GEOMETRY FINAL EXAM

1. Given $L(x,\xi)$, derive the Euler–Lagrange equations of the action

$$S[\gamma] = \int_a^b L(x,\dot{x})\,dt$$

for end-points fixed γ. When x^i and $p_i := \partial L/\partial \dot{x}^i$ can be used as coordinates for (x,ξ), prove its equivalence with Hamilton's equations

$$\dot{x}^i = \partial H/\partial p_i, \qquad \dot{p}_i = -\partial H/\partial x^i,$$

where $H(x,p) := \sum_i p_i \xi^i(x,p) - L(x,\xi(x,p))$ is the energy.

2. Prove Cartan's homotopy formula

$$L_X = \iota_X d + d\iota_X$$

on forms, and show that for a Hamiltonian system $\dot{y} = \nabla H(y)$ defined by a general symplectic form Ω we have $\dot{\Omega} = 0$.

3. Show that the second variation for $S[\gamma] = \frac{1}{2}\int_a^b \sum_{i,j} g_{ij}\dot{x}^i\dot{x}^j\,dt$ is

$$G_\gamma(\xi,\eta) = -\int_a^b \langle \triangledown_T^2 \xi + R(\xi,T)T, \eta\rangle\,dt$$

where γ is a geodesic and $T := d/dt = \gamma' = \sum_i \dot{x}^i\partial_i$. If $R_{ij} \ge (n-1)K\,g_{ij}$ for some constant $K > 0$, show that a curve with length $> \pi/\sqrt{K}$ is not shortest.

4. For the Hilbert–Einstein action $S[g^{ij}] = \int_S R\,d\sigma = \int_S R\sqrt{|g|}\,d^n x$, show that

$$\frac{1}{\sqrt{|g|}}\frac{\delta S}{\delta g^{ij}} = R_{ij} - \tfrac{1}{2}Rg_{ij}$$

for fast decayed g^{ij}. Determine its energy-momentum tensor T_{ij}. (If you do not know how to do it, give its general definition and prove $\triangledown_k T_i^k = 0$.)

5. For $A = \sum_i A_i(x)\,dx^i \in \Lambda^1(\mathfrak{g})$, where $\mathfrak{g} \subset M(n,\mathbb{R})$ is a matrix Lie algebra, define $\triangledown_i = \partial_i + A_i$ and curvature F by $F(X,Y)\psi := ([\triangledown_X, \triangledown_Y] - \triangledown_{[X,Y]})\psi$. Show

(1) F is a tensor and $F = dA + A \wedge A \in \Lambda^2(\mathfrak{g})$, i.e. $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$.
(2) $\triangledown_i B = \partial_i B + [A_i, B]$ for $B(x)$ a $\mathfrak{g}$-valued function,
(3) Bianchi's identity $d_A F = 0$, where $d_A \equiv D : \Lambda^k(\mathfrak{g}) \to \Lambda^{k+1}(\mathfrak{g})$ extends $\triangledown$.
(4) For a non-degenerate Killing form $\langle\,,\rangle$ of $\mathfrak{g}$, any extremal A for

$$S[A] = \int_S \langle F^{ij}, F_{ij}\rangle\,d\sigma$$

with fast decay satisfies the Yang–Mills equation $d_A^* F = 0$.

6. Show anything "significant" in the last two chapters of the book or in my class which you have "well prepared" but not on the above.

Date: pm 1:20 – 5:00, January 13, 2017 at AMB 101, A course by Chin-Lung Wang at NTU. Show your answers/computations/proofs in details. Each problem deserves 20 points.