_Γ is injective, by Theorem 4.5 we may restrict to the cuspidal subspace $S_k(\Gamma)\oplus\overline{S_k(\Gamma)}$. The key ingredient is the non-degeneracy of the Petersson inner product on the modular curve. We will define a pairing on the cohomology group, and hence a cup product. We show it coincides with the Petersson inner product, and conclude the proof for injectivity. These will be completed in the subsequent subsection.

Assuming the injectivity, we proceed to finish the proof. We reduce to the case Γ is torsion-free.

Lemma 4.7. Let $\Gamma' \trianglelefteq \Gamma \leqslant \mathrm{SL}_2(\mathbb{Z})$ be congruence subgroup. The inclusion $S_k(\Gamma) \subseteq S_k(\Gamma')$ induces an equality $S_k(\Gamma) = S_k(\Gamma')^{\Gamma/\Gamma'}$. Similarly, $M_k(\Gamma) = M_k(\Gamma')^{\Gamma/\Gamma'}$

Proof. For $\gamma \in \Gamma$, the action is given by $f \mapsto f|_k\gamma$. If $\gamma' \in \Gamma'$, then $f|_k\gamma\gamma' = f|_k(\gamma\gamma'\gamma^{-1}\gamma) = f|_k\gamma$, so the action passes to the quotient Γ/Γ'. The identity then follows from the definition. ∎

Retain the notation in the lemma. On the other hand, since Γ/Γ' is a finite group, from the inflation-restriction exact sequence and Lemma 2.3.(ii), we see the restriction induces an isomorphism

$$\mathrm{res}_{\Gamma|\Gamma'} : H^1(\Gamma, \mathbb{C}[X,Y]_{k-2}) \longrightarrow H^1(\Gamma', \mathbb{C}[X,Y]_{k-2})^{\Gamma/\Gamma'}$$

Lemma 4.8. The diagram

$$\begin{tikzcd} M_k(\Gamma) \oplus \overline{S_k(\Gamma)} \arrow[r, "\mathrm{ES}_\Gamma"] \arrow[d] & H^1(\Gamma, \mathbb{C}[X,Y]_{k-2}) \arrow[d, "\mathrm{res}_{\Gamma|\Gamma'}"] \\ M_k(\Gamma') \oplus \overline{S_k(\Gamma')} \arrow[r, "\mathrm{ES}_{\Gamma'}"] & H^1(\Gamma', \mathbb{C}[X,Y]_{k-2}) \end{tikzcd}$$

commutes, and the bottom arrow satisfies $\mathrm{ES}_{\Gamma'}(f|_k\gamma) = \gamma^{-1}.\mathrm{ES}_{\Gamma'}(f)$ for $\gamma' \in \Gamma'$.

Proof. The first is clear, where the second follows from the second identity in Lemma 4.1. ∎

Next we consider the parabolic cohomology. We have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & H^1_{\mathrm{par}}(\Gamma, \mathbb{C}[X,Y]_{k-2}) \arrow[r] \arrow[d, "\mathrm{res}"] & H^1(\Gamma, \mathbb{C}[X,Y]_{k-2}) \arrow[r, "\mathrm{res}"] \arrow[d, "\mathrm{res}"] & \prod_{c\in\mathbb{P}^1(\mathbb{Q})} H^1(\Gamma_c, \mathbb{C}[X,Y]_{k-2}) \arrow[d, "\mathrm{res}"] \\ 0 \arrow[r] & H^1_{\mathrm{par}}(\Gamma', \mathbb{C}[X,Y]_{k-2}) \arrow[r] & H^1(\Gamma', \mathbb{C}[X,Y]_{k-2}) \arrow[r, "\mathrm{res}"] & \prod_{c\in\mathbb{P}^1(\mathbb{Q})} H^1(\Gamma'_c, \mathbb{C}[X,Y]_{k-2}) \end{tikzcd}$$

Γ acts on the bottom-right product by conjugation, and the restriction map res by its left is Γ-equivariant. Γ (resp. Γ') acts on the factors on the upper-right (resp. bottom-right) trivially. These together implies that $H^1_{\mathrm{par}}(\Gamma', \mathbb{C}[X,Y]_{k-2})$ inherits a Γ/Γ'-action, and $\mathrm{res} : H^1_{\mathrm{par}}(\Gamma, \mathbb{C}[X,Y]_{k-2}) \to H^1_{\mathrm{par}}(\Gamma', \mathbb{C}[X,Y]_{k-2})$ induces an isomorphism

$$H^1_{\mathrm{par}}(\Gamma, \mathbb{C}[X,Y]_{k-2}) \xrightarrow{\ \sim\ } H^1_{\mathrm{par}}(\Gamma', \mathbb{C}[X,Y]_{k-2})^{\Gamma/\Gamma'}.$$

Hence, to prove that ES_Γ is an isomorphism, we can replace Γ by some $\Gamma(N)$ with $N \geqslant 3$ so that Γ is torsion-free. By Lemma 3.4, the sources and the targets have the same dimension in this case, so the injectivity implies ES_Γ is an isomorphism, and the image of cuspidal subspaces is exactly the parabolic cohomology. This finishes the proof modulo the injectivity. See [Hid06] for another proof using Laplacian and spectral theory for the unitary representation of $\mathrm{SL}_2(\mathbb{R})$ on the space $L^2(\Gamma\backslash\mathrm{SL}_2(\mathbb{R}))$.

4.4. **Petersson inner product.** For $f, g \in S_k(\Gamma)$, denote by $\langle f, g\rangle_{\mathrm{Pet}}$ their Petersson inner product, i.e.,

$$\langle f, g\rangle_{\mathrm{Pet}} = \frac{1}{\mathrm{vol}(Y(\Gamma))}\int_{Y(\Gamma)} f(z)\overline{g(z)}y^k\frac{dxdy}{y^2}.$$

Let B be the standard fundamental domain of $SL_2(\mathbb{Z})$. Then

$$\int_{Y(\Gamma)} f(z)\overline{g(z)}y^k\frac{dxdy}{y^2} = \sum_{\gamma\in\Gamma\backslash SL_2(\mathbb{Z})}\int_{\gamma B} f(z)\overline{g(z)}y^k\frac{dxdy}{y^2} = \sum_{\gamma\in\Gamma\backslash SL_2(\mathbb{Z})}\int_B f|_k\gamma(z)\overline{g|_k\gamma(z)}y^k\frac{dxdy}{y^2}.$$

Write $dz\wedge d\bar{z} = \frac{dz\wedge d\bar{z}}{-2i}$; then $y^k\frac{dx\wedge dy}{y^2} = \frac{-1}{(2i)^{k-1}}(z-\bar{z})^k\frac{dz\wedge d\bar{z}}{(z-\bar{z})^2}$ and hence

$$\langle f, g\rangle_{\mathrm{Pet}} = \frac{-1}{(2i)^{k-1}}\frac{1}{\mathrm{vol}(Y(\Gamma))}\sum_{\gamma\in\Gamma\backslash SL_2(\mathbb{Z})}\int_B f|_k\gamma(z)\overline{g|_k\gamma(z)}(z-\bar{z})^k\frac{dz\wedge d\bar{z}}{(z-\bar{z})^2}.$$

We use Stokes' theorem to compute the integral. The form $z\mapsto\left(\int_\infty^z f|_k\gamma(u)(u-\bar{z})^{k-2}du\right)\overline{g|_k\gamma(z)}d\bar{z}$ has differential $f|_k\gamma(z)\overline{g|_k\gamma(z)}(z-\bar{z})^k\frac{dz\wedge d\bar{z}}{(z-\bar{z})^2}$. By Stokes' theorem,

$$\int_B f|_k\gamma(z)\overline{g|_k\gamma(z)}(z-\bar{z})^k\frac{dz\wedge d\bar{z}}{(z-\bar{z})^2} = \int_{\partial B}\left(\int_\infty^z f|_k\gamma(u)(u-\bar{z})^{k-2}du\right)\overline{g|_k\gamma(z)}d\bar{z}.$$

Let $\alpha_1 = \infty\rightsquigarrow\mu_3$ and $\alpha_2 = \mu_3\rightsquigarrow i$; then $\partial B = \alpha_1+\alpha_2-S\alpha_2-T\alpha_1$. If C is a path and $\sigma\in SL_2(\mathbb{Z})$, then

$$\begin{aligned}\int_{\sigma C}\left(\int_\infty^z f|_k\gamma(u)(u-\bar{z})^{k-2}du\right)\overline{g|_k\gamma(z)}d\bar{z} &= \int_C\int_\infty^{\sigma z} f|_k\gamma(u)\overline{g|_k\gamma(\sigma z)}\,(u-\sigma\bar{z})^{k-2}\,\frac{d\sigma\bar{z}}{d\bar{z}}dud\bar{z}\\ &= \int_C\int_{\sigma^{-1}\infty}^z f|_k\gamma\sigma(u)\overline{g|_k\gamma\sigma(z)}\,(u-\bar{z})^{k-2}\,dud\bar{z}\\ &= \left(\int_C\int_\infty^z - \int_C\int_\infty^{\sigma^{-1}\infty}\right) f|_k\gamma\sigma(u)\overline{g|_k\gamma\sigma(z)}\,(u-\bar{z})^{k-2}\,dud\bar{z}\end{aligned}$$

so

$$\begin{aligned}\int_{C-\sigma C}\left(\int_\infty^z f|_k\gamma(u)(u-\bar{z})^{k-2}du\right)\overline{g|_k\gamma(z)}d\bar{z} &= \int_C\int_\infty^z\left(f|_k\gamma(u)\overline{g|_k\gamma(z)} - f|_k\gamma\sigma(u)\overline{g|_k\gamma\sigma(z)}\right)(u-\bar{z})^{k-2}dud\bar{z}\\ &\quad+\int_C\int_\infty^{\sigma^{-1}\infty} f|_k\gamma\sigma(u)\overline{g|_k\gamma\sigma(z)}\,(u-\bar{z})^{k-2}\,dud\bar{z}.\end{aligned}$$

Lemma 4.9. For $f, g\in S_k(\Gamma)$, one has

$$\langle f, g\rangle_{\mathrm{Pet}} = \frac{-1}{(2i)^{k-1}\,\mathrm{vol}(Y(\Gamma))}\sum_{\gamma\in\Gamma\backslash SL_2(\mathbb{Z})}\int_{\mu_3}^i\int_\infty^0 f|_k\gamma(z)\overline{g|_k\gamma(z)}(z-\bar{z})^{k-2}dzd\bar{z}.$$

Proof. From the last formula, we see

$$\begin{aligned}\sum_{\gamma\in\Gamma\backslash SL_2(\mathbb{Z})}\int_{\alpha_2-S\alpha_2}\int_\infty^z f|_k\gamma(u)\overline{g|_k\gamma(z)}(u-\bar{z})^{k_2}dud\bar{z} &= \sum_{\gamma\in\Gamma\backslash SL_2(\mathbb{Z})}\int_{\alpha_2}\int_\infty^{S\infty} f|_k\gamma S(z)\overline{g|_k\gamma S(z)}(u-\bar{z})^{k-2}dud\bar{z}\\ &= \sum_{\gamma\in\Gamma\backslash SL_2(\mathbb{Z})}\int_{\mu_3}^i\int_\infty^0 f|_k\gamma(u)\overline{g|_k\gamma(z)}(u-\bar{z})^{k-2}dud\bar{z}\end{aligned}$$

and

$$\sum_{\gamma\in\Gamma\backslash SL_2(\mathbb{Z})}\int_{\alpha_1-T\alpha_1}\int_\infty^z f|_k\gamma(u)\overline{g|_k\gamma(z)}(z-\bar{z})^{k_2}dud\bar{z} = \sum_{\gamma\in\Gamma\backslash SL_2(\mathbb{Z})}\int_{\alpha_1}\int_\infty^{T\infty} f|_k\gamma T(u)\overline{g|_k\gamma T(z)}(u-\bar{z})^{k-2}dud\bar{z} = 0.$$

The last integral is zero as $T\infty = \infty$. In the first integral, we use that $\gamma\mapsto\gamma\sigma$ is a bijection on the set $\Gamma\backslash SL_2(\mathbb{Z})$. $\square$

In the following subsections, we will equip ourselves with enough tools to compare $\langle f, g\rangle_{\mathrm{Pet}}$ with cup products of group cohomology. Then we complete the proof in (4.8).

4.5. Cup products.

Definition 4.10. The subgroup of **parabolic 1-cocycles**[4] is defined by

$$Z^1_P(\mathrm{PSL}_2(\mathbb{Z}), M) := \ker\left(Z^1(\mathrm{PSL}_2(\mathbb{Z}), M) \xrightarrow{\text{res}} Z^1(\langle T\rangle, M)\right).$$

In other words, a 1-cocycle is $u : \mathrm{PSL}_2(\mathbb{Z}) \to M$ parabolic if and only if $u(T) = 0$.

Let R be a unital ring in which 6 is invertible. Let M, N be two $R[\mathrm{PSL}_2(\mathbb{Z})]$-modules such that there is an $R[\mathrm{PSL}_2(\mathbb{Z})]$-homomorphism $\pi : M \otimes_R N \to R$. Define the pairing

$$\langle \cdot, \cdot \rangle_\pi : Z^1(\mathrm{PSL}_2(\mathbb{Z}), M) \times Z^1(\mathrm{PSL}_2(\mathbb{Z}), N) \longrightarrow R$$

as follows. If u, v are 1-cocycles, since $H^2(\mathrm{PSL}_2(\mathbb{Z}), R) = 0$ by Corollary 3.1.2, the 2-cocycle $\pi_*(u \cup v) \in Z^2(\mathrm{PSL}_2(\mathbb{Z}), R)$ is a 2-coboundary, i.e., $\pi_*(u \cup v) = \partial w$ for some $w : \mathrm{PSL}_2(\mathbb{Z}) \to R$. Set

$$\langle u, v\rangle_\pi = w(T).$$

We compute the pairing. For convenience, set $\rho = \pi_*(u \cup v)$. Then

$$\begin{aligned}
\rho(TS, S) &= w(S) - w(T) + w(TS)\\
\rho(S, S) &= w(S) - w(1) + w(S) = 2w(S)\\
\rho(TS, TS) &= w(TS) - w((TS)^2) + w(TS) = 2w(TS) - w((TS)^2)\\
\rho(TS, (TS)^2) &= w((TS)^2) + w(TS)
\end{aligned}$$

and thus

$$w(T) = \frac{1}{3}(\rho(TS, TS) + \rho(TS, (TS)^2)) + \frac{1}{2}\rho(S, S) - \rho(TS, S)$$

As a by-product, we see the pairing is independent of the choice of the 2-coboundary w.

(i) Suppose $u \in Z^1_P(\mathrm{PSL}_2(\mathbb{Z}), M)$. Then $u(T) = 0$ and

$$\rho(TS, S) = \pi_*(u(TS) \otimes TSv(S)) = \pi_*(-TSu(S) \otimes TSv(S)) = -\pi_*(u(S) \otimes v(S)) = \rho(S, S).$$

Here we use the identity $TSS = T$ to obtain $u(TS) = -TSu(S)$. Moreover, if $v = \partial n$ is a 1-coboundary, then

$$\begin{aligned}
\rho(x, y) = \pi_*(u(x) \otimes x(yn - n)) &= \pi_*(u(x) \otimes xyn) - \pi_*(u(x) \otimes xn)\\
&= \pi_*((u(xy) - xu(y)) \otimes xyn) - \pi_*(u(x) \otimes xn)\\
&= \pi_*\left(-x(u(y) \otimes yn) + u(xy) \otimes xyn - u(x) \otimes xn\right).
\end{aligned}$$

If we put $c(x) := -\pi_*(u(x) \otimes xn)$, then the above is $x.c(y) - c(xy) + c(x) = \partial c$. It follows from the definition that $\langle u, v\rangle_\pi = c(T) = -\pi_*(u(T) \otimes Tn) = 0$. In other words, the pairing $\langle u, v\rangle_\pi$ depends only on the class of $v \in H^1(\mathrm{PSL}_2(\mathbb{Z}), N)$ in this case.

(ii) Suppose $v \in Z^1_P(\mathrm{PSL}_2(\mathbb{Z}), N)$. Then $\rho(TS, S) = \rho(TS, (TS)^2)$ and $\langle u, v\rangle_\pi$ only depends on the class of $u \in H^1(\mathrm{PSL}_2(\mathbb{Z}), M)$. These are shown as (i).

[4]It is not so easy to define such a notion for general congruence subgroups, because different choice of representatives of cusps does not yield the same parabolic cocycles. This is very different from the situation for parabolic cohomology groups in §3.2.

4.6. **Pairing.** We construct a certain pairing that is useful for us. Let R be a unital ring. On R^2 consider an alternating pairing $\circ : R^2 \times R^2 \to R$ defined by

$$\begin{pmatrix} a \\ c \end{pmatrix} \circ \begin{pmatrix} b \\ d \end{pmatrix} := \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If we let $\mathrm{SL}_2(\mathbb{Z})$ on R^2 by left multiplication, then the pairing $\circ$ is clearly $\mathrm{SL}_2(\mathbb{Z})$-invariant. The pairing is naturally extended to the tensor power $T^n R^2$ of R^2:

$$\left(\begin{pmatrix} a_1 \\ c_1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} a_n \\ c_n \end{pmatrix}\right) \circ \left(\begin{pmatrix} b_1 \\ d_1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} b_n \\ d_n \end{pmatrix}\right) := \prod_{i=1}^{n} \begin{pmatrix} a_i \\ c_i \end{pmatrix} \circ \begin{pmatrix} b_i \\ d_i \end{pmatrix}$$

If $n!$ is invertible in R, the symmetric power $\mathrm{Sym}^n R^2$ can be viewed a subspace of $T^n R^2$ via

$$\begin{aligned} \mathrm{Sym}^n R^2 &\longrightarrow T^n R^2 \\ v_1 \otimes \cdots \otimes v_n &\longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \end{aligned}$$

so $\circ$ defines a pairing on $\mathrm{Sym}^n R^2$. For example, if $\begin{pmatrix} a \\ c \end{pmatrix}^{\otimes n}, \begin{pmatrix} b \\ d \end{pmatrix}^{\otimes n} \in \mathrm{Sym}^n R^2$ (elements in the quotient), then

$$\begin{pmatrix} a \\ c \end{pmatrix}^{\otimes n} \circ \begin{pmatrix} b \\ d \end{pmatrix}^{\otimes n} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{n}.$$

The following is standard, but we mention it for convenience.

Lemma 4.11. For $n \geqslant 1$, the map

$$\begin{aligned} \mathrm{Sym}^n R^2 &\longrightarrow R[X,Y]_n \\ \begin{pmatrix} a_1 \\ c_1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} a_n \\ c_n \end{pmatrix} &\longmapsto \prod_{i=1}^{n} (a_i X + c_i Y) \end{aligned}$$

is an $R[\mathrm{SL}_2(\mathbb{Z})]$-isomorphism.

Using the isomorphism, we've defined a pairing

$$\circ : R[X,Y]_{k-2} \times R[X,Y]_{k-2} \longrightarrow R$$

for $k \geqslant 2$ with $(k-2)! \in R^\times$. Again, we have

$$(aX + cY)^{k-2} \circ (bX + dY)^{k-2} = (ad - bc)^{k-2} \in R.$$

In the case $R = \mathbb{C}$, we have $(Xz + Y)^{k-2} \circ (X\bar{z} + Y)^{k-2} = (z - \bar{z})^{k-2}$, and this is exactly the point that this pairing is useful for us.

Let $\Gamma \leqslant \mathrm{SL}_2(\mathbb{Z})$ be of finite index and $R = \mathbb{R}, \mathbb{C}$. If $-I \in \Gamma$, we assume $k \geqslant 2$ is even; then $-I$ acts on $R[X,Y]_{k-2}$ trivially. Denote by $\overline{\Gamma}$ the image of Γ in $\mathrm{PSL}_2(\mathbb{Z})$. With this assumption, $R[X,Y]_{k-2}$ can be always viewed as an $R[\overline{\Gamma}]$-module. The above pairing extends to a pairing on the induced module $\mathrm{Ind}_{\overline{\Gamma}}^{\mathrm{PSL}_2(\mathbb{Z})} R[X,Y]_{k-2}$ naturally: for $f, g : \mathrm{PSL}_2(\mathbb{Z}) \to R[X,Y]_{k_2}$,

$$f \circ g := \sum_{\gamma \in \overline{\Gamma} \backslash \mathrm{PSL}_2(\mathbb{Z})} f(\gamma) \circ g(\gamma).$$

Consequently, using the result in the last subsection, we may define a pairing

$$\langle , \rangle := \langle , \rangle_\circ$$

on $Z^1(\mathrm{PSL}_2(\mathbb{Z}), \mathrm{Ind}_{\Gamma}^{\mathrm{PSL}_2(\mathbb{Z})} \mathbb{C}[X,Y]_{k-2})$. We impose the above condition on k throughout.

4.7. **Comparison.** For $f \in S_k(\Gamma)$, denote by $\phi_f^{z_0} \in Z^1(\mathrm{PSL}_2(\mathbb{Z}), \mathrm{Ind}_{\Gamma}^{\mathrm{PSL}_2(\mathbb{Z})} \mathbb{C}[X,Y]_{k_2})$ the 1-cocycle obtained under the map in Lemma 4.2 with basepoint $z_0 \in \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ postcomposed with the isomorphism in Lemma 2.6; explicitly, for $\gamma, \gamma' \in \mathrm{PSL}_2(\mathbb{Z})$,

$$\phi_f^{z_0}(\gamma)(\gamma') = I_f(\gamma' z_0, \gamma'\gamma z_0).$$

Denote $\phi_{\overline{f}}^{z_0} \in Z^1(\mathrm{PSL}_2(\mathbb{Z}), \mathrm{Ind}_{\Gamma}^{\mathrm{PSL}_2(\mathbb{Z})} \mathbb{C}[X,Y]_{k_2})$ similarly.

Theorem 4.12. For $f, g \in S_k(\Gamma)$, we have

$$\langle \phi_f^{\infty}, \phi_{\overline{g}}^{\mu_6} \rangle = c_{k,\Gamma} \langle f, g \rangle_{\mathrm{Pet}}$$

with $c_{k,\Gamma} = (2i)^{k-1} \operatorname{vol}(Y(\Gamma))$.

Proof. We compute $\rho = \circ^*(\phi_f^{\infty} \cup \phi_{\overline{g}}^{\mu_6})$. For $\alpha, \beta \in \mathrm{SL}_2(\mathbb{Z})$, one has

$$\begin{aligned}\rho(\alpha,\beta) = \sum_{\gamma \in \Gamma \backslash \mathrm{PSL}_2(\mathbb{Z})} \phi_f^{\infty}(\alpha)(\gamma) \circ \phi_{\overline{g}}^{\mu_6}(\beta)(\gamma\alpha) &= \sum_{\gamma \in \Gamma \backslash \mathrm{PSL}_2(\mathbb{Z})} \int_{\gamma\alpha\mu_6}^{\gamma\alpha\beta\mu_6} \int_{\gamma\infty}^{\gamma\alpha\infty} f(z)\overline{g(z)}(Xz+Y)^{k-2} \circ (X\overline{z}+Y)^{k-2} dz d\overline{z} \\ &= \sum_{\gamma \in \Gamma \backslash \mathrm{PSL}_2(\mathbb{Z})} \int_{\alpha\mu_6}^{\alpha\beta\mu_6} \int_{\infty}^{\alpha\infty} f|_k\gamma(z)\overline{g|_k\gamma(z)}(z-\overline{z})^{k-2} dz d\overline{z}.\end{aligned}$$

Since ϕ_f^{∞} is parabolic (while $\phi_{\overline{g}}^{\mu_6}$ is not), we only need to compute ρ for $(\alpha,\beta) = (S,S), (TS,TS), (TS,(TS)^2)$ (4.5).(i). But TS stabilizes μ_6, so $\rho(TS,TS) = \rho(TS,(TS)^2) = 0$.

$$\begin{aligned}\rho(S,S) &= \sum_{\gamma \in \Gamma \backslash \mathrm{PSL}_2(\mathbb{Z})} \int_{\mu_3}^{\mu_6} \int_{\infty}^{0} f|_k\gamma(z)\overline{g|_k\gamma(z)}(z-\overline{z})^{k-2} dz d\overline{z} \\ &= \left(\sum_{\gamma \in \Gamma \backslash \mathrm{PSL}_2(\mathbb{Z})} \int_{\mu_3}^{i} \int_{\infty}^{0} + \sum_{\gamma \in \Gamma \backslash \mathrm{PSL}_2(\mathbb{Z})} \int_{i}^{\mu_6} \int_{\infty}^{0} \right) (\cdots) \\ &= \left(\sum_{\gamma \in \Gamma \backslash \mathrm{PSL}_2(\mathbb{Z})} \int_{\mu_3}^{i} \int_{\infty}^{0} + \sum_{\gamma \in \Gamma \backslash \mathrm{PSL}_2(\mathbb{Z})} \int_{Si}^{S\mu_3} \int_{S0}^{S\infty} \right) (\cdots) = 2 \sum_{\gamma \in \Gamma \backslash \mathrm{PSL}_2(\mathbb{Z})} \int_{\mu_3}^{i} \int_{\infty}^{0} f|_k\gamma(z)\overline{g|_k\gamma(z)}(z-\overline{z})^{k-2} dz d\overline{z}\end{aligned}$$

By (4.5) and Lemma 4.9, we find

$$\langle \phi_f^{\infty}, \phi_{\overline{g}}^{\mu_6} \rangle = - \sum_{\gamma \in \Gamma \backslash \mathrm{PSL}_2(\mathbb{Z})} \int_{\mu_3}^{i} \int_{\infty}^{0} f|_k\gamma(z)\overline{g|_k\gamma(z)}(z-\overline{z})^{k-2} dz d\overline{z} = (2i)^{k-1} \operatorname{vol}(Y(\Gamma)) \langle f, g \rangle_{\mathrm{Pet}}.$$

$\square$

4.8. **Injectivity of Eichler-Shimura map.** We use the isomorphism in Lemma 2.6 as an identify the cohomology of $\mathrm{PSL}_2(\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z})$. We prove the injectivity first. By Lemma 4.4, it suffices to show $\mathrm{sh}^{-1}_{\mathrm{SL}_2(\mathbb{Z})|\Gamma} \circ \mathrm{rES}_{\Gamma}$ is injective. Let $f \in S_k(\Gamma)$ with $2\mathrm{sh}^{-1}_{\mathrm{SL}_2(\mathbb{Z}|\Gamma)}(\mathrm{rES}_{\Gamma}(f)) = [\phi_f^{\infty} + \phi_{\overline{f}}^{\infty}] = 0$. Since ϕ_f^{∞} is a parabolic cocycle, from (4.5).(i) and Theorem 4.12 we have

$$0 = \langle \phi_f^{\infty}, \phi_f^{\infty} + \phi_{\overline{f}}^{\infty} \rangle = \underbrace{\langle \phi_f^{\infty}, \phi_f^{\infty} \rangle}_{=0 \text{ by supercommutativity}} + \langle \phi_f^{\infty}, \phi_{\overline{f}}^{\infty} \rangle = \langle \phi_f^{\infty}, \phi_{\overline{f}}^{\mu_6} \rangle = c_{k,\Gamma} \langle f, f \rangle_{\mathrm{Pet}}.$$

Since $\langle\,,\rangle_{\mathrm{Pet}}$ is non-degenerate, we conclude that $f = 0$ in $S_k(\Gamma)$.

4.9. **Hecke actions on cohomology.** Let $\Gamma_1, \Gamma_2 \leqslant SL_2(\mathbb{Z})$ be a congruence subgroup. Recall the double coset operator $[\Gamma_1\alpha\Gamma_2]$ defines a correspondence depicted in the following way.

$$\begin{tikzcd} \alpha^{-1}\Gamma_1\alpha\cap\Gamma_2 \arrow[r, "x\mapsto\alpha x\alpha^{-1}", "\sim"'] \arrow[d] & \Gamma_1\cap\alpha\Gamma_2\alpha^{-1} \arrow[d] \\ \Gamma_2 & \Gamma_1 \end{tikzcd} \rightsquigarrow \begin{tikzcd} Y(\alpha^{-1}\Gamma_1\alpha\cap\Gamma_2) \arrow[r, "x\mapsto\alpha x", "\sim"'] \arrow[d] & Y(\Gamma_1\cap\alpha\Gamma_2\alpha^{-1}) \arrow[d] \\ Y(\Gamma_2) & Y(\Gamma_1) \end{tikzcd}$$

If $\Gamma_2 = \bigsqcup_{\gamma\in A}(\alpha^{-1}\Gamma_1\alpha\cap\Gamma_2)\gamma$, then $\Gamma_1\alpha\Gamma_2 = \bigsqcup_{\gamma\in A}\Gamma_1\alpha\gamma$ and

$$\Gamma_2\tau \mapsto \sum_{\gamma\in A}(\alpha^{-1}\Gamma_1\alpha\cap\Gamma_2)\gamma\tau \mapsto \sum_{\gamma\in A}(\Gamma_1\cap\alpha\Gamma_2\alpha^{-1})\alpha\gamma \mapsto \sum_{\gamma\in A}\Gamma_1\alpha\gamma.$$

For an $R[SL_2(\mathbb{Z})]$-module M and $\alpha\in GL_2(\mathbb{Q})^+$, define the operator $T_\alpha : H^1(\Gamma_1, M)\to H^1(\Gamma_2, M)$ as

$$\begin{tikzcd} T_\alpha : H^1(\Gamma_1, M) \arrow[r, "\mathrm{res}"] & H^1(\Gamma_1\cap\alpha\Gamma_2\alpha^{-1}, M) \arrow[r, "\det(\alpha)\mathrm{conj}_\alpha", "\sim"'] & H^1(\alpha^{-1}\Gamma_1\alpha\cap\Gamma_2, M) \arrow[r, "\mathrm{cores}"] \arrow[d, equal] & H^1(\Gamma_2, M) \\ & & H^1((\Gamma_1\cap\alpha\Gamma_2\alpha^{-1})^\alpha, M) & \end{tikzcd}$$

If $u : \alpha^{-1}\Gamma_1\alpha\cap\Gamma_2\to M$ is a 1-cocycle, then

$$\mathrm{cores}(u)(g) = \sum_{a\in A} a^{-1}.u(aga_g^{-1})$$

where $a_g\in A$ is the unique element such that $aga_g^{-1}\in\alpha^{-1}\Gamma_1\alpha\cap\Gamma_2$ (c.f. [NSW13, p.48]). Hence, for a 1-cocycle $u : \Gamma_1\to M$, we see

$$T_\alpha(u)(g) = \sum_{a\in A} a\det(\alpha).\mathrm{conj}_\alpha(u)(aga_g^{-1}) = \sum_{a\in A}(\alpha a)^\iota u(\alpha aga_g^{-1}\alpha^{-1}).$$

where for a matrix $\delta\in M_2(\mathbb{C})$, put $\delta^\iota = \mathrm{adj}\,\delta$. If we put $\beta_\alpha = \alpha\gamma$, then $B := \{\beta_\alpha\}_{\alpha\in B}$ is a set of representatives of $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$, and

$$T_\alpha(u)(g) = \sum_{\beta\in B}\beta^\iota.u(\beta g\beta_g^{-1})$$

where $\beta_g\in B$ is the unique element such that $\beta g\beta_g^{-1}\in\Gamma_1$.

Lemma 4.13. There is a commutative diagram

$$\begin{tikzcd} M_k(\Gamma_1)\oplus\overline{S_k(\Gamma_1)} \arrow[r, "{[\Gamma_1\alpha\Gamma_2]}"] \arrow[d, "ES_{\Gamma_1}"] & M_k(\Gamma_2)\oplus\overline{S_k(\Gamma_2)} \arrow[d, "ES_{\Gamma_2}"] \\ H^1(\Gamma_1, \mathbb{C}[X,Y]_{k-2}) \arrow[r, "T_\alpha"] & H^1(\Gamma_2, \mathbb{C}[X,Y]_{k-2}) \end{tikzcd}$$

Proof. The corresponding 1-cocycle for $f\in M_k(\Gamma_1)$ is $\phi_f : \gamma\mapsto I_f(z_0, \gamma z_0)$. By Lemma 4.1, we have

$$\begin{aligned} \phi_{[\Gamma_1\alpha\Gamma_2]f}(\gamma) &= \sum_{\beta\in B} I_{f|_k\beta}(z_0, \gamma z_0) = \sum_{\beta\in B}\beta^\iota.I_f(\beta z_0, \beta\gamma z_0) \\ &= \sum_{\beta\in B}\beta^\iota.\left(I_f(\beta z_0, z_0) + I_f(z_0, \beta\gamma\beta_\gamma^{-1}z_0) + I_f(\beta\gamma\beta_\gamma^{-1}z_0, \beta\gamma\beta_\gamma^{-1}\beta_\gamma z_0)\right) \\ &= T_\alpha(\phi_f)(\gamma) + \sum_{\beta\in B}\beta^\iota(I_f(\beta z_0, z_0) + \beta\gamma\beta_\gamma^{-1}I_f(z_0, \beta_\gamma z_0)) \\ &= T_\alpha(\phi_f)(\gamma) - (1-\gamma)\sum_{\beta\in B}\beta^\iota.I_f(z_0, \beta z_0). \end{aligned}$$

The last term is a coboundary, so this finishes the proof. □

Let χ be a Dirichlet character modulo N. Consider the usual action of $\Gamma_0(N)$ on $\mathbb{C}[X,Y]_{k-2}$ but twisted by χ, in the sense that

$$\gamma.f(X,Y) = \chi(d)f((X,Y)\gamma), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

We denote this module by $\mathbb{C}[X,Y]^{\chi}_{k-2}$. Note that the $\Gamma_1(N)$-modules $\operatorname{Res}^{\Gamma_0(N)}_{\Gamma_1(N)} \mathbb{C}[X,Y]^{\chi}_{k-2}$ are the same for all Dirichlet characters χ.

Lemma 4.14. Let G be a group and $H \trianglelefteq G$ be such that G/H is finite abelian. If M is a G-module, for $\chi \in \widehat{G/H}$ denote by M^{χ} the G-module defined by $g._{\gamma} m = \chi(g \bmod H)g.m$. Then

$$\bigoplus_{\chi} H^{\bullet}(G, M^{\chi}) \xrightarrow{\text{res } G|H} H^{\bullet}(H, M)$$

is an isomorphism.

Proof. By the general theory, we have $H^p(H,M) = \bigoplus_{\chi} H^p(H,M)^{\chi}$, where

$$H^p(H,M)^{\chi} = \{u \in H^p(H,M) \mid \operatorname{conj}_g u = \chi(g \bmod H)u \text{ for all } g \in G\}.$$

As H-modules, we have $H^p(H,M) = H^p(H,M^{\chi})$. It is clear that $H^p(H,M)^{\chi} = H^p(H,M^{\chi})^{G/H}$ as sets, so

$$H^{\bullet}(H,M) = \bigoplus_{\chi} H^{\bullet}(H,M^{\chi})^{G/H}.$$

Since G/H is finite, the inflation-restriction sequence shows that res : $H^{\bullet}(G,M^{\chi}) \to H^{\bullet}(H,M^{\chi})^{G/H}$ is an isomorphism. This finishes the proof. □

Consider the Eichler-Shimura isomorphism for $\Gamma_1(N)$

$$M_k(\Gamma_1(N)) \oplus \overline{S_k(\Gamma_1(N))} \longrightarrow H^1(\Gamma_1(N), \mathbb{C}[X,Y]_{k-2})$$

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, we know $[\Gamma_1(N)\gamma\Gamma_1(N)]$ is the diamond operator $\langle d \rangle$. On the other hand, by definition the action of T_{γ} on $H^1(\Gamma_1(N))$ is $\operatorname{conj}_{\gamma}$, so $\gamma \mapsto T_{\gamma}$ is simply the action map of $\Gamma_0(N)$ on $H^1(\Gamma_1(N), \mathbb{C}[X,Y]_{k-2})$. Let χ be a Dirichlet character modulo N. From Lemma 4.13 and Lemma 4.14, taking χ-eigen part yields

$$M_k(N,\chi) \oplus \overline{S_k(N,\chi)} \longrightarrow H^1(\Gamma_0(N), \mathbb{C}[X,Y]^{\chi}_{k-2})$$

From the definition we see T_{α} preserves the parabolic subspace, i.e., $T_{\alpha}(H^1_{\text{par}}(\Gamma_1, M)) \subseteq H^1_{\text{par}}(\Gamma_2, M)$. Hence the Eichler-Shimura map also induces an isomorphism

$$S_k(N,\chi) \oplus \overline{S_k(N,\chi)} \longrightarrow H^1_{\text{par}}(\Gamma_0(N), \mathbb{C}[X,Y]^{\chi}_{k-2}).$$

Since the diamond operators and the Hecke operators commute, by Lemma 4.13 the above isomorphisms are all Hecke-equivariant. We record this as a

Theorem 4.15. Let $N \geqslant 1$ and $\chi \in \widehat{(\mathbb{Z}/N)^{\times}}$. Then the Eichler-Shimura map induces Hecke-equivariant isomorphisms

$$\begin{array}{ccc} M_k(N,\chi) \oplus \overline{S_k(N,\chi)} & \longrightarrow & H^1(\Gamma_0(N), \mathbb{C}[X,Y]^{\chi}_{k-2}) \\ \subseteq\uparrow & & \subseteq\uparrow \\ S_k(N,\chi) \oplus \overline{S_k(N,\chi)} & \longrightarrow & H^1_{\text{par}}(\Gamma_0(N), \mathbb{C}[X,Y]^{\chi}_{k-2}). \end{array}$$

Also, the real Eichler-Shimura $S_k(\Gamma_1(N)) \xrightarrow{\sim} H^1_{\text{par}}(\Gamma_0(N), \mathbb{R}[X,Y]_{k-2})$ is a Hecke-equivariant isomorphism.

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