GEOMETRY 2016 MIDTERM EXAM - I

There are **6 problems**. Each problem deserves **20 points**.
Show your answers/computations/proofs in details.
You may work on each part separately by assuming other parts.

1. (a) (10 pts) Prove that a curve in $\mathbb{R}^3$ lies on a sphere of radius R if and only if its curvature κ and torsion τ satisfy

$$R^2 = \frac{1}{\kappa^2}\left(1 + \frac{(d\kappa/d\ell)^2}{(\tau\kappa)^2}\right).$$

(b) (10 pts) For a curve r in $\mathbb{R}^3$ with general parameter, show that

$$\kappa = \frac{|[r', r'']|}{|r'|^3}, \qquad \tau = -\frac{(r', r'', r''')}{|[r', r'']|^2}.$$

2. Let $r : [0, L] \to \mathbb{R}^n$ be a C^∞ curve parameterized by arc length $\ell \in [0, L]$. Assume that r is $n-1$ regular in the sense that $r'(\ell), r''(\ell), \ldots r^{(n-1)}(\ell) \neq 0$ for any ℓ.

(a) (10 pts) Show that a Frenet unitary frame $\mathbf{T} = (\mathbf{t}_1, \ldots, \mathbf{t}_n)^T$ can be defined so that $\mathbf{t}_1 = r'$ and

$$\mathbf{T}' = \begin{pmatrix} & \kappa_1 & & & \\ -\kappa_1 & & \kappa_2 & & \\ & -\kappa_2 & & \ddots & \\ & & \ddots & & \kappa_{n-1} \\ & & & -\kappa_{n-1} & \end{pmatrix} \mathbf{T},$$

where all the other entries are zero.

(b) (10 pts) Conversely, given $\kappa_1, \ldots, \kappa_{n-1} \in C^\infty([0, L])$, show that up to Euclidean motion there is a unique curve r so that its Serret–Frenet frame takes the above form.

(You get partial credit for solving the cases $n = 2, 3$.)

3. Let $S \subset \mathbb{R}^3$ be a non-singular surface with parameterization $r(u, v)$, and let $N : S \to S^2$ be its Gauss map. Denote the two fundamental forms by $d\ell^2 = Edu^2 + 2Fdudv + Gdv^2$ and $Ldu^2 + 2Mdudv + Ndv^2$.

(a) (5 pts) If the two principal curvatures of a connected surface S coincide $\lambda_1 = \lambda_2 = \lambda$ at all points, show that λ is a constant and S is a portion of a sphere or a plane.

(b) (7 pts) Show that the matrix for dN_p, $p \in S$, in the basis r_1, r_2 of T_pS is given by

$$\frac{1}{EG - F^2}\begin{pmatrix} MF - LG & LF - ME \\ NF - MG & MF - NE \end{pmatrix}.$$

Deduce that the Gaussian curvature K equals $(LN - M^2)/(EG - F^2)$.

(c) (8 pts) Gauss' famous Theorema Egregium says that K depends only on the first fundamental form $d\ell^2$. Prove it by any method you know. For example, one way to do it is to show that when $F = 0$, K can be written in terms of $A = \sqrt{E}$ and $B = \sqrt{G}$ as

$$K = -\frac{1}{AB}\left[\left(\frac{A_2}{B}\right)_2 + \left(\frac{B_1}{A}\right)_1\right].$$

Another way is to prove it under isothermal coordinates.

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4. Consider the surface S given by the graph of $z = f(x, y)$ over $\mathbb{R}^2$.

(a) (10 pts) Show that the mean curvature is given by

$$H = \operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right).$$

(b) (5 pts) Let $B \subset \mathbb{R}^2$ be an open ball, $h \in C^\infty(\overline{B})$, $h|_{\partial B} = 0$, and $A_h(t)$ be the area of the surface S_t defined by $z = f(x, y) + th(x, y)$. Show that $A_h'(0) = 0$ for all such h if and only if $H = 0$ over B.

(c) (5 pts) If $H = 0$ over B, is that true S has minimal area among all S_t for t small?

5. Three models of two dimensional hyperbolic geometry.

(a) (4 pts) Lobachevsky plane: Let $L^2 \subset \mathbb{R}^{1,2}$ be the upper half of the space-like surface $t^2 - x^2 - y^2 = 1$, and $d\ell^2$ be the positive metric induced from $\mathbb{R}^{1,2}$. Show that

$$d\ell^2 = d\chi^2 + \sinh\chi\, d\varphi^2$$

in the pseudo-spherical coordinates (χ, φ).

(b) (4 pts) Poincaré model: Compute the stereographic projection $L^2 \to \mathbb{D}^2$ from the south pole $(0, 0, -1)$ onto the unit disk $\mathbb{D}^2$, and show that the induced metric is given by

$$d\ell^2 = \frac{4|dw|^2}{(1-|w|^2)^2}.$$

(c) (4 pts) Klein model: Construct a Möbius transformation $M : \mathbb{H} \cong \mathbb{D}^2$, where $\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$, and show the induced metric is given by

$$d\ell^2 = \frac{dx^2 + dy^2}{y^2}.$$

(d) (4 pts) Determine the (direct) isometry group in one of the above three models.

(e) (4 pts) Determine the set of all geodesics in one of the three models of hyperbolic plane. In particular show that given a point p not in a geodesic γ there are infinitely many geodesics passing through p and disjoint from γ.

6. Theorem L: *A surface S with $d\ell^2 = e^\varphi\,|dz|^2$ and constant K is locally isometric to an open subset of either S_R^2, $\mathbb{R}^2$ or L_R^2.* Prove it by the following steps. Let $\psi(z) := \varphi_{zz} - \frac{1}{2}\varphi_z^2$.

(a) (4 pts) Show that ψ is analytic.

(b) (4 pts) Under an analytic coordinate change $z = f(w)$ we get $d\ell^2 = e^{\widetilde{\varphi}}\,|dw|^2$ and the corresponding $\widetilde{\psi}(w)$. Show that

$$\widetilde{\varphi}(w) = \varphi(z) + \log|f'(w)|^2, \qquad \widetilde{\psi}(w) = (f'(w))^2\psi(z) + \left(\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right)(w).$$

(c) (4 pts) Denote the above RHS term (the Schwarzian derivative) by $S(z; w) \equiv S(f; w)$. Show that $S(z; w) = -S(w; z)(f')^2$, and for any two independent solutions g_1, g_2 of the ODE $g''(z) + I(z)g(z) = 0$ we have $S(g_1/g_2; z) = 2I(z)$.

(d) (4 pts) Show that there exists analytic f such that $\widetilde{\psi} = 0$ in (b). Then conclude that

$$e^{-\widetilde{\varphi}/2} = a|w|^2 + bw + \overline{b}\overline{w} + c$$

for some $a, c \in \mathbb{R}$, $b \in \mathbb{C}$.

(e) (4 pts) Complete the proof of Theorem L by a further change of coordinate.