GEOMETRY 2016 MIDTERM EXAM - II

1. [Lie derivative] (20 pts)

(a) Define $L_\xi T$ of a tensor field T along a vector field ξ and compute $L_\xi T^{i_1\cdots i_p}_{j_1\cdots j_q}$.

(b) Show that $L_\xi \eta = [\xi, \eta]$ for vector field η and $L_\xi\, d = dL_\xi$ on differential forms.

2. [Cartan d and Hodge $*$ in n dimensional Riemannian space] (20 pts)

(a) Define $* : \Lambda^p \to \Lambda^{n-p}$ by

$$(*T)_{i_{p+1}\cdots i_n} = \tfrac{1}{p!}\sqrt{g}\,\epsilon_{i_1\cdots i_n}\,T^{i_1\cdots i_p}.$$

Show that $*^2 = (-1)^{p(n-p)}$ and $T \wedge *S = \{T, S\}\, d\sigma$.

(b) On Ω^p_U, forms supported in a bounded region U, with $\langle \omega_1, \omega_2\rangle := \int_U \omega_1 \wedge *\omega_2$, show that the adjoint of d is given by $\delta := (-1)^{np-n+1} * d*$ and $\delta^2 = 0$.

(c) Let $\triangle = d\delta + \delta d$. Show that $\triangle$ is self-adjoint and it commutes with d, δ and $*$.

(d) $\triangle\omega = 0$ if and only if $d\omega = 0$ and $\delta\omega = 0$. If furthermore $\omega = d\eta$ then $\omega = 0$.

3. [Invariant metric on classical Lie groups] (20 pts)

(a) For a matrix group G with $X \in \mathfrak{g}$, show that R_X define by $R_X(A) = -XA$ for $A \in G$ is right invariant. Also $[R_X, R_Y] = R_{[X,Y]}$ and $[L_X, R_Y] = 0$.

(b) Show that a left invariant metric $\langle\, ,\rangle$ defined by a Killing form is bi-invariant and $\Omega(L_X, L_Y, L_Z) := \langle [L_X, L_Y], L_Z\rangle$ is a 3-form with $d\Omega = 0$.

(c) With $\langle\, ,\rangle$ in (b), determine ∇^{LC} and all geodesics through $e \in G$.

4. [Geodesic normal coordinates] (20 pts)

(a) Show that $\exp_p : \mathbb{R}^n \cong T_p \to U$ defined by geodesics $\xi \mapsto \gamma_\xi(1)$ is invertible near p, and Γ^k_{ij} and $\partial_k g_{ij}$ all vanish at p.

(b) For a surface U with polar coordinates (ρ, θ) induced from T_p, show that $F = \langle \partial_\rho, \partial_\theta\rangle = 0$. Also $G(0, \theta) = 1$, $\sqrt{G}_\rho(0, \theta) = 0$ and $K = -\sqrt{G}_{\rho\rho}/\sqrt{G}$.

5. [Pseudo-Riemannian space with Levi-Civita connection] (20 pts)

(a) Prove the two Bianchi identities $R^i_{[jkl]=0}$ and $R^i_{j[kl;m]=0}$.

(b) Show that $\nabla_i R^i_m = \frac{1}{2}\partial_m R$. If $R_{ij} = \lambda g_{ij}$, when can we deduce that λ is a constant?

(c) For $n = 3$ show that R_{ijkl} is determined by R_{ij}.

6. [Gauss–Bonnet theorem on a surface] (20 pts)

(a) Let α be a piecewise smooth closed curve bounding a region Ω in a surface. Prove that $\int_\Omega K\, dA = \theta_\alpha$ where θ_α is the holonomy angle along the curve.

(b) Prove the local Gauss–Bonnet Theorem

$$2\pi = \sum_{\text{outer angles}} \alpha_j + \int_{\partial\Omega} k_g\, d\ell + \int_\Omega K\, dA.$$

Date: pm 1:20 – 5:00, December 2, 2016 at AMB 101, A course by Chin-Lung Wang at NTU. Show your answers/computations/proofs in details. You may work on each part separately by assuming other parts.