Vanishing Theorems: A Discussion of Techniques

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Contents

Abstract

Cohomology groups most frequently arise as obstructions to solving global analytic problems, eg. the Mittag-Leffler problem. In this article, our main purpose is to describe several methods to reach vanishing theorems which are useful tools to eliminate cohomology groups and have played a central role in algebraic geometry. We also

0 Introduction

Cohomology groups most frequently arise as obstructions to solving global analytic problems, eg. the Mittag-Leffler problem. In this article, our main purpose is to describe several methods to reach vanishing theorems which are useful tools to eliminate cohomology groups and have played a central role in algebraic geometry.

§1 is devoted to using harmonic theory and some differential geometry formalism to prove Kodaira-Nakano vanishing theorem, which states that let X be a compact complex manifold, the higher cohomology groups $H^q(X, \Omega^p(\mathcal{L})) = 0$ for all $q > 0$ provided that $\mathcal{L} \to X$ is positive and $p + q > \dim X$. The proof can be found in [Gri78] or [Aki]. This is a generalization of the original statement of Kodaira in 1953, which said $H^q(X, \omega_X \otimes \mathcal{L}) = 0$ under the same assumption. Moreover, we introduce Serre vanishing theorem by generaralizing the line bundle in Kodaira-Nakano vanishing theorem to arbitrary coherent sheaf, which said that $H^q(X, \mathcal{O}(\mathcal{L}^\mu) \otimes \mathcal{F}) = 0$ for all $q > 0$ provided that $\mu >> 0$ where $\mathcal{L} \to X$ is a positive line bundle, and $\mathcal{F}$ any coherent sheaf.

In §2, we begin with Hörmander's L^2 existence theorem ([Hör]) to obtain solutions of $\bar{\partial}$-equations and introduce the multiplier ideal sheaf $\mathcal{I}(\varphi)$. Let $\mathcal{L} \to X$ be a line bundle equipped with a singular hermitian metric of weight $e^{-2\varphi}$, where φ is a plurisubharmonic function. Then $\mathcal{I}(\varphi)$ is the sheaf of germs of holomorphic functions f such that $|f|^2 e^{-2\varphi}$ is integrable. Using the L^2 existence theorem, we get an acyclic resolution of $\mathcal{O}(K_X \otimes \mathcal{L}) \otimes \mathcal{I}(\varphi)$, and see Nadel vanishing theorem [Nad]: $H^q(X, \mathcal{O}(K_X \otimes \mathcal{L}) \otimes \mathcal{I}(\varphi)) = 0$ for all $q > 0$. The reference of this section is §5 of [Dem].

Instead of discussing line bundles with semipositivity conditions, we give a more flexible notion for algebraic purpose, the notion of numerical effectivity in §3, and show that Nadel vanishing theorem is a generalization of Kawamata-Viehweg vanishing theorem([Kaw] or [Vie]): $H^q(X, K_X + \lceil D \rceil) = 0$ for $q > 0$, provided that X is a projective algebraic manifold, and D is a nef and big $\mathbb{Q}$-divisor. We discuss also the original (more algebraic) proof of K-V vanishing theorem by applying the concept of a cyclic cover ([Esn]). This can be viewed as application of the generalization of Kodaira vanishing theorem to de Rham log complexes.

In §4 we discuss Deligne's theory of mixed Hodge structures([Del II] or [Del III]). To extend classical Hodge theory on compact nonsingular manifolds, we construct mixed Hodge structure on varieties with normal crossings and nonsingular quasi-projective varieties. Deligne's technique is to express the cohomology of the variety as the abutment of a degenerate spectral sequence ($E_\infty = E_2$) whose E_2 term is the cohomology of some smooth projective manifold. Thus the E_2 term has a Hodge structure and induces a mixed Hodge structure on the cohomology of our desired variety(see §4, §5 in [Gri]).

In the final §5, we apply the concept of a cyclic cover Y([Esn]) along some smooth divisor on X where X is a quasi-projective space with dimension n. Using the De-Rham complexes we mentioned in §4, one can correlate the cohomology groups $H^p(X, \Omega_X \otimes \mathcal{A})$ with the cohomology groups $H^k(Y \backslash D, \mathbb{C})$. Finally by some projection formula and topological vanishing theorem, we conclude that combining some mixed Hodge structure on cohomology groups and the cyclic cover imply Kodaira vanishing theorem.

1 Kodaira-Nakano Vanishing Theorem

In this section the remarkable result, Kodaira-Nakano vanishing theorem, which provides the most useful condition such that the higher cohomology groups are zero, was discussed. We use the Hodge decomposition of cohomology of a Kähler manifold with value in a vector bundle, together with some differential geometry formalism to give the proof of the vanishing theorem. Furthermore, we reach Serre vanishing theorem by applying Kodaira-Nakano vanishing theorem to $\mathbb{P}^N$ and using

Hilbert Syzygy theorem.

1.1 Harmonic theory and Kodaira-Nakano vanishing theorem

Definition 1.1 Let X be a compact complex manifold, $\mathcal{E}$ a holomorphic vector bundle on X equipped with a hermitian metric and let $D = D' + \bar{\partial}$ be the unique metric connection compatible with the holomorphic structure on $\mathcal{E}$.

a) If A, B are differential operators acting on the algebra $\oplus_{p,q}\Lambda^{p,q}(\mathcal{E})$, their graded commutator is defined by $[A, B] = AB - (-1)^{ab}BA$, where a, b are the degrees of A and B respectively.

b) If moreover, X has a hermitian metric, let A^* be the adjoint operator of A and define $\triangle_A := [A, A^*]$ the associated Laplace operator of A.

Theorem 1.2 Hodge Decomposition Theorem (Harmonic Theory) *Let X be a compact complex manifold, $\mathcal{E}$ a holomorphic vector bundle, both equipped with hermitian metrics, then $\triangle_{\bar{\partial}}$ is an elliptic operator. Let $\mathcal{H}^{p,q} := \ker(\triangle_{\bar{\partial}}\big|_{\Lambda^{p,q}(\mathcal{E})})$*

a) $\dim\mathcal{H}^{p,q} < \infty$,

b) Define $\mathcal{H} : \Lambda^{p,q}(\mathcal{E}) \to \mathcal{H}^{p,q}(\mathcal{E})$ to be the orthogonal projection to $ker\triangle_{\bar{\partial}}$. Then there exists an unique operator, the Green operator, $G_{\bar{\partial}} : \Lambda^{p,q}(\mathcal{E}) \to \Lambda^{p,q}(\mathcal{E})$, satisfying $G_{\bar{\partial}}(\mathcal{H}^{p,q}(\mathcal{E})) = 0, \bar{\partial}G_{\bar{\partial}} = G_{\bar{\partial}}\bar{\partial}$, and $\bar{\partial}^*G_{\bar{\partial}} = G_{\bar{\partial}}\bar{\partial}^*$ such that $I = \mathcal{H} + \triangle_{\bar{\partial}}G_{\bar{\partial}}$ on $\Lambda^{p,q}(\mathcal{E})$.

Remark 1.3 In the above theorem, $\bar{\partial}$ can be replaced by D' or D.

Proof . For the proof, see [Wel].

Proposition 1.4 Basic commutation relations(Hodge-Kähler identities) *Let (X, ω) be a compact Kähler manifold and let L be the operator defined by $Lu = \omega \wedge u$ and $\Lambda = L^*$. Then*

$$[\bar{\partial}^*, L] = \sqrt{-1}D', \quad [\bar{D}'^*, L] = -\sqrt{-1}\bar{\partial},$$

$$[\Lambda, \bar{\partial}] = -\sqrt{-1}D'^*, \quad [\Lambda, D'] = \sqrt{-1}\bar{\partial}^*.$$

Proof . For the proof, see [Gri78]. □

Proposition 1.5 Bochner-Kodaira-Nakano identity *If (X, ω) is a compact Kähler manifold, then the Laplace operators $\triangle_{D'}$ and $\triangle_{\bar{\partial}}$ acting on $\mathcal{E}$-valued forms satisfying the identity*

$$\triangle_{\bar{\partial}} = \triangle_{D'} + [\sqrt{-1}\Theta(\mathcal{E}), \Lambda].$$

Moreover, given $u \in \Lambda^{p,q}(\mathcal{E})$, we have

$$\|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2 \geq \int_X \langle [\sqrt{-1}\Theta(\mathcal{E}), \Lambda]u, u\rangle dV_\omega.$$

This inequality is known as the Bochner-Kodaira-Nakano inequality.

Proof .

i) From Proposition 1.4, we have $\bar{\partial}^* = -\sqrt{-1}[\Lambda, D']$, hence

$$\triangle_{\bar{\partial}} = -\sqrt{-1}[\bar{\partial}, [\Lambda, D']].$$

Using the Jacobi Identity, we get

$$\begin{aligned} [\bar{\partial}, [\Lambda, D']] &= [\Lambda, [D', \bar{\partial}]] + [D', [\bar{\partial}, \Lambda]] \\ &= [\Lambda, \Theta(\mathcal{E})] + \sqrt{-1}[D', D'^*], \end{aligned}$$

and

$$\triangle_{\bar{\partial}} = [\sqrt{-1}\Theta(\mathcal{F}), \Lambda].$$

ii) Given $u \in \Lambda^{p,q}(\mathcal{E})$, we have $\langle D', u\rangle = \|D'u\|^2 + \|D'^* u\|^2 \geq 0$. Plug $\triangle_{D'} = \triangle_{\partial} + [\sqrt{-1}\Theta(\mathcal{E}), \Lambda]$ into this formula, we get

$$\|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2 \geq \int_X \langle [\sqrt{-1}\Theta(\mathcal{E}), \Lambda]u, u\rangle dV_\omega.$$

□

Theorem 1.6 Dolbeault Theorem *For X a complex manifold, $H^q(X, \Omega^p(\mathcal{L})) \simeq H^{p,q}_{\bar{\partial}}(X, \mathcal{L})$.*

Proof . For a proof, see [Gri78].

Theorem 1.7 Kodaira-Nakano Vanishing Theorem *Let X be a compact complex manifold with dimension n. If $\mathcal{L} \to X$ is a positive line bundle, then $H^q(X, \Omega^p(\mathcal{L}) = 0$ for $p + q > n$.*

Proof . Since $\mathcal{L} \to X$ is positive, the associated curvature form Θ is a positive (1,1)-form. Let the metric on X be the one given by $\omega = (\sqrt{-1}/2\pi)\Theta$. By Theorem 1.2 and Theorem 1.7,

$$H^q(X, \Omega^p \otimes \mathcal{L}) \simeq \mathcal{H}^{p,q}(\mathcal{L}).$$

It suffices to show there are no nonzero harmonic forms of degree larger than n. Define Let $\eta \in \mathcal{H}^{p,q}(\mathcal{L}), \Theta := D^2 = D'\bar{\partial} + \bar{\partial}D'$, where D is the metric connection on $\mathcal{L}$. Use Hodge Identities,

$$[\Lambda, \bar{\partial}] = -\sqrt{-1}D'^*, [\Lambda, D'] = \sqrt{-1}\bar{\partial}^*$$

we get

$$\begin{aligned} \sqrt{-1}(\Lambda\Theta\eta, \eta) &= \sqrt{-1}(\Lambda\bar{\partial}D'\eta, \eta) \\ &= \sqrt{-1}((\bar{\partial}\Lambda - \sqrt{-1}D'^*)D'\eta, \eta) \\ &= (D'^*, D'\eta, \eta) \\ &= (D'\eta, D'\eta) \geq 0 \end{aligned}$$

since $\bar{\partial}\eta = 0$ implies $\Theta\eta = \bar{\partial}D'\eta$ and $(\bar{\partial}\Lambda D'\eta, \eta) = (\Lambda D'\eta, \bar{\partial}^*\eta) = 0$.

Similarly,

$$\sqrt{-1}(\Theta\Lambda\eta, \eta) \leq 0.$$

Therefore,

$$\sqrt{-1}([\Lambda, \Theta]\eta, \eta) \geq 0.$$

But $\Theta = 2\pi/\sqrt{-1}L$ and

$$\begin{aligned} \sqrt{-1}([\Lambda\Theta]\eta, \eta) &= 2\pi([\Lambda, L]\eta, \eta) \\ &= 2\pi(n - p - q)\|\eta\| \geq 0. \end{aligned}$$

Thus $p+q \geq n \Rightarrow \eta = 0$ $\square$

Remark 1.8 Use this vanishing theorem, Kodaira has shown that: If $\mathcal{L}$ is a line bundle on a compact complex manifold, then $\mathcal{L}$ is positive if and only if $\mathcal{L}$ is ample. (i.e. sections of $\mathcal{L}^N$ give an embedding into $\mathbb{P}^r$ for large N.)

One of the most important application of Theorem 1.2 is Kodaira-Serre duality.

Definition 1.9 Let X be a compact complex manifold with $\dim X = n$ and $\mathcal{E}$ a holomorphic vector bundle, both equipped with hermitian metrics, we ca define

a) the star operator

$$*_{\mathcal{E}} : \Lambda^{p,q}(\mathcal{E}) \longrightarrow \Lambda^{n-p,n-q}(\mathcal{E})$$

by requiring, for $\eta, \psi \in \Lambda^{p,q}(\mathcal{E})$,

$$(\eta, \psi) = \int_X \eta \wedge *_{\mathcal{E}} \psi.$$

b)

$$\bar{*}_{\mathcal{E}} : \Lambda^{p,q}(\mathcal{E}) \longrightarrow \Lambda^{n-p,n-q}(\mathcal{E}^*)$$

by setting

$$\bar{*}_{\mathcal{E}}(\varphi \otimes e) = *(\varphi) \otimes \tau(e),$$

where $\varphi \in \Lambda^{p,q}, e \in \mathcal{E}$, and $\tau : \mathcal{E} \to \mathcal{E}^*$ is the unique conjugate-linear isometry such that $\tau(v)(w) = (w, v)$ for all $v, w \in \mathcal{E}$.

Proposition 1.10 Kodaira-Serre Duality Theorem *Let X be a compact, Kähler manifold of dimension n, $\mathcal{E}$ a holomorphic vector bundle over X. Then*

$$H^q(X, \Omega^p(\mathcal{E})) = H^{n-q}(X, \Omega^{n-p}(\mathcal{E}^*)).$$

Proof . Observe that We have the following commutative diagram, which proves the result immediately.

$$\begin{array}{ccc}
\Lambda^{p,q}(\mathcal{E}) & \xrightarrow{\bar{*}_{\mathcal{E}}} & \Lambda^{n-p,n-q}(\mathcal{E}^*) \\
\downarrow & & \downarrow \\
\mathcal{H}^{p,q}_{\bar{\partial}}(\mathcal{E}) & \xrightarrow{\bar{*}_{\mathcal{E}}} & \mathcal{H}^{n-p,n-q}_{\bar{\partial}}(\mathcal{E}^*) \\
\wr || & & \wr || \\
H^{p,q}_{\bar{\partial}}(\mathcal{E}) & \longrightarrow & H^{n-p,n-q}_{\bar{\partial}}(\mathcal{E}^*) \\
\wr || & & \wr || \\
H^q(X, \Omega^p(\mathcal{E})) & \longrightarrow & H^{n-q}(X, \Omega^{n-p}(\mathcal{E}^*))
\end{array}$$

□

Corollary 1.11 *Let X be a compact, complex manifold, $\mathcal{L} \to X$ a negative line bundle, then $H^q(X, \Omega^p(L)) = 0$ for $p+q < n$.*

Proof . Use Kodaira-Serre duality theorem to dualize Theorem 1.7. □

Corollary 1.12 *Let $H = \mathbb{P}^{n-1} \subset \mathbb{P}^n$ the hyperplane divisor, we have $H^q(\mathbb{P}^n, \mathcal{O}(kH) = 0$ for $q \geq 1, k \in \mathbb{N}$.*

Proof . We have

$$\begin{array}{rcl}
H^q(\mathbb{P}^n, \mathcal{O}(kH) & = & H^q(\mathbb{P}^n, \Omega^n(kH - K_{\mathbb{P}^n}) = 0 \\
& = & H^q(\mathbb{P}^n, \Omega^n((k+n+1)H)) \\
& = & 0
\end{array}$$

for $q \geq 1, k \in \mathbb{N}$ by Theorem 1.7. □

1.2 Serre vanishing theorem

Theorem 1.13 Hilbert Syzygy Theorem *If M is an A-module of finite type and if*

$$0 \to F \to E_{n-1} \to \ldots \to E_1 \to E_0 \to M \to 0$$

is an exact sequence of A-module where E_k's are all free, then F is also free.

Proof . For a proof, see [Kob], p143. □

Lemma 1.14 *Let Y be a subset of X, $\mathcal{F}$ a sheaf of abelian groups on Y, and $j : Y \to X$ the inclusion map. Then $H^i(Y, \mathcal{F}) = H^i(X, j_*\mathcal{F})$ where $j_*\mathcal{F}$ is the extension of $\mathcal{F}$ by zero outside Y.*

Proof. If $(I^\cdot, d)$ is an injective resolution of $\mathcal{F}$ on Y,then $(j_*\mathcal{I}^\cdot, j_*d)$ is also an injective resolution of $j_*\mathcal{F}$ on X. Thus

$$H^i(Y, \mathcal{F}) = H^i_d(Y, I^\cdot) = H^i_d(X, j_*\mathcal{I}^\cdot) = H^i(X, j_*\mathcal{F}).$$

$\square$

The following vanishing theorem for the cohomology of arbitrary coherent sheaf is less precise but broader in scope than theorem 1.7.

Theorem 1.15 Serre vanishing theorem *Let X be a projective manifold, $\mathcal{L} \to X$ a positive line bundle and $\mathcal{F}$ any coherent sheaf, then there is an integer $\mu_0 > 0$ such that $H^q(X, \mathcal{O}(\mathcal{L}^\mu) \otimes \mathcal{F}) = 0$ for all $\mu \geq \mu_0$ and $q \geq 1$.*

Proof. Since $\mathcal{L}$ is positive, there exists some μ_0 and a closed embedding $\iota_{\mathcal{L}^\mu} : X \to \mathbb{P}^r$ for some r such that $\iota^*(\mathcal{O}(H)) = \mathcal{O}(\mathcal{L}^\mu)$ where H is the hyperplane divisor on $\mathbb{P}$. If $\mathcal{F}$ is coherent on X, $\iota_*\mathcal{F}$ is also coherent. By applying Lemma 1.14, we may assume $X = \mathbb{P}^n$ and $\mathcal{O}(\mathcal{L}^\mu) = \mathcal{O}(H)$ in the following. We want to prove the theorem by using descending induction on i. For $i > r$, we have $H^i(X, \mathcal{O}(H) \otimes \mathcal{F}) = 0$ since X can be covered by $r + 1$ open affine spaces. Since $\mathcal{F}$ is coherent, we have the exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \oplus_{i=1}^m \mathcal{O}_X \longrightarrow \mathcal{F} \longrightarrow 0$$

where $m \in \mathbb{Z}$, and $\mathcal{G}$ is a coherent sheaf by Theorem 1.13. We twist the above formula by $\mathcal{O}(H)$ and get a long exact sequence

$$\begin{array}{lll} \to & H^i(X, \mathcal{G} \otimes \mathcal{O}(H)) & \to H^i(X, \oplus_{i=1}^m \mathcal{O}_X \otimes \mathcal{O}(H)) \to H^i(X, \mathcal{F} \otimes \mathcal{O}(H)) \\ \to & H^{i+1}(X, \mathcal{G} \otimes \mathcal{O}(H)) & \to \ldots. \end{array}$$

Note that

$$\begin{array}{rcl} H^i(X, \oplus_{i=1}^m \mathcal{O}_X \otimes \mathcal{O}(H)) & = & \oplus_{i=1}^m H^i(X, \mathcal{O}(H)) = 0 \\ \textit{and} \quad H^{i+1}(X, \mathcal{G} \otimes \mathcal{O}(H)) & = & 0, \end{array}$$

by Corollary 1.12 and induction hypothesis, thus we get $H^i(X, \mathcal{F} \otimes \mathcal{O}(H)) = 0$ $\square$

Corollary 1.16 Nakano vanishing theorem *Let X a compact, complex manifold and $\mathcal{L} \to X$ a positive line bundle. Then for any holomorphic vector bundle $\mathcal{E}$, there exists μ such that*

$$H^q(\mathcal{E}, \mathcal{O}(\mathcal{L}^\mu \otimes \mathcal{E})) = 0 \quad \textit{for } q > 0, \mu \geq \mu_0.$$

Proof . This is a special case of Theorem 1.16. $\square$

Remark 1.17 We can also prove Theorem 1.16 by a method similar to the proof of Theorem 1.7.

2 Nadel Vanishing Theorem

We start from the L^2 existence theorem and introduce the multiplier ideal sheaf $\mathcal{I}(\varphi)$. At last, we reach not only Nadel vanishing theorem, but also obtain a quantitative nature about the solution of $\bar{\partial}$-equation.

2.1 Hörmander's L^2 estimate

Theorem 2.1 L^2 existence Theorem *Let (X, ω) be a Kähler manifold. Here X is not necessarily compact, but we assume the geodesic distance δ_ω is complete on X. Let $\mathcal{E}$ be a hermitian vector bundle of rank r over X with a positive definite curvature operator $A = A^{p,q}_{\mathcal{E},\omega} = [\sqrt{-1}\Theta(\mathcal{E}), \Lambda_\omega]$ on $\Lambda^{p,q}T^*_X \otimes \mathcal{E}, q \geq 1$. Then for any form $g \in L^2(X, \Lambda^{p,q}T^*_X \otimes \mathcal{E})$ satisfying $\bar{\partial} g = 0$ and $\int_X < A^{-1}g, g > dV_\omega < \infty$, there exists $f \in L^2(X, \Lambda^{p,q-1}T^*_X \otimes \mathcal{E})$ such that $\bar{\partial} f = g$ and*

$$\int_X |f|^2 dV_\omega \leq \int_X < A^{-1}g, g > dV_\omega$$

Proof . Since δ_ω is complete, there exists cut-off functions $\psi_\nu$with arbitrary large support such that $|d\psi_\nu| \leq 1$. It follows that every form $u \in L^2(X, \Lambda^{p,q}T^*_X \otimes \mathcal{E})$ such that $\bar{\partial} u \in L^2, \bar{\partial}^* u \in L^2$ in the distribution sense is a limit of a sequence of smooth forms u_ν with compact support such that $u_\nu \to u, \bar{\partial} u_\nu \to \bar{\partial} u, \bar{\partial}^* u_\nu \to \bar{\partial}^* u$ in L^2. Therefore Ker$\bar{\partial}$ is weakly (hence strongly) closed and

$$L^2(X, \Lambda^{p,q} T_X^* \otimes \mathcal{E}) = Ker\bar{\partial} \oplus (Ker\bar{\partial})^\perp.$$

According to the decomposition, write $v = v_1 + v_2$ for all $v \in \mathcal{D}^{p,q} \otimes \mathcal{E})$. Use Cauchy-Schwartz inequality, $(\text{Ker}\bar{\partial})^\perp \subset \text{Ker}\bar{\partial}^*$, and Proposition 1.5 we get

$$|| < g, v > ||^2 = || < g, v_1 > ||^2 \le (\int_X < A^{-1} g, g > dV_\omega)(\int_X < A v_1, v_1 > dV_\omega),$$

$$\int_X < A v_1, v_1 > dV_\omega \le \|\bar{\partial} v_1\|^2 + \|\bar{\partial}^* v_1\|^2 = \ |\bar{\partial} v_1\|^2 = \|\bar{\partial} v\|^2$$

Combining these inequalities, we have

$$|| < g, v > ||^2 \le (\int_X < A^{-1} g, g > dV_\omega) \|\bar{\partial} v\|^2$$

for every smooth (p, q)-form v with compact support. Define $T : \bar{\partial}^*(\mathcal{D}^{p,q} \otimes \mathcal{E}) \to \mathbb{C}$, $w = \bar{\partial} v \mapsto < v, g >$. T is a well-defined continuous linear functional on the range of $\bar{\partial}^*$ with norm $\le (\int_X < A^{-1} g, g > dV_\omega)^{1/2}$. By Hahn-Banach theorem, there exists some $f \in L^2(X, \Lambda^{p,q} T_X^* \otimes \mathcal{E})$ with $\|f\| \le \int_{\mathcal{E}} < A^{-1} g, g > dV_\omega$ for every v such that $< v, g > = < \bar{\partial} v, f >$ for every v, hence $\bar{\partial} f = g$ in the distribution sense. □

We want to show that L^2 existence theorem can be applied to fairly general manifolds,e.g. weakly pseudoconvex manifolds.

Definition 2.2 A function $u : \Omega \to [-\infty, +\infty[$ defined on an open subset $\Omega \subset \mathbb{C}^n$ is said to be plurisubharmonic (psh for short) if

a) u is upper semicontinuous;

b) for every complex line $L \subset \mathbb{C}^n$, $u\Big|_{\Omega \cap L}$ is subharmonic on $\Omega \cap L$ (i.e. for all $a \in \Omega$ and $\xi \in \mathbb{C}^n$ with $|\xi| < d(a, \partial\Omega)$, the function u satisfies the mean value inequality $u(a) \le \frac{1}{2\pi} \int_0^{2\pi} u(a + \xi e^{\sqrt{-1}\theta}) d\theta$

The set of psh functions on Ω is denoted by Psh(Ω)

Definition 2.3 ψ is said to be an exhaustion function if for every $c<0$ the sublevel set $X_c=\{x\in X:\psi(x)\leq c\}$ is relatively compact.

Definition 2.4 We say that a complex manifold X is weakly pseudoconvex if there exists a smooth psh exhaustion function ψ on X.

Theorem 2.5 *Let (X,ω) be a Kähler manifold, $dimX=n$. Assume X is weakly pseudoconvex. Let $\mathcal{L}$ be a hermitian line bundle and let $\gamma_1(x)\leq\ldots\leq\gamma_n(x)$ be the curvature eigenvalues(i.e. the eigenvalues of $\sqrt{-1}\Theta(\mathcal{L})$ with respect to the metric ω) at x. Assume that the curvature is positive (i.e. $\gamma_1\geq 0$ everywhere). Then for any form $g\in L^2(X,\Lambda^{p,q}T_X^*\otimes\mathcal{L})$ satisfying $\bar\partial g=0$ and $\int_X(\gamma_1+\ldots+\gamma_q)^{-1}|g|^2dV_\omega<\infty$, there exists $f\in L^2(X,\Lambda^{p,q-1}T_X^*\otimes\mathcal{E})$ such that $\bar\partial f=g$ and*

$$\int_X|f|^2dV_\omega\leq\int_X(\gamma_1+\ldots+\gamma_q)^{-1}|g|^2dV_\omega$$

Proof .We first show that every weakly pseudoconvex Kähler manifold carries a complete Kähler metric : let $\psi\geq 0$ be a psh exhaustion function and set

$$\omega_\epsilon=\omega+\epsilon\sqrt{-1}\partial\bar\partial\psi^2=\omega+2\sqrt{-1}\epsilon(\psi\partial\bar\partial\psi+\partial\psi\wedge\bar\partial\psi).$$

Then each ω_ϵ defines a Kähler metric on X with $|d\psi|_{\omega_\epsilon}\leq 1/\epsilon$ and $|\psi(x)-\psi(y)|\leq\epsilon^{-1}\delta_{\omega_\epsilon}(x,y)$. It follows that the geodesic balls are relative compact, hence δ_{ω_ϵ} is complete for every $\epsilon>0$. Apply Theorem 2.1 to each δ_{ω_ϵ} and pass to the limit. Since $\langle Au,u\rangle\geq(\gamma_1+\ldots+\gamma_q)|u^2|$, we have $\langle A^{-1}u,u\rangle\leq(\gamma_1+\ldots+\gamma_q)^{-1}|u|^2$, hence

$$\int_X|f|^2dV_\omega\leq\int_X(\gamma_1+\ldots+\gamma_q)^{-1}|g|^2dV_\omega.$$

□

An important observation is that Theorem 2.5 still applies when the metric on $\mathcal{E}$ is singular with positive curvature in the sense of currents.

Definition 2.6 A singular metric on a line bundle $\mathcal{L}$ is a metric which is defined in any trivilization $\theta:\mathcal{L}\Big|_U\overset{\simeq}{\to}U\times\mathbb{C}$ by

$$\|\xi\|=|\theta(\xi)|e^{varphi(x)},\quad x\in U,\xi\in\mathcal{L}_x$$

where $\varphi \in L^1_{loc}(U)$ is an arbitrary function, called the weight of the metric with respect to the trivialization θ.

Corollary 2.7 *Let (X, ω) be a Kähler manifold, $dimX = n$. Assume X is weakly pseudoconvex. Let $\mathcal{E}$ be a holomorphic line bundle equipped with a singular metric whose local weight are denoted by $\varphi \in L^1_{loc}$. Suppose that $\sqrt{-1}\Theta(\mathcal{E}) = 2\sqrt{-1}\partial\bar{\partial}\varphi \geq \epsilon\omega$ for some $\epsilon > 0$. Then for any form $g \in L^2(X, \Lambda^{p,q}T^*_X \otimes \mathcal{E})$ satisfying $\bar{\partial}g = 0$, there exists $f \in L^2(X, \Lambda^{p,q-1}T^*_X \otimes \mathcal{E})$ such that $\bar{\partial}f = g$ and*

$$\int_X |f|^2 e^{-2\phi} dV_\omega \leq \frac{1}{q\epsilon} \int_X |g|^2 e^{-2\phi} dV_\omega.$$

Proof . Using convolution of psh functions by smoothing kernel, the metric can be made smooth and the solutions for the smooth metric have limits satisfying the desired estimates. □

2.2 Nadel's vanishing theorem

Now we introduce the concept of multiplier ideal sheaf $\mathcal{I}(\varphi)$. The ideal $\mathcal{I}(\varphi)$ plays an important role between analytic geometry and algebraic geometry because it converts an analytic object into an algebraic one, and takes care of singularities in a very efficient way.

Definition 2.8 Let φ be a psh function on an open subset $\Omega \subset X$. Define the multiplier ideal subsheaf $\mathcal{I}(\varphi) \subset \mathcal{O}_\Omega$ by $\mathcal{I}(\varphi)(U) = \{f \in \mathcal{O}_\Omega(U) : |f|^2 e^{-2\varphi}$ is integrable with respect to Lebesgue measure in some local coordinates near x for every $x \in U\}$.

We see the zero variety $V(\mathcal{I}(\varphi))$ is the set of points in a neighborhood on which $e^{-2\varphi}$ is not integrable and such points occur only if φ has logarithmic poles.

Definition 2.9 A psh function φ is said to have a logarithmic pole of coefficient γ

at a point $x \in X$ if the Lelong number

$$\nu(\varphi, x) := \liminf_{z \to x} \frac{\phi(z)}{\log|z-x|}$$

is nonzero and $\nu(\varphi, x) = \gamma$.

Lemma 2.10 *Let φ be a psh function on an open set Ω and let $x \in \Omega$*

a) If $\nu(\varphi, x) < 1$, then $e^{-2\phi}$ is integrable in a neighborhood of x; in particular, $\mathcal{I}(\varphi)_x = \mathcal{O}_{\Omega,x}$

b) If $\nu(\varphi, x) \geq n + s$ for some integer $s \geq 0$, then $e^{-2\varphi} \geq C|z-x|^{-2n-2s}$ in a neighborhood of x and $\mathcal{I}(\phi)_x \subset m_{\Omega,x}^{s+1}$, where $m_{\Omega,x}$ is the maximal ideal of $\mathcal{O}_{\Omega,x}$

c) The zero variety $V(\mathcal{I}(\varphi)$ of $\mathcal{I}(\varphi)$ satisfies

$$E_n(\varphi) \subset V(\mathcal{I}(\varphi) \subset E_1(\varphi)$$

where $E_c(\varphi) = x \in X : \nu(\varphi, x) \geq c$ is the c-sublevel set of Lelong numbers of varphi.

Proof . The proofs are mostly elementary calculation and we omit them (for details, see [Dem]). □

Lemma 2.11 Krull lemma *Let R be a noetherian ring, I be an ideal of R, and $\mathcal{E} \subset N$ be finitely-generated R-modules, then for all $n > 0$, there exists an $n' > n$ such that $\mathcal{E} \cap I^{n'} N \subset I^n \mathcal{E}$.*

Proof . For the proof, see [Ati].

Proposition 2.12 *For any psh function φ on $\Omega \in X$, the sheaf $\mathcal{I}(\varphi)$ is a coherent sheaf of ideal over Ω.*

Proof . Since the result is local, we may assume that Ω is the unit ball in $\mathbb{C}^n$. Let E be the set of all holomorphic functions f on Ω such that $\int_\Omega |f|^2 e^{-2\varphi}\, d\lambda < +\infty$, then the set E generates a coherent ideal sheaf $\mathcal{J} \subset \mathcal{O}_\Omega$ We claim: $\mathcal{J} = \mathcal{I}(\varphi)$. $\mathcal{J} \subset \mathcal{I}(\varphi)$ is clear; in order to prove the equality, by Krull lemma we need only check that $\mathcal{J}_x + \mathcal{I}(\varphi)_x \cap m_{\Omega,x}^{s+1} = \mathcal{I}(\varphi)_x$ for every integer s. Let $f \in \mathcal{I}(\varphi)_x$ be defined in a neighborhood V of x and let θ be a cut-off function with support in V such that $\theta = 1$ in a neighborhood of x. We solve the equation $\bar{\partial} u = g := \bar{\partial}(\theta f)$ by Theorem 2.1, where F is the trivial line bundle $\Omega \times \mathbb{C}$ equipped with the strictly psh weight

$$\tilde{\varphi}(z) = \varphi(z) + (n+s)\log|z-x| + |z|^2.$$

We get a solution u such that $\int_\Omega |u|^2 e^{-2\varphi}|z-x|^{-2(n+s)} d\lambda < \infty$, thus $F = \theta f - u$ is holomorphic, $F \in E$ and $f_x - F_x = u_x \in \mathcal{I}(\varphi)_x \cap m_{\Omega,x}^{s+1}$. □

The multiplier ideal sheaves are functorial with respect to direct images of sheaves by modifications.

Proposition 2.13 *Let $\mu : X' \to X$ be a modification of nonsingular complex manifolds (i.e. a proper generically 1 : 1 holomorphic map), and let φ be a psh function on X. Then*

$$\mu_\star(\mathcal{O}(K_{X'}) \otimes \mathcal{I}(\varphi \circ \mu)) = \mathcal{O}(K_{X'}) \otimes \mathcal{I}(\varphi)$$

Proof .Let $n=$ dim$X=$ dimX' and let $S \subset X$ be an analytic subset such that $\mu : X' \backslash S' \to X \backslash S$ is a biholomorphism. Recall that $\mathcal{O}(K_X) \otimes \mathcal{I}(\varphi)$ is the sheaf of holomorphic n-forms f on open sets $U \subset X$ such that$\sqrt{-1}^{n^2} f \wedge \bar{f} e^{-2\varphi}$ in $L^1_{loc}(U)$. Since φ is locally bounded from above and $f \in L^2_{loc}(U)$, we may even consider f as a form defined only on $U \backslash S$, then use Hartogs' extension theorem to extend f through S. The change of variable formula yields

$$\int_U \sqrt{-1}^{n^2} f \wedge \bar{f} e^{-1\varphi} = \int_{\mu^{-1}(U)} \sqrt{-1}^{n^2} \mu^\star f \wedge \overline{\mu^\star f} e^{-2\varphi\circ\mu},$$

hence $f \in \Gamma(U, \mathcal{O}(K_X) \otimes \mathcal{I}(\varphi))$ iff $\mu^\star f \in \Gamma(\mu^{-1}(U), \mathcal{O}(K_{X'}) \otimes \mathcal{I}(\varphi \circ \mu))$ □

Definition 2.14 A psh function $u \in Psh(X)$ will be said to have analytic singularities if u can be written locally as

$$u = \frac{\alpha}{2}\log(|f_1|^2 + \ldots + |f_N|^2) + v,$$

where $\alpha \in \mathbb{R}_+$, v is a locally bounded function and the f_j are holomorphic functions. If X is algebraic, we say u has algebraic singularities if u can be written as above on sufficiently small Zariski open sets, with $\alpha \in \mathbb{Q}_+$ and f_j algebraic.

Remark 2.15 In some special cases, we can explicitly express $\mathcal{I}(\varphi)$. Consider that φ has the form $\varphi = \sum \alpha_j \log|g_j|$ where $D_j = g_j^{-1}(0)$ are nonsingular irreducible divisors with normal crossings, then $\mathcal{I}(\varphi)$ is the sheaf of function h such that $|h|^2 \prod |g_j|^{-2\alpha_j}$ is locally integrable. Since locally g_j can be taken to be coordinate functions from a local coordinate system $(z_1, \ldots, z_n)$, the condition is that h is divisible by $\prod |g_j|^{m_j}$ where $m_j - \alpha_j > -1$ for each j, i.e. $m_j \geq \lfloor \alpha_j \rfloor (integer)$. Hence

$$\mathcal{I}(\varphi) = \mathcal{O}(-\lfloor D \rfloor) = \mathcal{O}(-\sum \lfloor \alpha_j \rfloor D_j)$$

In general, consider the analytic singularities $\varphi \sim \frac{\alpha}{2}\log(|f_1|^2 + \ldots + |f_N|^2)$ near the poles. We may assume that the (f_j) are generators of the integrally closed ideal sheaf $\mathcal{J} = \mathcal{J}(\varphi/\alpha)$, defined as the sheaf of holomorphic functions h such that $|h| \leq C\exp(\varphi/\alpha)$. First, one computes a smooth modification $\mu : \tilde{X} \to X$ of X such that $\mu^\star\mathcal{J}$ is an invertible sheaf $\mathcal{O}(-D)$ associated with a normal crossing divisor $D = \sum \lambda_j D_j$, where (D_j) are the components of the exceptional divisor of $\tilde{X}$ (take the blow-up X' of X with respect to the ideal $\mathcal{J}$ so that the pull-back of $\mathcal{J}$ to X' becomes an invertible sheaf $\mathcal{O}(-D')$, then blow up again by[Hir64] to make X' smooth and D' have normal crossings). Now, we have $K_{\tilde{X}} = \mu^\star K_X + R$ where $R = \sum \rho_j D_j$ is the zero divisor of the Jacobian function J_μ of the blow-up map. By Theorem 2.12, we get

$$\mathcal{I}(\varphi) = \mu_\star\left(\mathcal{O}(K_{\tilde{X}} - \mu^* K_X) \otimes \mathcal{I}(\varphi \circ \mu)\right) = \mu_*\left(\mathcal{O}(R) \otimes \mathcal{I}(\varphi \circ \mu)\right).$$

Now, $(f_j \circ \mu)$ are generators of the ideal $\mathcal{O}(-D)$, hence

$$\varphi \circ \mu \sim \alpha \sum \lambda_j \log |g_j|$$

where g_j are local generators of $\mathcal{O}(-D_j)$. Thus we are reduced to computing multiplier ideal sheaves in the case where the poles are given by a $\mathbb{Q}$-divisor with normal crossings $\sum \alpha\lambda_j D_j$. We obtain $\mathcal{I}(\varphi \circ \mu) = \mathcal{O}(-\sum \lfloor \alpha\lambda_j \rfloor D_j)$, hence

$$\mathcal{I}(\varphi) = \mu_* \mathcal{O}_{\tilde{X}} \left(\sum (\rho_j - \lfloor \alpha\lambda_j \rfloor) D_j \right).$$

We conclude this section with Nadel vanishing theorem which might be the most central results of analytic and algebraic geometry.

Theorem 2.16 Nadel Vanishing Theorem *Let (X, Ω) be a Kähler weakly psudoconvex manifold, and let $\mathcal{L}$ be a holomorphic line bundle over X equipped with a singular hermitian metric h of weight φ. Assume that $\sqrt{-1}\Theta_h(\mathcal{L}) \geq \epsilon\omega$ foe some continuous positive function ϵ on X. Then*

$$H^q(X, \mathcal{O}(K_X \otimes \mathcal{L}) \otimes \mathcal{I}(h)) = 0 \textit{ for all } q \geq 1$$

Proof . Let $\mathcal{L}^q := \Lambda^{n,q}_{L^2_h} \otimes \mathcal{L}$ be the sheaf of germs of (n,q)-forms u with values in $\mathcal{L}$ and with measurable coefficients such that both $|u|^2 e^{-2\varphi}$ and $|\bar{\partial} u|^2 e^{-2\varphi}$ are locally integrable.

$$0 \to \mathcal{O}(K_X \otimes \mathcal{L}) \otimes \mathcal{I}(h) \xrightarrow{\bar{\partial}} \Lambda^{n,0}_{L^2_h} \otimes \mathcal{L} \xrightarrow{\bar{\partial}} \Lambda^{n,1}_{L^2_h} \otimes \mathcal{L} \xrightarrow{\bar{\partial}} \ldots \to 0$$

Then by Corollary 2.7, we get $(\mathcal{L}^{\cdot}, \bar{\partial})$ is a resolution of the sheaf $\mathcal{O}(K_X \otimes \mathcal{L})$. Since each sheaf $\mathcal{L}^q$ is a $\mathcal{C}^\infty$-module, $\mathcal{L}^q$ is an acyclic resolution. Let ψ be a smooth psh exhaustion function on X and apply Corollary 2.7 globally on X, with the original metric on $\mathcal{L}$ multiplied by the factor $e^{\chi \circ \psi}$, where χ is a convex increasing function of arbitrary fast growth at infinity. This factor is to ensure the convergence of the integral at infinity. Therefore, we conclude that $H^q(\Gamma(X, \dot{\mathcal{L}})) = 0$ (i.e. $H^q(X, \mathcal{O}(K_X \otimes \mathcal{L}) \otimes \mathcal{I}(h)) = 0$) for all $q \geq 1$. $\square$

3 Kawamata-Viehweg Vanishing Theorem

3.1 A proof as a special case of Nadel vanishing theorem

In this section, we claim that Nadel vanishing theorem can be seen as a generalization of Kawamata-Viehweg vanishing theorem. First we introduce a more flexible notion suitable for algebraic purposes, the notion of numerical effectivity.

Definition 3.1 A holomorphic line bundle $\mathcal{L}$ over a projective manifold X is said to be numerically effective (nef for short), if $\mathcal{L} \cdot C = \int_C c_1(\mathcal{L}) \geq 0$ for every curve $C \subset X$

Lemma 3.2 Nakai-Moishezon ampleness criterion *A line bundle $\mathcal{L}$ is ample if and only if $\mathcal{L}^p \cdot Y = \int_C c_1(\mathcal{L})^p > 0$ for every p-dimensional subvariety.*

Proof . For a proof, see [Har].

Proposition 3.3 *Let $\mathcal{L}$ be a line bundle on a projective algebraic manifold X, on which an ample line bundle $\mathcal{A}$ and a hermitian metric ω are given. The following properties are equivalent:*

a) $\mathcal{L}$ is nef;

b) For any integer $k \geq 1$, the line bundle $\mathcal{L}^k \otimes \mathcal{A}$ is ample;

c) for every $\epsilon \geq 0$, there is a smooth metric h_ϵ on $\mathcal{L}$ such that $\sqrt{-1}\Theta_{h_\epsilon}(\mathcal{L}) \geq -\epsilon\omega$.

Proof . *a)* $\Rightarrow$ *b)* If $\mathcal{L}$ is nef and A is ample then clearly $\mathcal{L}^k \otimes \mathcal{A}$ satisfies the Nakai-Moishezon criterion, hence $\mathcal{L}^k \otimes \mathcal{A}$ is ample.

b) $\Rightarrow$ *c)* Statement c) is independent of the choice of the hermitian metric, so we may select a metric $h_\mathcal{A}$ on $\mathcal{A}$ with positive curvature and set $\omega = \sqrt{-1}\Theta(\mathcal{A})$. If $\mathcal{L}^k \otimes \mathcal{A}$ is ample, the bundle has a metric $h_{\mathcal{L}^k \otimes \mathcal{A}}$ of positive curvature, then the metric $h_\mathcal{L} = (h_{\mathcal{L}^k \otimes \mathcal{A}} \otimes h_A^{-1})^{1/k}$ has the curvature

$$\sqrt{-1}\Theta(\mathcal{L}) = \frac{1}{k}(\sqrt{-1}\Theta(\mathcal{L}^k \otimes \mathcal{A}) - \sqrt{-1}\Theta(\mathcal{A})) \geq -\frac{1}{k}\sqrt{-1}\Theta(\mathcal{A});$$

we can make the negative part smaller than $\epsilon\omega$ by taking k large enough.

$c) \Rightarrow a)$ Under hypothesis c), we get $\mathcal{L} \cdot C = \int_C \frac{\sqrt{-1}}{2\pi}\Theta_{h_\epsilon}(L) \geq -\epsilon 2\pi \int_C \omega$ for every curve C and every $\epsilon > 0$, hence $\mathcal{L} \cdot C \geq 0$ and $\mathcal{L}$ is nef. $\square$

Definition 3.4 If $\mathcal{L}$ is a line bundle, the Kodaira-Iitaka dimension $\kappa(\mathcal{L})$ is the supremum of the rank of the canonical maps for $m \geq 1$

$$\begin{array}{rlll} \Phi_m : & X \backslash B_m & \longrightarrow P(V_m^*), \\ & x & \longmapsto H_x = \{\sigma \in V_m : \sigma(x) = 0\}, \end{array}$$

with $V_m = H^0(X, \mathcal{L}^m)$ and $B_m = \cap_{\sigma \in V_m} \sigma^{-1}(0) =$ base locus of V_m. In case $V_m = \{0\}$ for all $m \geq 1$, we set $\kappa(\mathcal{L}) = -\infty$. A line bundle is said to be big if $\kappa(\mathcal{L}) = \dim X$.

Lemma 3.5 Serre-Siegel lemma *Let $\mathcal{L}$ be any line bundle on a compact complex manifold. Then we have*

$$h^0(X, \mathcal{L}^m) \leq O(m^{\kappa(\mathcal{L})}) \text{ for } m \geq 1,$$

and $\kappa(\mathcal{L})$ is the smallest constant for which this estimate holds

Proof . The proof is a rather elementary consequence of Schwartz lemma, and we omit it.

Proposition 3.6 *Let X be a projective algebraic variety, $\mathcal{L}$ a nef line bundle, and $\mathcal{E}$ an arbitrary holomorphic vector bundle. Then*

$$h^q(X, \mathcal{O}(\mathcal{E} \otimes \mathcal{L}^k) = O(k^{n-q}) \text{ for all } q \geq 0, \text{ as } k \to +\infty$$

Proof . We proceed by induction on $n = \dim X$. If $n = 1$, the result follows by using Rieman-Roch Theorem. Since $\mathcal{L}$ is nef, there exists some k_0 such that for each $k \geq k_0$, we can find a ample divisor A such that $H^q(X, \mathcal{O}(\mathcal{E} \otimes \mathcal{L}^k \otimes \mathcal{O}(A)) = 0$ for all $q \geq 1$ by Corollary 1.16. Let $\mathcal{A} = \mathcal{O}(A)$. The exact sequence

$$0 \to \mathcal{O}(\mathcal{L}^k) \to \mathcal{O}(\mathcal{L}^k) \otimes \mathcal{A}) \to \mathcal{O}(\mathcal{L}^k \otimes \mathcal{A})\Big|_A \to 0$$

twisted by $\mathcal{O}(\mathcal{E})$ implies

$$H^q(X, \mathcal{O}(\mathcal{E} \otimes \mathcal{L}^k)) \simeq H^{q-1}\left(A, \mathcal{O}(\mathcal{E}|_A \otimes (\mathcal{L}^k \otimes \mathcal{A})|_A)\right),$$

and we easily conclude by induction since $\dim A = n-1$. $\square$

Proposition 3.7 *Let (X, ω) be a compact Kähler manifold. Then $\kappa(\mathcal{L}) = \dim X \Rightarrow$ there exists $\epsilon > 0, h$ (possibly singular) metric on $\mathcal{L}$ such that $\sqrt{-1}\Theta_h(\mathcal{L}) \geq \epsilon\omega$.*

Proof . If $\kappa(L) = n$, then $h^0(X, k\mathcal{L}) \geq ck^n$ for $k \geq k_0$ and $c > 0$ by Lemma 3.5. Fix a nonsingular ample divisor A on X. The exact cohomology sequence

$$0 \to H^0(X, \mathcal{L}^k \otimes \mathcal{O}(-A)) \to H^0(X, \mathcal{L}^k) \to H^0(A, \mathcal{L}^k|_A)$$

where $h^0(A, \mathcal{L}^k|_A) = O(k^{n-1})$ by Lemma 3.6 shows that $\mathcal{L}^k \otimes \mathcal{O}(-A)$ has nonzero sections for k large. If D is the divisor associated to such a section, then $\mathcal{O}(\mathcal{L}^k) = \mathcal{O}(A + D)$. Select a smooth metric on A such that $\frac{\sqrt{-1}}{2\pi}\Theta(A) \geq \varepsilon_0\omega$ for some $\varepsilon \geq 0$ and take the singular metric on $\mathcal{O}(D)$ with weight function $\varphi_D = \sum \alpha_j \log|g_j|$. Then the metric with weight $\varphi_{\mathcal{L}} = \frac{1}{k}(\varphi_A + \varphi_D)$ on $\mathcal{L}$ yields

$$\frac{\sqrt{-1}}{2\pi}\Theta(\mathcal{L}) = \frac{1}{k}\left(\frac{\sqrt{-1}}{2\pi}\Theta(A) + [D]\right) \geq (\varepsilon_0/k)\omega.$$

$\square$

Proposition 3.8 *If $\mathcal{L}$ is nef, then $\mathcal{L}$ is big (i.e. $\kappa(\mathcal{L}) = n$) if and only if $\mathcal{L}^n > 0$. Moreover, if $\mathcal{L}$ is nef and big, then for every $\delta > 0$, $\mathcal{L}$ has a singular metric $h = e^{-2\varphi}$ such that $\max_{x\in X}\nu(\varphi, x) \leq \delta$ and $\sqrt{-1}\Theta_h(\mathcal{L}) \geq \varepsilon\omega$ for some $\varepsilon > 0$. The metric h can be chosen to be smooth on the complement of a fixed divisor D, with logarithmic poles along D.*

Proof . (i) By Proposition 3.6 and the Riemann-Roch Theorem, we have $h^0(X, \mathcal{L}^k) = \chi(X, \mathcal{L}^k) + o(k^n) = k^n L^n/n! + o(k^n)$.

(ii) If $\mathcal{L}$ is big, Proposition 3.7 shows that there is a singular metric h_1 on $\mathcal{L}$ such that

$$\frac{\sqrt{-1}}{2\pi}\Theta_{h_1}(\mathcal{L}) = \frac{1}{k}(\frac{\sqrt{-1}}{2\pi}\Theta(\mathcal{A}) + [D])$$

with a positive line bundle $\mathcal{A}$ and an effective divisor D. Now, for every $\varepsilon > 0$, there is a smooth metric h_ϵ on $\mathcal{L}$ such that $\frac{\sqrt{-1}}{2\pi}\Theta_{h_\varepsilon}(L) \geq -\varepsilon\omega$, where $\omega = \frac{\sqrt{-1}}{2\pi}\Theta(\mathcal{A})$. The convex combination of metrics $h'_\varepsilon = h_1^{k\varepsilon}h_\varepsilon^{1-k\varepsilon}$ is a singular metric with poles along D which satisfies

$$\frac{\sqrt{-1}}{2\pi}\Theta_{h'_\varepsilon}(L) \geq \varepsilon(\omega + [D]) - (1-k\varepsilon)\varepsilon\omega \geq k\varepsilon^2\omega.$$

Its Lelong numbers are $\varepsilon\nu(D,x)$ and they can be made smaller than δ by choosing $\varepsilon > 0$ small. □

Definition 3.9 Let X be a projective algebraic manifold. We say a $\mathbb{Q}$-divisor D on X is big and nef if its associated line bundle $\mathcal{O}(D)$ is big and nef.

Definition 3.10 Given a divisor $D = \sum \alpha_j D_j$ on X where D_j are irreducible hypersurfaces. we define $\lfloor D \rfloor = \sum_j \lfloor \alpha_j \rfloor D_j$; $\lceil D \rceil = \sum_j \lceil \alpha_j \rceil D_j$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x, and $\lceil x \rceil$ is the least integer greater than or equal to x.

Theorem 3.11 Kawamata-Viehweg vanishing theorem *Let X be a projective algebraic manifold. Given $D = \sum_j \alpha_i D_j$ a nef and big $\mathbb{Q}$-divisor with normal crossings, then*

$$H^q(X, K_X + \lceil D \rceil) = 0 \text{ for } q \geq 0$$

Proof . Denote $\mathcal{L} = \mathcal{O}(D)$, $D' = \lfloor D \rfloor - D$. Since $\kappa(\mathcal{L}) = n$, by proof of Proposition 3.7, there is a singular hermitian metric on $\mathcal{L}$ such that the corresponding weight $\varphi_{\mathcal{L},0}$ has algebraic singularities and

$$\sqrt{-1}\Theta_0(\mathcal{L}) = 2\sqrt{-1}\partial\bar{\partial}\varphi_{\mathcal{L}} \geq \epsilon_0\omega$$

for some $\epsilon_0 > 0$. On the other hand, since $\mathcal{L}$ is nef, there are smooth metrics given by weights $\varphi_{\mathcal{L},\epsilon}$ such that $\frac{\sqrt{-1}}{2\pi}\Theta_\epsilon(\mathcal{L}) \geq -\epsilon\omega$ for every $\epsilon > 0, \omega$ being a Kähler metric on X. Let $\varphi_{D'} = \sum \alpha_j \log|g_j|$ be the weight of the singular metric on $\mathcal{O}(D')$,where $\alpha_j \in \mathbb{Q}^+, g_j$ is the generator of the ideal of D_j. We define a singular metric on $\mathcal{O}(\lfloor D \rfloor)$ by

$$\varphi_{\mathcal{O}(\lceil D \rceil)} = (1-\delta)\varphi_{\mathcal{L},\epsilon} + \delta\varphi_{\mathcal{L},0} + \varphi_D,$$

with $\epsilon << \delta << 1, \delta$ rational. Then $\varphi_{\mathcal{O}(\lceil D \rceil)}$ has algebraic singularities, and by taking δ small enough we find $\mathcal{I}(\varphi_{\mathcal{L}}) = \mathcal{I}(\varphi_{D'}) = \mathcal{I}(D')$. In fact, $\mathcal{I}(\varphi_{\mathcal{L}}) = 0$ by taking integer parts of D', and adding $\delta\varphi_{L,0}$ does not change the integer part of the rational numbers involved when δ is small. Then

$$\begin{aligned} \sqrt{-1}\partial\bar{\partial}\varphi_{\mathcal{O}(\lceil D \rceil)} &= ((1-\delta)\varphi_{\mathcal{L},\epsilon} + \delta\sqrt{-1}\partial\bar{\partial}\varphi_{\mathcal{L},0} + \sqrt{-1}\partial\bar{\partial}\varphi'_D) \\ &\geq (-(1-\delta)\epsilon\omega + \delta\epsilon_0\omega + [D']) \geq \tfrac{\delta\epsilon}{m}\omega, \end{aligned}$$

if we choose $\epsilon \leq \delta\epsilon_0$. Applying Nadel's theorem thus implies the desired vanishing result for all $q \geq 1$. □

3.2 A proof using cyclic cover

Definition 3.12 Let X be a scheme, and $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebra. There is an unique scheme Y, and a morphism $f : Y \to X$, such that for every open affine $V \subset X, f^{-1}(V) \cong Spec\mathcal{A}(V)$, and for every inclusion $U \subset V$, the morphism $f^{-1}(U) \hookrightarrow f^{-1}(V)$ corresponds to the restriction homomorphism $\mathcal{A}(V) \hookrightarrow \mathcal{A}(U)$. The scheme X is called **Spec**$\mathcal{A}$

Definition 3.13 Let X be a complex projective manifold and D an effective and normal crossing divisor. Suppose for some line bundle $\mathcal{L}$ and some integer $N \geq 0$, we have

$$\mathcal{L}^N = \mathcal{O}_X(D).$$

Let $s \in H^0(X, \mathcal{L}^N)$ be a section whose zero divisor is D. Then the dual section $\check{s} : \mathcal{L}^{-N} \to \mathcal{O}_X$ defines a $\mathcal{O}_X$-algebra structure on $\mathcal{A}' = \oplus_{i=0}^{N-1}\mathcal{L}^{-i}$. i.e. $\mathcal{A}' =$

$\oplus_{i=0}^{\infty}\mathcal{L}^i/I$ where I is the sheaf generated locally by $\{\check{s}(l) : l$ local section of $\mathcal{L}^{-N}\}$. Let $\pi' : Y' = \mathbf{Spec}_X(\mathcal{A}') \to X$ be the natural map, $n : Y \to Y'$ the normalization, and $\pi = \pi' \circ n$. We will call Y the cyclic cover obtained by taking the N-th root out of D.

Let $E = \sum \nu_j E_j$ be an effective divisor, $\mathcal{M}$ and $\mathcal{L}$ invertible sheaves such that for some large $N > 0$ such that $\mathcal{L}^N = \mathcal{M} \otimes \mathcal{O}(E)$.

Lemma 3.14 *Define* $\mathcal{L}^{(i)} = \mathcal{L}^i(-[\frac{i}{N}D]) = \mathcal{L}^i \otimes \theta_X(-[\frac{i}{N}D])$ *and* $\mathcal{A} = \oplus_{i=0}^{N-1}(\mathcal{L}^{(i)})^{-1}$, *then* $\pi_*\mathcal{O}_Y = \mathcal{A}$.

Proof . For any open subvariety X_0 in X with $\mathrm{codim}_X(X - X_0) \geq 2$ and for $Y_0 = \pi^{-1}(X_0)$ consider the induced morphisms

$$\begin{tikzcd} Y_0 \arrow[r, "\iota'"] \arrow[d, "\pi_0"'] & Y \arrow[d, "\pi"] \\ X_0 \arrow[r, "\iota"] & X \end{tikzcd}$$

Since Y is normal one has $\iota'_*\mathcal{O}_{Y_0} = \mathcal{O}_Y$ and $\pi_*\mathcal{O}_Y = \iota_*\pi_{0_*}\mathcal{O}_{Y_0}$. Note that $\mathbf{Spec}\mathcal{A} \to X$ is finite and $\mathbf{Spec}\mathcal{A}$ is normal. Thus $\pi_{0_*}\mathcal{O}_{Y_0} = \mathcal{A}|_{X_0}$ and the lemma follows since $\mathcal{A}$ is locally free. $\square$

Lemma 3.15 *Let* $\tilde{D} = \pi^{-1}(D)_{red}$, *we have* $\pi^*\Omega_X^a(\log D) \subset \Omega_Y^a(\log \tilde{D})$, *and moreover this inclusion is an isomorphism at all points* p *where* π *is finite and* $p \notin \pi^{-1}(Sing(D_{red}))$.

Proof . It is enough to consider the case $a = 1$. Since both sheaves are subsheaves of the meromorphic differential forms having poles along D', we may check the statement locally at $p \in U$. We choose regular parameters $x_1, \ldots, x_n$ at $\pi(p)$ of X, such that D is defined by $z_1^{\nu_1} \cdots z_r^{\nu_r} = 0$. Hence $\Omega_X^1(\log D)$ is generated by $\frac{dz_1}{z_1}, \cdots, \frac{dz_r}{z_r}, dz_{r+1}, \cdots, dz_n$ over $\mathcal{O}_X$.

Since $\tilde{D}$ has only transversal intersections as singularities we may choose regular parameters $w_1, \dots, w_n$ of p at U, such that for $i = 1, \dots, r$ and some units u_i we have $\pi^* z_i = \prod_j w_j^{\eta_{ij}} \dot{u}_i$. One sees immediately that $\pi^* \frac{dz_i}{z_i} = \sum_j \eta_{ij} \frac{dw_j}{w_j} + u_i^{-1} du_i$ is an element of $\Omega^1_Y(\log \tilde{D})$. If $\pi(p)$ belongs to the regular locus of D we have $r = 1$ and if moreover π is finite, then $\pi^* \frac{dz_1}{z_1}, \pi^* dz_2, \dots, \pi^* dz_n$ generates $\Omega^1_Y(\log \tilde{D})$ in U. $\square$

Lemma 3.16 *With $\pi : Y \to X$ as above, we have*

$$\pi_* \Omega^a_Y(\log \tilde{D}) = \Omega^a_X(\log D) \otimes \pi_* \mathcal{O}_Y$$

and

$$\pi_* \Omega^a_Y \subset \oplus_{i=1}^{N-1} \left(\Omega^a_X(\log D) \otimes \mathcal{L}^{(i)^{-1}} \right) \oplus \Omega^a_Y.$$

Proof . (i)By Lemma 3.15, the former formula follows from projection formula directly. (ii) We define

$$C_D = \oplus_j \Omega^{p-1}_{E_j}(\log(E_j \cap \cup_{k \neq j} E_k)) \oplus \Omega^{p-1}_B(\log(B \cap \cup_j E_j))$$

and similarly for $C_{\tilde{D}}$. We have the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^p_X \arrow[r] \arrow[d] & \Omega^p_X(\log D) \arrow[r] \arrow[d, shift right] & C_D \arrow[d] \\
0 \arrow[r] & \Omega^p_Y \arrow[r] & f_* \Omega^p_Y(\log \tilde{D}) \arrow[r] \arrow[u, shift right] & f_* C_{\tilde{D}}
\end{tikzcd}$$

Applying Lemma 3.15 to the component of D, we find that η is injective, and hence the second statement follows. $\square$

Theorem 3.17 *Let X be a regular projective variety, $\mathcal{F}$ an invertible sheaf with $\kappa(\mathcal{F}) \geq m$ and D an effective divisor with regular components intersecting transversally. Then*

$$H^0(X, \Omega^p_X(\log D) \otimes \mathcal{F}^{-1}) = 0 \textit{ for } p < m.$$

Proof . Observe that any section of the above sheaf defines an inclusion of $\mathcal{F}$ as a subsheaf of $\Omega^p_X(\log D)$. Since $\kappa(\mathcal{F}) \geq m$, some power $\mathcal{F}^\nu$ has $m+1$ sections

$s_0, \ldots, s_m$ such that $\frac{s_1}{s_0}, \cdots, \frac{s_m}{s_0}$ are algebraic independent rational functions. Let $\pi : Y \to X$ be the morphism obtained by taking the ν-th root out of one of the s_i. Lemma 3.15 gives $\pi^*\mathcal{F}$ as subsheaf of $\Omega^p_Y(\log \tilde{D})$. Repeating this construction we may assume $\nu = 1$. We denote the p-forms induced by s_i under the inclusion of $\mathcal{F}$ in $\Omega^p_X(\log D)$ also by s_i. Then $ds_i = 0$ for $i = 0, \ldots, m$ and $0 = ds_i = s_0 \wedge d(\frac{s_i}{s_0})$. This means that $d(\frac{s_1}{s_0}), \cdots, d(\frac{s_m}{s_0})$ span a rank m subsheaf of Ω^1_X over some open set, which sheaf annihilates $\mathcal{F}$ under the exterior product. This is only possible if $p \geq m$. $\square$

Theorem 3.18 *Let notations as above. Assume that $\mathcal{M} = \mathcal{O}_X(B)$ for some prime divisor B and that $D = B + E$ is a divisor with regular components and transversal intersections. If $\kappa(\mathcal{L}^{(i)}) \geq m$ for $i = 1, \ldots, N-1$, then*

$$H^p(X, (\mathcal{L}^{(i)})^{-1}) = 0 \text{ for } p < m.$$

Proof . Let $\pi : Y \to X$ the cyclic cover obtained by taking n-th root out of D. Then the symmetry of the Hodge numbers on X together with Lemma 3.14 gives

$$h^p(\mathcal{O}_X) + \sum_{i=1}^{N-1} h^p((\mathcal{L}^{(i)})^{-1}) = h^p(\mathcal{O}_Y) = h^0(\Omega^p_Y),$$

where $h^p(\cdot)$ denotes the dimension of $H^p(\cdot)$. Using Lemma 3.16 one obtains the inequality

$$h^0(\Omega^p_Y) \leq h^0(\Omega^p_X), + \sum_{i=1}^{N-1} h^0(\Omega^p_X(\log D) \otimes \mathcal{L}^{(i)-1}).$$

Using Theorem 3.17 and the symmetry of the Hodge numbers of X, we see that $H^p(X, (\mathcal{L}^{(i))^{-1}}) = 0$, $\forall p < m$. $\square$

Another proof of Kawamata-Viehweg Theorem. Note that the cover Y of X has at most rational singularity, hence the statement of Theorem 3.11 is compatible with blowing up. Since $\kappa(\mathcal{O}(D)) = n$, we can find $\mathcal{H}$ ample sheaf and B normal crossing divisor such that $\mathcal{O}(D)^N = \mathcal{H} \otimes \mathcal{O}(B)$ for some N. Since D is nef, $\mathcal{H} \otimes \mathcal{O}(D)^\nu$ is ample for all $\nu \geq 0$. Replacing N by $N+\nu$, we may assume N is larger than the multiplicities of the components of B and replacing N, $\mathcal{L}$, $\mathcal{H}$, B by μN, $\mathcal{L}^{\mu}$, $\mathcal{H}^{\mu}$, μB such that $\mathcal{H}$ is very ample. Let H be the divisor with $\mathcal{O}(H)=\mathcal{H}$ and $D=B+H$. Then $\mathcal{L}^{(1)}=\mathcal{L}$ and thus we get $H^p(X,\mathcal{L}\otimes\mathcal{O}(K_X))=0$. Note that since $\mathcal{O}(D)$ is contained in $\mathcal{O}(\lceil D\rceil)$, $\lceil D\rceil$ is also big and nef. Replacing D by $\lceil D\rceil$, we have

$$K^p(X, K_X+\lceil D\rceil)=0, \quad \forall p>0.$$

$\square$

4 Hodge Theory

For a general algebraic variety X (singular, non-compact), the singular cohomology $H^k(X,\mathbb{C})$ does not admits functorial Hodge structures. However, Deligne has shown that it always admits functorial "mixed Hodge structures". In the following, we discuss 2 special cases of Deligne's results, namely for X a normal crossing variety and for X a nonsingular quasi-projective variety.

Let $K^{\cdot}$ be a differential complex with differential operator D, (i.e. $K^{\cdot}$ is an abelian group and $D: K^{\cdot}\to K^{\cdot}$ is a group homomorphism such that $D^2=0$.)

Definition 4.1 A filtered complex $(F^pK^{\cdot},d)$ is a decreasing sequence of subcomplexes

$$K^{\cdot}=F^0K^{\cdot}\supset F^1K^{\cdot}\supset\ldots\supset F^nK^{\cdot}\supset F^{n+1}K^{\cdot}=0$$

The associated graded complex to a filtered complex $(F^pK^{\cdot},d)$ is the complex

$$Gr^{\cdot}_K=\oplus_{p\geq 0}Gr^pF^{\cdot}$$

where

$$Gr^pK^{\cdot}=\frac{F^pK^{\cdot}}{F^{p+1}K^{\cdot}}$$

and the differential is the obvious ons. The filtration $F^pK^\cdot$ on K also induces a filtration on the cohomology by

$$F^pH^q(K^\cdot) = \frac{F^pZ^q}{F^pB^q}.$$

The associated graded cohomology is

$$GrH^\cdot(K^\cdot) = \oplus_{p,q} Gr^pH^q(K^\cdot)$$

where

$$Gr^pH^q(K^\cdot) = \frac{F^pH^q(K^\cdot)}{F^{p+1}H^q(K^\cdot)}$$

Remark 4.2 For notational reasons, we sometimes extend the filtration to negative indices by defining $K_p = K$ for $p \leq 0$

Definition 4.3 A spectral sequence is a sequence $\{E_r, d_r\}(r \geq 0)$ of bigraded groups

$$E_r = \oplus_{p,q\geq 0} E_r^{p,q}$$

together with differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}, d_r^2 = 0$ such that $H^\cdot(E_r) = E_{r+1}$. In practice, we will always have $E_r = E_{r+1} = \ldots$ for $r \geq r_0$; we call this the limit group E_∞ and if E_∞ is equal to the associated graded group of some filtered group H, then we say that the spectral sequence converges to H.

Proposition 4.4 *Let $K^\cdot$ be a filtered complex. Then there exists a spectral sequence E_r with*

$$\begin{aligned} E_0^{p,q} &= F^pK^{p+q}/F^{p+1}K^{p+q} = Gr_pK^{p+q}, \\ E_1^{p,q} &= H^{p+q}(Gr^pK^\cdot), \text{ and} \\ E_\infty^{p,q} &= Gr^p(H^{p+q}(K^\cdot)). \end{aligned}$$

The last statement is usually written as

$$E_r \Rightarrow H^\cdot(K^\cdot)$$

and we say the spectral sequence converges to $H^\cdot(K^\cdot)$.

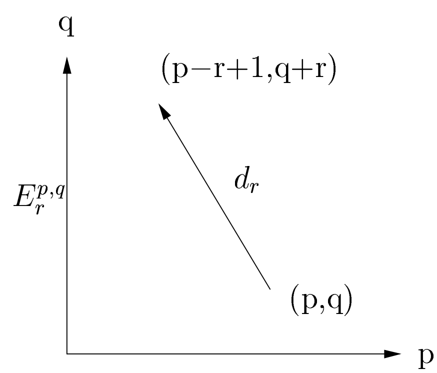
Give a double complex $K^{\cdot,\cdot} = \oplus_{p,q\geq 0} K^{p,q}$, with differentials $d : K^{p,q} \to K^{p+1,p}$ and $\delta : K^{p,q} \to K^{p,q+1}$ satisfying $d^2 = 0, \delta^2 = 0, d\delta = \delta d$. The associated single complex $(K^{\cdot}, D)$ is defined by $K^n = \oplus, D = d + (-1)^p \delta$.

Theorem 4.5 *Give a double complex $K = \oplus_{p,q\geq 0} K^{p,q}$, there is a spectral sequence $\{E_r, d_r\}$ converging to the total cohomology $H_D(K)$ such that each E_r has a bigrading with*

$$d_r : E_r^{p,q} \to E_r^{p-r+1,q+r}$$

and

$$\begin{aligned} E_1^{p,q} &= H_d^{p,q}(K) \\ E_2^{p,q} &= H_\delta^{p,q} H_d(K) \end{aligned}$$



furthermore, the associated graded complex of the total cohomology is given by

$$GH_D^n(K) = \oplus_{p+q=n} E_\infty^{p,q}(K)$$

Definition 4.6 Hodge structure Let $H_{\mathbb{R}}$ be a finite dimensional vector space, containing a lattice $H_{\mathbb{Z}}$, and let $H = H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. A Hodge structure of weight m consists of a direct sum decomposition

$$H = \oplus_{p+q=m} H^{p,q}, \text{ with } H^{q,p} = \bar{H}^{p,q}.$$

To each Hodge structure of weight m, we may associate the Hodge filtration

$$H \supset \ldots \supset F^{p-1} \supset F^p \supset F^{p+1} \supset \ldots \supset 0 \text{ with } F^p = \oplus_{i\geq p} H^{i,m-i}$$

Since $H^{p,q} = F^p \cap \bar{F}^q$ and $F^p \oplus \bar{F}^{m-p} \xrightarrow{\simeq} H$ for all p, one has $1:1$ correspondence between Hodge structures and Hodge filtrations.

Definition 4.7 A mixed Hodge structure on H consists of two filtrations,

$$0 \subset \ldots \subset W_{m-1} \subset W_m \subset W_{m+1} \subset \ldots \subset H,$$

the weight filtration which shall be defined over $\mathbb{Q}$, and

$$H \supset \ldots \supset F^{p-1} \supset F^p \supset F^{p+1} \supset \ldots \supset 0$$

the Hodge filtration, such that the filtration induced by the latter on $Gr_m(W_*) = W_m/W_{m-1}$ defines a Hodge structure of weight m, for each m the induced filtration on $Gr_m(W_*)$ is given by

$$F^p(Gr(W_m)) = (W_m \cap F^p)/(W_{m-1} \cap F^p).$$

Definition 4.8 A morphism between two mixed Hodge structures $\{H, W_m, F^p\}, \{H', W'_m, F'_p\}$ is a rationally defined linear map $\varphi : H \to H'$, such that $\varphi(W_m) \subset W'_m$ and $\varphi(F^p) \subset F'^p$. More generally, a rationally defined linear map $\varphi : H \to H'$ is called a morphism of mixed Hodge structures of type (r, r) if $\varphi(W_m) \subset W'_{m+2r}, \varphi(F^p) \subset F'^{p+r}$, for all p and m.

Definition 4.9 Let V be a compact, Kähler manifold, and $D \subset V$ a smooth divisor. Applying Poincaré duality to the homology mapping

$$H_p(D) \xrightarrow{i} H_p(V)$$

induced by the inclusion $D \subset V$, one obtains the Gysin map

$$H^q(D) \xrightarrow{\gamma} H^{q+2}(V)$$

Remark 4.10 Both the Poincaré duality isomorphisms and i are morphisms of Hodge structures, γ is also a morphism of type $(1, 1)$.

Proposition 4.11 *Let $(K^{\cdot}, d)$ be a finite dimensional complex with a mixed Hodge structure, and such that the differential d is a morphism of mixed Hodge structure of type (r, r) for some r. Then the induced filtration on the cohomology determine a mixed Hodge structure.*

Lemma 4.12 *If $\varphi \in A^{p,q}(V)$ is a d-exact form, V is a compact, Kähler manifold, then we have both $\varphi = \partial\eta'$ for some $\eta' \in A^{p-1,q}$ with $\bar{\partial} = 0$; and $\varphi = \bar{\partial}\eta''$ for some $\eta'' \in A^{p,q-1}$ with $\partial\eta'' = 0$.*

Proof . Use Hodge decomposition theorem and $\triangle_d = 2\triangle_\partial$, we know $\varphi = \triangle_\partial G_\partial \varphi$, where G_∂ is the Green's operator for ∂ which commutes with ∂. Then $\varphi = \partial\eta'$ where $\eta' = \partial^* G_\partial \varphi$ has type $(p-1, q)$; another case is similar. □

Remark 4.13 The use of the above lemma comes up in the principle of two types: If $[\varphi] \in H^m(V, \mathbb{C})$ can be represented by $\varphi' \in A^{p',q'}(V)$, and also by $\varphi'' \in A^{p'',q''}(V)$ with $p' \neq p''$, then $[\varphi] = 0$

4.1 Varieties with normal crossings

According to Hironaka, a suitable modification of an arbitrary variety is a variety with normal crossings. Thus We consider a compact analytic space V,which can be realized as a union of coordinate hyperplanes

$$\{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} | z_1 \cdot z_2 \ldots z_k = 0\}$$

We assume that globally $V = D_1 \cup \ldots \cup D_N$, where the D are compact Kähler manifolds meeting transversely.

Theorem 4.14 *On $H^{\cdot}(V, \mathbb{C})$, there exists a mixed Hodge structure, which is functorial for holomorphic mappings between compact analytic spaces satisfying the above conditions.*

Proof .

step 1. For each index set $I = (i_1, \ldots, i_q) \subset \{1, \ldots, N\}$, we define

$$\begin{array}{rcl} D_I &=& D_{i_1} \cap \ldots \cap D_{i_q} \\ D^{[q]} &=& \dot{\sqcup}_{|I|=q} D_I \text{ (disjoint union)} \end{array}$$

Each $D^{[q]}$ is a compact Kähler manifold, and we can define

$$A^{p,q} = A^p(D^{[q]}),$$

where $A^{\cdot}(D^{[q]})$ is the usual de Rham complex, and

$$\begin{array}{rcl} d &:& A^{p,q} \to A^{p+1,q} \ d = \text{the exterior derivative}) \\ \delta &:& A^{p,q} \to A^{p,q+1} \end{array}$$

by the formula $(\delta\varphi)_{(j_1,\ldots,j_{q+1})} = \sum_{l=1}^{q+1} \varphi_{(j_1,\ldots,\hat{j}_l,\ldots,j_{q+1})}\Big|_{D_{(j_1,\cdots,j_{q+1})}}$. Therefore $\{A^{p,q}, d, \delta\}$ is a double complex.

step 2. We want to show that $H_D^{\cdot}(A^{\cdot}) \cong H^{\cdot}(V, \mathbb{C})$ There are sheaves $\mathcal{A}^{p,q}$ on V with $\Gamma(V, \mathcal{A}^{p,q}) = A^{p,q}$ and $H^r(V, \mathcal{A}^{p,q}) = 0$ for $r > 0$ (by partition of unity). Setting $\mathcal{A}^n = \oplus_{p+q=n} \mathcal{A}^{p,q}$, we consider the complexes of sheaves

$$0 \to \mathbb{C}_V \to \mathcal{A}^0 \xrightarrow{D} \mathcal{A}^1 \xrightarrow{D} \mathcal{A}^2 \to \ldots$$

The usual sheaf-theoretical proof of de Rham theorem on manifolds will apply if we can prove the Poincaré lemma for D. Let $A^{p,q}(U) = \Gamma(U, \mathcal{A}^{p,q})$,where U is a small open neighborhood on V. By the usual d-Poincaré lemma, we have $H_d^p(A^{\cdot,q}) = 0$ for $p > 0$ and $H_d^0(A^{\cdot,q}) = H^0(q\text{-fold intersections})$. Thus in the spectral sequence for $\{A^{p,q}(U), d, \delta\}$, $E_0^{p,q} = A^{p,q}$, $d_0 = d$, $E_1^{p,q} = H_d^p(A^{\cdot,q}) = 0$ for $p > 0$ and $E_1^{0,q} \cong \mathbb{C}^{\binom{n}{q}}$. The d_1 map $\delta : E_1^{0,q} \to E_1^{0,q+1}$ is given by the formula as occurs in the coboundary operator for a simplex, and consequently

$$\begin{array}{rcl} E_2^{p,q} &=& 0 \text{ if } p+q > 0, \text{ and} \\ E_2^{0,0} &\cong& \mathbb{C} \cong H^0(U, \mathbb{C}). \end{array}$$

Therefore $E_2 = E_\infty$ and we have the Poincaré lemma for D.

step 3. We define a weight filtration W_n and a Hodge filtration F^p on $A^{\cdot,\cdot}$ by

$$\begin{array}{rcl} W_n &=& \oplus_{r \le n} A^{r,\cdot}, \\ F^p &=& \oplus_{\substack{r,s \\ s \ge p}} (r,s)\text{-forms on } D^{[q]}. \end{array}$$

Consider the spectral sequence $(A^{\cdot,\cdot}, d, \delta)$. We have $E_1^m := E_1^{m,\cdot} \cong \oplus_q H_d^m(D^{[q]})$ and consequently has a Hodge structure of pure weight m induced by the above Hodge filtration. Moreover, $d_1 = \delta : E_1^m \to E_1^m$ is a morphism of pure weight m. Therefore the E_2^m term has a Hodge structure of pure weight m. claim : $E_2 = E_\infty$

Assume this holds, we have

$$\begin{array}{rcl} H^k(V, \mathbb{C}) & \cong & H_D^k(A^{\cdot}) \\ & = & E_\infty^{k,0} \oplus E_\infty^{k-1,1} \oplus \ldots \oplus E_\infty^{0,k} \\ & = & E_2^{k,0} \oplus E_2^{k-1,1} \oplus \ldots \oplus E_2^{0,k} \end{array}$$

Thus $H^{\cdot}(V, \mathbb{C})$ has a mixed Hodge structure.

step 4. To prove the claim, it suffices to show $d_2 = d_3 = \ldots = 0$. Given $[\alpha] \in E_2^{m,q}$, by decomposing this class into type, we may assume $[\alpha]$ is represented by a closed C^∞ form α which has type (r, s) with $r + s = m$. Since $d_1[\alpha] = 0$, we have $\delta\alpha = d\beta$ for some $\beta \in A^{m-1,q+1}$. Applying lemma (), we may write

$$\delta\alpha = d\beta', \delta\alpha = d\beta'',$$

where β' has type $(r, s-1)$ and β'' has type $(r-1, s)$. But $[\delta\beta'] = [\delta\beta'']$ in E_2^{m-1} which has a Hodge structure of pure weight $m-1$. Thus $d_2([\alpha]) = 0$. The proof for $d_i = 0$, $i \geq 3$ is similar. $\square$

Definition 4.15 Given a complex of sheaves $(\mathcal{K}^{\cdot}, d) = \{\mathcal{K}^0 \xrightarrow{d} \mathcal{K}^1 \xrightarrow{d} \mathcal{K}^2 \to \ldots\}$. The cohomology sheaf $\mathcal{H}^q(\mathcal{K}^{\cdot})$ is the sheaf coming from the presheaf

$$U \mapsto \frac{Ker\{d : \mathcal{K}^p(U) \to \mathcal{K}^{p+1}(U)\}}{d\mathcal{K}^{q-1}(U)}$$

We note that essentially by definition, the cohomology sheaves $\mathcal{H}^q(\mathcal{K}^{\cdot}) = 0$ for all $q > 0 \Leftrightarrow$ the Poincaré lemma holds for the complex of sheaves $(\mathcal{K}^{\cdot}, d)$.

Definition 4.16 A map $j : \mathcal{L}^{\cdot} \to \mathcal{K}^{\cdot}$ between complexes of sheaves is a quasi-isomorphism if it induces an isomorphism on cohomology sheaves: $j^*\mathcal{H}^q(\mathcal{L}^{\cdot}) \xrightarrow{\simeq} \mathcal{H}^q(\mathcal{K}^{\cdot}), \forall q \geq 0$.

4.2 Mixed Hodge structure for nonsingular quasi-projective space

Let X be a smooth, quasi-projective algebraic variety over $\mathbb{C}$. According to [Hir], we may find a smooth compactification $\bar{X}$ of$X$. Thus X is a smooth projective variety on which there is a divisor$D$ with normal crossings, such that $X = \bar{X}\backslash D$. Locally D is given by $\{z \in \mathbb{C}^n : |z_i| < \epsilon, z_1 z_2 \dots z_k = 0\}$. It will simplify matters notationally to assume that globally $D = D_1 \cup \dots \cup D_N$, where the D_i are smooth divisors meeting transversely.

Theorem 4.17 *The cohomology $H^*(X)$ has a functorial mixed Hodge structure.*

Proof .

step 1. By a neighborhood at infinity, we will mean an open set $U \subset X$, and $\bar{U}$ is the polycylinder $|z_i| \le \epsilon$ so that $U = \bar{U}\backslash \bar{U} \cap D$.

Definition 4.18 The C^∞ log complex $A^{\cdot}(U, \log D)$ is the complex of C^∞ forms $\varphi \in A^{\cdot}(U)$ such that $z_1 \dots z_k \varphi$ and $z_1 \dots z_k d\varphi$ are C^∞ in $\bar{U}$.

Lemma 4.19 $A^{\cdot}(U, \log D) = A^{\cdot}(\bar{U})\{\frac{dz_1}{z_1}, \dots, \frac{dz_k}{z_k}\}$

Definition 4.20 The C^∞ log complex $A^{\cdot}(X, \log D)$ on X is the subcomplex of $A^{\cdot}(X)$ consisting of all φ which are in $A^{\cdot}(U, \log D)$ for all neighborhood U at infinity.

To give the global description, we consider the line bundles $[D_i] \to \bar{X}$ and choose sections $\sigma_i \in \Gamma(\bar{X}, \mathcal{O}([D_i]))$ with $(\sigma_i) = D_i$ and fiber metrics in $[D_i]$. Setting

$$\begin{aligned} \eta_i &= \tfrac{1}{2\pi\sqrt{-1}}\partial \log |\sigma_i|^2, \\ \omega_i &= \bar{\partial}\eta_i \end{aligned}$$

Lemma 4.21 $A^{\cdot}(X, \log D) = A^{\cdot}(\bar{X})\{\eta_1, \dots, \eta_N\}$.

Definition 4.22 On $A^{\cdot}(X, \log D)$ we define the weight filtration $W_l = W_l(A^{\cdot}(X, \log D))$ to be those forms φ such that locally at infinity $\varphi \in A^{\cdot}(\bar{U})\{\frac{dz_{i_1}}{z_{i_1}}, \dots, \frac{dz_{i_l}}{z_{i_l}}\}$, i.e. φ involves at most l $\frac{dz_i}{z_i}$'s.

Note that the definitions of the log complex and weight filtration are local around a point $z \in X$. Thus we may define complexes of sheaves on $\bar{X}$

$$\begin{array}{l} \mathcal{A}^{\cdot}(\log D), \\ \mathcal{W}_l = W_l(\mathcal{A}^{\cdot}(\log D)) \end{array}$$

such that $A^{\cdot}(X, \log D) = \Gamma(\bar{X}, \mathcal{A}^{\cdot}(\log D))$ and similarly for W_l. By the usual partition of unity argument, these sheaves have no higher cohomology.

Given that $D = D_1 \cup \ldots \cup D_N$, we shall use the following notations:

$$\begin{array}{rcl} I & = & \{i_1, \ldots, i_k\} \subset \{1, \ldots, N\} \text{ is an index set ;} \\ D_I & = & D_{i_1} \cap \ldots \cap D_{i_k}, |I| = k; \\ D^{[k]} & = & \dot{\sqcup}_{|I|=k} D_I \text{ (disjoint union).} \end{array}$$

Definition 4.23 The Poincaré residue operator $R^{[k]} : \mathcal{W}_k(\mathcal{A}^{\cdot}(\log D)) \to \mathcal{A}^{\cdot-k}(D^{[k]})$ is defined by $R^{[k]}(\alpha \wedge \frac{dz_{i_1}}{z_{i_1}} \wedge \ldots \wedge \frac{dz_{i_k}}{z_{i_k}}) = \alpha\Big|_{D_I}$

Lemma 4.24 *a) $R^{[k]}$ is well-defined and $R^{[k]}(\mathcal{W}_{k-1}) = 0$.*

b) $R^{[k]}$ commutes with d, ∂, and $\bar{\partial}$.

Lemma 4.25 *The induced mappings*

$$\begin{array}{rcl} R^{[k]} & : & \mathcal{H}^{\cdot}_d(\mathcal{W}_k/\mathcal{W}_{k-1}) \to \mathcal{H}^{\cdot-k}_d(\mathcal{A}^{\cdot}(D^{[k]})) \\ R^{[k]} & : & \mathcal{H}^{\cdot}_{\bar{\partial}}(\mathcal{W}_k/\mathcal{W}_{k-1}) \to \mathcal{H}^{\cdot-k}_{\bar{\partial}}(\mathcal{A}^{\cdot}(D^{[k]})) \end{array}$$

are isomorphisms

Proof . It suffices to show the Poincaŕe lemma for Ker$R^{[k]}$. Let $z' = (z_{N+1}, \ldots, z_n)$. Given

$$\varphi = \sum_I \varphi \frac{dz_I}{z_I} + \beta$$

a representative in $\mathcal{W}_k/\mathcal{W}_{k-1}$ satisfying $d\varphi = 0$, where $I \subset \{1, \ldots, N\}$, β does not involve dz_I. Write $\beta = \gamma \wedge \bar{d}z + \delta$, where $\delta \equiv 0(dz', d\bar{z}')$. Then

$$d\varphi = 0 \quad \Rightarrow \quad \left\{ \begin{array}{ll} \frac{\partial \varphi_I}{\partial \bar{z}_i} & = 0 \text{ for } i = 1, \ldots, n. \\ d_{z'}\delta & = 0, \end{array} \right.$$

an so $\delta = d_{z'}\theta$ for some θ by the usual Poincare lemma. Now by subtracting $d\theta$, we can write

$$\varphi = \sum_I \alpha'_I \wedge \frac{dz_I}{z_I} + \sum_I \beta'_I \wedge d\bar{z}_I,$$

where $\beta' \equiv 0(dz', d\bar{z}')$. Again,

$$\begin{aligned} d\varphi = 0 \quad \Rightarrow & \left\{ \begin{array}{lcl} d'_z\beta' & = & 0 \\ d'_z\alpha' & = & 0 \end{array} \right. \\ \Rightarrow & \left\{ \begin{array}{lcl} \beta' & = & d'_z\theta' \equiv \sum_I \psi' \wedge \frac{dz_I}{z_I} + \sum_I \psi'' \wedge d\bar{z}_I (\text{mod exact forms}), \\ \alpha'_I & = & d'_z\theta''. \end{array} \right. \end{aligned}$$

Therefore, subtracting $d(\theta'' \wedge \frac{dz_I}{z_I})$, we have

$$\begin{array}{l} \varphi \equiv \sum_I \tau_I \wedge d\bar{z}_I \wedge \frac{dz_I}{z_I}, \text{ where } d'_z\tau_I = 0 \\ \Rightarrow \varphi \equiv \sum_I \rho_I(z_I, z'_I) \wedge d\bar{z}_I \wedge \frac{dz_I}{z_I} \text{ mod exact forms.} \end{array}$$

Observe that $\bar{\partial}(rho(z_I, z'_I)d\bar{z}_I) = 0$, and by the $\bar{\partial}$-poincaré lemma, $\rho_I d\bar{z}_I = \bar{\partial}\xi_I$. Without loss of generality, we may assume

$$\varphi = \sum_I \partial\xi_I \wedge \frac{dz_I}{z_I},$$

where $\xi_I\Big|_{D_I} = o$ since $\varphi \in KerR^{[k]}$. Thus applying the usual Poincaré lemma gives finally that φ is exact. $\square$

step 2. It will be proved that the canonical map

$$H^*(A^\cdot(X, \log D)) \to H^*(A^\cdot(X)) \simeq H^*(X, \mathbb{C})$$

is an isomorphism.

Given $\mathcal{L}^\cdot$ a complex of sheaves on a space Y with the cohomology groups $H^q(Y, \mathcal{L}^p) = 0$ for $q \geq 1$, then we can define a spectral sequence E_r with

$$\begin{array}{rcl} E_0^{p,q} & = & C^p(\underline{\mathcal{U}}, \mathcal{L}^q) \\ E_1^{p,q} & = & H_\delta^p(\underline{\mathcal{U}}, \mathcal{L}^q)) \\ & = & \left\{ \begin{array}{ll} 0 & \text{for } p \neq 0 \\ \Gamma(Y, \mathcal{L}^q) & \text{for } p = 0 \end{array} \right. \\ E_2^{p,q} & = & H_d^q H_\delta^p(\underline{\mathcal{U}}, \mathcal{L}^\cdot)) \\ & = & \left\{ \begin{array}{ll} 0 & \text{for } p \neq 0 \\ H_d^q(\Gamma(Y, \mathcal{L}^\cdot)) & \text{for } p = 0 \end{array} \right. \end{array}$$

and $E_\infty = E_2$. $\square$

Proposition 4.26 *Given two complexes of sheaves $\mathcal{L}^\cdot, \mathcal{K}^\cdot$ with $H^q(Y, \mathcal{L}^p) = H^q(Y, \mathcal{K}^p) = 0$ for $q > 0$, and a morphism $\psi : \mathcal{L}^\cdot \to \mathcal{K}^\cdot$ of complexes of sheaves such that the induced cohomology sheaf map $\psi_* : \mathcal{H}^*(\mathcal{L}^\cdot) \to \mathcal{H}^*(\mathcal{K}^\cdot)$ is an isomorphism, then the global cohomology map*

$$\frac{ker\{\Gamma(Y, \mathcal{L}^p) \to \Gamma(Y, \mathcal{L}^{p+1})\}}{d\Gamma(Y, \mathcal{L}^{p-1})} \to \frac{ker\{\Gamma(Y, \mathcal{K}^p) \to \Gamma(Y, \mathcal{K}^{p+1}\}}{d\Gamma(Y, \mathcal{K}^{p-1})}$$

is also an isomorphism.

Apply this principle to $Y = \bar{X}, \mathcal{L}^\cdot = \mathcal{A}^\cdot(\log D)$, and $\mathcal{K}^\cdot = j_*\mathcal{A}^\cdot(\log D))$, where $j : X \hookrightarrow \bar{X}$ is the inclusion mapping. Note that $\Gamma(\bar{X}, \mathcal{A}^\cdot(\log D)) = A^\cdot(X, \log D)$ and $\Gamma(\bar{X}, j_*\mathcal{A}^\cdot) = A^\cdot(X)$. Step 2 of the proof of Theorem 4.17 consequently follows from the following lemma.

Lemma 4.27 *The induced mapping*

$$\mathcal{H}^\cdot(\mathcal{A}^\cdot(\log D)) \to \mathcal{H}^\cdot(j_*\mathcal{L}^\cdot(X))$$

is an isomorphism.

Proof . The question is local in a neighborhood U at infinity. Let $\mathbb{C}\{\frac{dz_1}{z_1}, \dots, \frac{dz_k}{z_k}\} = \mathbb{C}\{\{\frac{dz_i}{z_i}\}\}$ be the free differential graded algebra generated by $\frac{dz_1}{z_1}, \dots, \frac{dz_k}{z_k}$ and having differential $d = 0$. There is a commutative triangle

$$\begin{tikzcd} A^\cdot(U, \log D) \arrow[rr, "i"] & & A^\cdot(U) \\ & \mathbb{C}\{\frac{dz_1}{z_1}, \cdots, \frac{dz_k}{z_k}\} \arrow[ul, "\mu"] \arrow[ur, "\nu"'] & \end{tikzcd}$$

Then we have $i \circ \mu = \nu$. Since U is homotopic to $(\triangle^*)^k \times \triangle^{n-k}$, and the cohomology on $(\triangle^*)^k$ has as basis the forms $\frac{dz_I}{z_I}$, where $I \subset \{1, \dots, k\}$, use Künneth formula, we see ν_* is an isomorphism on cohomology. Thus it suffices to show μ_* is an isomorphism on cohomology. There is an obvious weight filtration $W_l(\mathbb{C}\{\{\frac{dz_i}{z_i}\}\})$ such that $\mu(W_l(\mathbb{C}\{\{\frac{dz_i}{z_i}\}\}) \subset W_l(A^{\cdot}(U, \log D))$. Consider the commutative triangle

$$\begin{tikzcd} \frac{W_l(A^{\cdot}(U,\log D))}{W_{l-1}(A^{\cdot}(U,\log D))} \arrow[rr, "R^{[l]}"] & & A^{\cdot -k}(D^{[l]}, \bar{U}) \\ & \frac{W_l(\mathbb{C}\{\{\frac{dz_i}{z_i}\}\})}{W_{l-1}(\mathbb{C}\{\{\frac{dz_i}{z_i}\}\})} \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

where β is the Poincaré map. we have $\beta = R^{[k]} \circ \alpha$, β_* is an isomorphism on cohomology and $R^{[k]}_*$ is also an isomorphism by Lemma 4.25. Thus α_* is also an isomorphism on cohomology. Using induction on l, it follows

$$\mu : W_l(\mathbb{C}\{\{\frac{dz_i}{z_i}\}\}) \longrightarrow W_l(\Omega^{\cdot}(U, \log D))$$

induces an isomorphism on cohomology. For $l = k$, we obtain our lemma. $\square$

Corollary 4.28 *The residue map $R^{[k]} : W_k/W_{k-1} \to A^{\cdot -k}(D^{[k]})$ induces isomorphisms on both d and $\bar{\partial}$ cohomology.*

step 3. On the C^∞ log complex $A^{\cdot}(X, \log D)$ we have defined the weight filtration W_l, and we now define the Hodge filtration by

$$F^p A^{\cdot}(X, \log D) = \oplus_{i \geq p} A^{i,\cdot}(X, \log D).$$

The weight and Hodge filtrations induce filtrations on the cohomology $H^{\cdot}(A^{\cdot}(\log D)) \simeq H^{\cdot}(X, \mathbb{C})$, and it is to be proved that this gives a mixed Hodge structure.

Proposition 4.29 *The weight filtration $W_l(H^{\cdot}(X, \mathbb{C}))$ is defined over $\mathbb{Q}$.*

Proof. Take a closed form $\varphi \in A^{\cdot}(X, \log D)$, then $R^{[n]}\varphi = 0$. Conversely if $R^{[n]}\varphi = 0$ in cohomology, then there exists a $\theta \in A^{\cdot}(X, \log D)$ such that $\varphi - d\theta \in W_{n-1}$.

Repeating this argument, we get

$$W_l(H^\cdot(X,\mathbb{C})) = \{\varphi \in H^\cdot(X,\mathbb{C}) : R^{[n]}\varphi = \ldots = R^{[l+1]} = 0\}$$

where on the right side, $R^{[\nu]}\varphi$ are taken in $H^\cdot(D[\nu],\mathbb{C})$. Therefore $W_l(H^\cdot(X,\mathbb{C}))$ is defined over $\mathbb{Q}$. $\square$

Now consider the filtration $W^{-l} = W_l(A^\cdot(X, \log D))$ on the complex. Thus we have the decreasing filtration $\ldots \subset W^{-l} \subset W^{-l+1} \subset \ldots$, and we may consider the spectral sequence of a filtered complex. Accordingly, there is a spectral sequence E_r such that E_∞ is the associated graded complex to the filtration on $H(X,\mathbb{C})$ and $E_1 = H^*(W^{-l}/W^{-l+1})$.

Lemma 4.30 *The Hodge filtration F^p induces Hodge structure of pure type $k+l$ on $H^k(W^{-l}/W^{-l+1})$.*

Proof . The Poincaré residue operator induces

$$F^p(W^{-l}/W^{-l+1}) \xrightarrow{R^{[l]}} F^{p-1}(A^\cdot(D^{[l]})).$$

Applying Corollary 4.28, it follows

$$H^p(W^{-l}/W^{-l+1}) \cong H^{p-1}(D^{[l]}).$$

is a morphism of Hodge structures of type $(-l,-l)$ and $H^k(W^{-l}/W_{-l+1})$ is of pure type $k+l$. $\square$

Lemma 4.31 *The mapping $d_1 : E_1 \to E_1$ is a morphism of Hodge structures.*

Proof . Using the isomorphism

$$E_1 \cong \oplus_I H^*(D_I) \text{ note } I \text{ might be an empty set },$$

we want to show d_1 is given by the Gysin mappings

$$H^*(D_{I_1 \ldots I_l}) \xrightarrow{\gamma} H^{*+2}(D_{I_1 \ldots I_l})$$

Given $\varphi \in H^{\cdot}(W^{-l}/W^{-l+1})$, we may represent it by a form $\varphi \in W_l(A^{\cdot}(\log D))$ with $R^{[l]}d\varphi = dR^{[l]}\varphi = 0$. i.e. $d\varphi \in W_{l-1}$. Since d_1 is the coboundary map arising from the exact sequence of complexes

$$0 \to W_{l-1}/W_{l-2} \to W_l/W_{l-2} \to W_l/W_{l-1} \to 0,$$

we have

$$d_1\varphi = R^{[l-1]}d\varphi.$$

Writing $\varphi = \sum_{|I|=l} \varphi_I \wedge \eta_I$, and $\eta_I = \eta_{i_1} \wedge \ldots \wedge \eta_{i_l}$. Then it suffices to show

$$\sum_{|I|=l} \int_{D^{[l]}} \varphi_I \wedge \psi = \int_{D^{[l-1]}} R^{[l-1]} d\varphi \wedge \psi$$

for all closed form ψ,

$$d\varphi = \sum_{|I|=l} (d\varphi_I \wedge \eta_I + (-1)^{i+l-1} \sum_j \varphi_I \wedge \omega_{i_j} \wedge \eta_{I-i_j})$$

$$\begin{aligned}
& \int_{D^{[l-1]}} R^{[l-1]} d\varphi \\
= & \sum_{\substack{|I|=l \\ i_j \in I}} \int_{D_{I\setminus\{i_j\}}} (-1)^{j-1} d\varphi_I \wedge \eta_{i_j} \wedge \psi + (-1)^{j+l-1} \varphi_I \wedge \omega_{i_j} \wedge \psi \\
= & \sum_{\substack{|I|=l \\ i_j \in I}} \int_{D_{I\setminus\{i_j\}}} \{(-1)^{j-1} d\varphi_I \wedge \eta_{i_j} \wedge \psi + (-1)^{j-1} d(\varphi_I \wedge \eta_{i_j} \wedge \psi) \\
= & \sum_{\substack{|I|=l \\ i_j \in I}} \int_{D_{I\setminus\{i_j\}}} (-1)^{j-1} d(\varphi_I \wedge \eta_{i_j} \wedge \psi) \\
= & \sum_{\substack{|I|=l \\ i_j \in I}} \int_{\partial D_{I\setminus\{i_j\}}} (-1)^{j-1} \varphi_I \wedge \eta_{i_j} \wedge \psi \\
= & \sum_{|I|=l} \int_{D^{[l]}} \varphi_I \wedge \psi
\end{aligned}$$

by Cauchy residue formula. □

Corollary 4.32 *The weight and Hodge filtrations on $A^{\cdot}(\log D)$ induces a mixed Hodge structure on E_2.*

step 4. The main remaining step is to show that the spectral sequence in question degenerates at E_2. In case D is smooth. the weight filtration is just $W_0 \subset W_1$ and therefore $E_2 = E_\infty$. The crucial case is when $D = D_1 \cup D_2$, and we shall show $d_2 = 0$ — this will suffices to make clear how the general argument goes. Let

$\alpha \in E_1^{-2} = H^\cdot(W_2/W_1) \cong H^{\cdot -2}(D_1 \cap D_2)$. We may assume α is a closed $C^\infty(p,q)$ form on $D_1 \cap D_2$. Let $\tilde{\alpha}$ be a C^∞ extension of α, then $A' = \eta_1 \wedge \eta_2 \wedge \tilde{\alpha}$ gives a form in $W_2(A^\cdot(\log D))$ with $R^{[2]}A' = \alpha$. Since $R^{[2]}dA' = dR^{[2]}A' = 0$, we have $dA' \in W_1(A^\cdot(\log D))$. If $d_1(\alpha) = 0$, there exist forms β_i 0n D such that

$$\begin{array}{lll} R^{[1]}(dA')_{D_1} & = \eta_2 \wedge d\tilde{\alpha} + \omega_2 \wedge \tilde{\alpha}|_{D_1} & = d\beta_1 \\ R^{[1]}(dA')_{D_2} & = \eta_1 \wedge d\tilde{\alpha} + \omega_1 \wedge \tilde{\alpha}|_{D_2} & = d\beta_2 \end{array}$$

Moreover, we may choose the β_i to have type $(p+q+1,0)+\cdots+(p+1,q)$. Setting $\tilde{B}' = -(\eta_1 \wedge \tilde{\beta}_1 + \eta_2 \wedge \tilde{\beta}_2)$ where $\tilde{\beta}_i$ are C^∞ extensions of β_i. We find the relations

$$\begin{array}{rl} R^{[2]}(A'+B') = & \alpha \\ R^{[2]}(d(A'+B')) = & R^{[1]}(d(A'+B')) = 0 \\ d(A'+B') \in & F^{p+2}A^\cdot(\log D) \end{array}$$

Then we repeat the same argument using $\bar{\eta}_1, \bar{\eta}_2$. This leads to $A''.B''$ satisfying

$$\begin{array}{rl} R^{[2]}(A''+B'') = & \alpha \\ R^{[2]}(d(A''+B'')) = & R^{[1]}(d(A''+B'')) = 0 \\ d(A''+B'') \in & F^{q+2}A^\cdot(\log D) \end{array}$$

Since $\deg(d(A'+B'))= \deg(d(A''+B'')) = p+q+3$, the above equations say exactly that $d_2\alpha \in E_2^0$ has total degree $p+q+3$ and is in $F^{p+2}(E_2^0) \cap F^{q+2}(E_2^0) = 0$ since E_2 has a mixed Hodge structure. Thus $d_2\alpha = 0$.

step 5. Given a morphism $Y \xrightarrow{f} X$, we may find a diagram

$$\begin{array}{ccc} Y & \hookrightarrow & \bar{Y} \\ f\downarrow & & \downarrow\bar{f} \\ X & \hookrightarrow & \bar{X} \end{array}$$

according to Hironaka, and $\bar{f}^* : A^\cdot(X, \log D) \to A^\cdot(Y, \log D)$ commutes with the weight and Hodge filtrations. This implies functoriality.

Remark 4.33 Given smooth completion $\bar{X}_1, \bar{X}_2$ of X, there exists a smooth completion $\bar{X}_3$ and a diagram according to [Hir].

$$\begin{tikzcd} & X_3 \arrow[dl] \arrow[dr] & \\ X_1 & \cup & X_2 \\ & X \arrow[ul, phantom, sloped, "\subset"] \arrow[ur, phantom, sloped, "\supset"] & \end{tikzcd}$$

, thus independence of the smooth compactification follows.

Corollary 4.34 *The Hodge numbers* $h^{p,q}(H^n(X)) = 0$ *unless* $0 \le p, q \le n, n \le p + q \le 2n$.

Corollary 4.35 *Let* $\bar{X}$ *be a compactification of* X. *Then the image of* $H^n(\bar{X}) \to H^n(X)$ *is* $W_n(H^n(X))$

Corollary 4.36 *Let* $X, \bar{X}$ *be as in Theorem 4.17 and furthermore* $\bar{X}$ *is a Kähler manifold, then* $H^k(X, \mathbb{C}) = \oplus_{p+q=k} H^q_{\bar\partial}(\bar{X}, \Omega(\log D))$

Proof. By theorem 4.17, it suffices to show $H^k_d(\bar{X}, A^{\cdot}(\log D)) = \oplus_{p+q=k} H^q_{\bar\partial}(\bar{X}, \Omega(\log D))$. Consider the double complex $\{\mathcal{A}^{\cdot,\cdot}, \partial, \bar\partial\}$

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
 & \mathcal{A}^{0,1}(\log D) \arrow[r, "\partial"] \arrow[u, "\bar\partial"'] & \mathcal{A}^{1,1}(\log D) \arrow[r, "\partial"] \arrow[u, "\bar\partial"'] & \mathcal{A}^{2,1}(\log D) \arrow[r, "\partial"] \arrow[u, "\bar\partial"'] & \dots \\
 & \mathcal{A}^{0,0}(\log D) \arrow[r, "\partial"] \arrow[u, "\bar\partial"'] & \mathcal{A}^{1,0}(\log D) \arrow[r, "\partial"] \arrow[u, "\bar\partial"'] & \mathcal{A}^{2,0}(\log D) \arrow[r, "\partial"] \arrow[u, "\bar\partial"'] & \dots \\
0 \to \mathbb{C}_{\bar{X}} \arrow[r] & \Omega^0(\log D) \arrow[r, "\partial"] \arrow[u, "\bar\partial"'] & \Omega^1(\log D) \arrow[r, "\partial"] \arrow[u, "\bar\partial"'] & \Omega^2(\log D) \arrow[r, "\partial"] \arrow[u, "\bar\partial"'] & \dots
\end{tikzcd}$$

we see that

$$E_2^{p,q} = H^p_\partial(H^q_{\bar\partial}(\mathcal{A}^{\cdot,\cdot}(\log D))).$$

Since $\bar{X}$ is compact and Kähler, $\triangle_\partial = \triangle_{\bar\partial}$, we have $d_1 = 0$ and therefore

$$E_\infty^{p,q} = E_1^{p,q} = H^q_{\bar\partial}(\mathcal{A}^{p,\cdot}(\log D)) = H^{p,q}_{\bar\partial}(\mathcal{A}^{\cdot,\cdot}(\log D)).$$

Use Dolbeault Theorem for log complexes, we have

$$H_{\bar\partial}^{p,q}(\mathcal{A}^{\cdot,\cdot}(\log D)) = H_{\bar\partial}^{q}(\bar X, \Omega_{\bar X}^{p}(\log D)).$$

Hence

$$\begin{aligned} H_d^k(\mathcal{A}^{\cdot}(\log D)) &= E_\infty^{k,0} + \ldots + E_\infty^{0,k} \\ &= \oplus_{p+q=k} H^k(\bar X, \Omega^p(\log D)) \end{aligned}$$

□

5 Mixed Hodge Structure ⇒ Kodaira Vanishing Theorem

In this section, we uses the concept of cyclic cover and the decomposition of the cohomology of a projective space to reach the Kodaira Vanishing Theorem.

Let X projective space, D some effective divisor. Consider $\pi : Y \to X$ the cyclic cover obtained by taking N-th root out of D as we mentioned in §4.

Theorem 5.1 *A complex analytic manifold X of complex dimension k, bianalytically embedded as a closed subset of $\mathbb{C}^n$ has the homotopy type of a k-dimensional CW-complex.*

Proof . For the proof, see [Mil].

Corollary 5.2 *If $X \in \mathbb{C}^n$ is a nonsingular affine algebraic variety in complex n-space with real dimension $2k$, then $H^i(X, \mathbb{C}) = 0$ for $i > n$.*

Proof . This is trivial from theorem().

Theorem 5.3 Kodaira Vanishing Theorem *Let X be a complex, projective manifold and $\mathcal{A}$ be an ample invertible sheaf. Then*

$$H^b(X, \mathcal{O}(K_X) \otimes \mathcal{A}) = 0.$$

Proof . Since $\mathcal{A}$ is ample, there exists some N such that $\mathcal{A}^N = \mathcal{O}_X(D)$, where D is an effective and nonsingular divisor. Let Y the cyclic cover obtained by taking the

n-th root of D. Define $\tilde{D} = (\pi^* D)_{red}$, we have that $\pi : Y \to X$ is unramified outside of D and

$$\begin{array}{rcl} \pi^*\Omega^a_X(\log D) & = & \Omega^a_Y(\log \tilde{D}), \\ \pi_*\mathcal{O}_Y & = & \oplus_{i=0}^{N-1}\mathcal{L}^{-i} \end{array}$$

where $\Omega^a_X(\log D)$ denotes the sheaf of differential a-forms with logarithmic poles along $\tilde{D}$. By projection formula,

$$\begin{array}{rcl} \pi_*\Omega^n_Y(\log \tilde{D}) & = & \Omega^a_x(\log D) \otimes \pi_*\mathcal{O}_Y \\ & = & \oplus_{i=0}^{N-1}\Omega^n(\log D) \otimes \mathcal{L}^{-i} \\ & = & \oplus_{i=1}^{N-1}\Omega^n_X \otimes \mathcal{L}^{N-i}. \end{array}$$

Observe that $Y \xrightarrow{\pi} X$ is a finite morphism and $H^q(Y, \Omega^n_Y(\log \tilde{D})) = 0$ by the existence of partition of unity. Therefore there exists Leray spectral sequence,

$$E_2^{p,q} = H^q(X, R^p\pi_*\Omega^n_Y(\log \tilde{D})) \Rightarrow H^{p+q}(Y, \Omega^n_Y(\log \tilde{D})).$$

where $R^p\pi_*\Omega^n_Y(\log \tilde{D})$ is the p-th direct image sheaf associated to the presheaf $U \mapsto H^p(\pi^{-1}(U), \Omega^n_Y(\log \tilde{D}))$. Since $R^p\pi_*\Omega^n_Y(\log \tilde{D}) = 0$, we get

$$H^i(X, \pi_*\Omega^n_Y(\log \tilde{D})) = H^i(Y, \Omega^n_Y(\log \tilde{D})).$$

Since we have

$$H^k(Y\backslash\tilde{D}, \mathbb{C}) = \oplus_{p+q=k}H^q(Y, \Omega^p_Y(\log \tilde{D}))$$

and $Y\backslash\tilde{D}$ is affine, it holds that $H^k(Y\backslash D, \mathbb{C}) = 0$ for $k > n$, and therefore

$$0 = H^b(Y, \Omega^n_Y(\log \tilde{D}) = \oplus_{i=0}^{N-1}H^q(X, \Omega^n_X \otimes \mathcal{L}^{N-i})$$

□

Remark 5.4 We can see from the proof that we do not really need the mixed Hodge structure of the cohomology of a quasi-projective space but only use cohomology groups of de Rham log complexes to decompose it.

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