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特殊拉格朗日球面存在性問題之探討

On the Existence Problem of Special Lagrangian Spheres

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On the Existence of Special Lagrangian Spheres

本論文係邱詩凱君（R01221031）在國立臺灣大學數學系完成之碩士學位論文，於民國 103 年 7 月 14 日承下列考試委員審查通過及口試及格，特此證明

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中文摘要

在 Seidel 的博士論文 [Sei97] 中，他與他的指導教授 Donaldson 證明，若一緊緻凱勒流形 (compact Kähler manifold) 擁有一個尋常退化 (ordinary degeneration)，則此凱勒流形內存在拉格朗日球面 (Lagrangian sphere)。這個結果引發以下的延伸問題：如果此凱勒流形為一卡拉比 -丘流形 (Calabi-Yau manifold)，我們是否能夠在其中找出一個特殊拉格朗日球面 (special Lagrangian sphere)？透過文獻回顧，我們將探討特殊拉格朗日子流形 (special Lagrangian submanifolds) 的基本知識，以及球面的切叢 (the cotangent bundle of sphere) 上的瑞奇平坦度量 (Ricci-flat metrics)。在論文的最後，我們透過均曲率流 (mean curvature flow) 來探討一維的情形。

Abstract

In his PhD thesis[Sei97], Paul Seidel and his advisor Simon K. Donaldson gave two proofs showing that a vanishing cycle in a Kähler manifold admitting an ordinary degeneration can be chosen to be *Lagrangian*. This gives rise to the question whether the vanishing cycle is *special Lagrangian* if the manifold is Calabi-Yau. We investigate this problem by reviewing the geometric aspect of special Lagrangian manifolds and the Ricci-flat metrics on the noncompact local model, namely the cotangent bundle of sphere. Finally, we approach this problem in dimension one through mean curvature flow.

Contents

Chapter 1

Introduction

In his thesis[Sei97], Paul Seidel and his advisor Simon K. Donaldson showed that if a compact Kähler manifold admits an ordinary degeneration, then it contains a Lagrangian sphere. This motivates us to ask further that if the manifold is in fact Calabi-Yau, is it possible to find a special Lagrangian sphere in it? If this is the case, then by a theorem of McLean [McL96], this special Lagrangian sphere is rigid, provided that $n \geq 2$. This is a significant property in algebraic geometry.

We investigate this problem in the following way. In Chapter 2 we review the special Lagrangian geometry and McLean's theorem on special Lagrangian deformations. The geometric structures on the local moduli space, due to [Hit97], is then reviewed. In Chapter 3 we follow Stenzel's approach to contruct a Ricci-flat metric on the cotangent bundle of a sphere. In Chapter 4, we review the two methods to contruct a Lagrangian sphere in a compact Kähler manifold admitting an ordinary degeneration. Finally, in Chapter 5 we discuss possible approaches to the main problem we are concerned about, and describe some elemenetary results in the one dimensional case.

Chapter 2

Special Lagrangian Geometry

2.1 Definitions and Basic Results

The main reference of the following contents on special Lagrangian geometry is [Joy03].

Let (M, g) be a Riemannian manifold. To each oriented k-plane $V \subset T_xM$, we can assign a volume form vol_V, which is a k-form on V.

Definition 2.1.1. *Let (M, g) be a Riemannian manifold and let ϕ be a closed k-form on M. We say that ϕ is a calibration on M if for every oriented k-plane V on M, we have $\phi|_V \leq \mathrm{vol}_V$. Here $\phi|_V = \alpha \cdot \mathrm{vol}_V$ for some $\alpha \in \mathbb{R}$ and by $\phi|_V \leq \mathrm{vol}_V$ we mean $\alpha \leq 1$. A k dimensional submanifold N of M is said to be calibrated by ϕ if for each $x \in N$ we have $\mathrm{vol}_{T_xN} = \phi|_{T_xN}$.*

Proposition 2.1.2. *Let (M, g) be a Riemannian manifold, ϕ a calibration and N a submanifold calibrated by ϕ. Then N is volume-minimizing among its homology class.*

Definition 2.1.3. *The tuple (X^n, J, ω, Ω) is a Calabi-Yau manifold if (X^n, J, ω) is a Kähler manifold of complex dimension n and Ω is a covariantly constant $(n, 0)$-form such that*

$$\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} (\frac{i}{2})^n \Omega \wedge \bar{\Omega}.$$

By standard linear algebra, it can be shown that if (X, J, ω, Ω) is a Calabi-Yau manifold, then the real part of Ω, Re Ω, is a calibration. From the calibrated geometry point of view, special Lagrangian submanifolds are the ones that is calibrated by Re Ω.

Definition 2.1.4. *Let (X, J, ω, Ω) be a Calabi-Yau manifold. A submanifold calibrated by* Re Ω *is called a special Lagrangian submanifold, or SL n-fold for short.*

A special Lagrangian submanifold is indeed a Lagrangian submanifold with respect to the Kähler form, with an additional feature:

Proposition 2.1.5. *Let (X^n, J, ω, Ω) be Calabi-Yau, and let L be a real n-dimensional submanifold of X. Then L is a special Lagrangian submanifold of X if and only if*

$$\text{Im } \Omega\big|_L = 0, \ \omega\big|_L = 0.$$

The proof of this propostion can be found in section III.1 in the foundational paper [HL82].

Proposition 2.1.6. *Let (X, J, ω) be a Kähler manifold and let L be a Lagrangian submanifold of L. Then*

$$NL \simeq T^*L,$$

where NL denote the normal bundle of L in X.

Proof. Since L is Lagrangian, J is an isomorphism between NL and TL. Now composing J with the musical isomorphism $\flat$ yields the desired isomorphism. □

2.2 McLean's Theorem

Theorem 2.2.1 ([McL96])**.** *Let (X, J, ω, Ω) be a Calabi-Yau manifold, let L be a compact special Lagrangian submanifold of L, and let $V \in C^\infty(NL)$. Then V is a deformation vector field to a normal deformation through special Lagrangian submanifolds if and only if the 1-form $(JV)^\flat$ is harmonic.*

We give a detailed proof here, following [Mar02]. In the following, we always write down the immersion $f : L \to X$ explicitly for clarity. In fact, we only require f to be an immersion. The proof begins with a local observation.

Lemma 2.2.2. *On $\mathbb{C}^n$, we have $g_0 = \sum_{j=1}^n dx_j \otimes dx_j + dy_j \otimes dy_j$, $\omega_0 = \sum_{j=1}^n dx_j \wedge dx_j$, $J_0 = i\cdot$, and $\Omega_0 = dz_1 \wedge \ldots \wedge dz_n$, making $(\mathbb{C}^n, J_0, \omega_0, \omega_0, \Omega_0)$ a noncompact Calabi-Yau* *manifold. Let $V \subset \mathbb{C}^n$ be a special Lagrangian n-plane. Then for all $\xi \in V^\perp$, we have*

$$(\iota(\xi)\omega_0)|_V = (J_0(\xi))^\flat,$$
$$(\iota(\xi)\mathrm{Im}\ \Omega_0)|_V = \star(J_0(\xi))^\flat,$$

where $\star$ is the Hodge star operator on V.

Proof. Since the equations are $\mathrm{SU}(n)$-invariant, and since $\mathrm{SU}(n)$ acts transitively on the set of all SL n-planes in $\mathbb{C}^n$, we may assume that $V = \mathbb{R}^n$, the real slice of $\mathbb{C}^n$ generated by $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$. Then for $k = 1, \dots, n$,

$$\begin{aligned}(\iota(\tfrac{\partial}{\partial y_k})\omega_0)\big|_V &= (-dx_k)|_V \\ &= (J_0\tfrac{\partial}{\partial y_k})^{\flat_0},\end{aligned}$$

and

$$\begin{aligned}(\iota(\tfrac{\partial}{\partial y_k})\mathrm{Im}\ \Omega_0)\big|_V &= (-1)^k dx_1 \wedge \ldots \wedge \widehat{dx_k} \wedge \ldots \wedge dx_n \\ &= -\star_0 dx_k \\ &= \star_0(J_0\tfrac{\partial}{\partial y_k})^{\flat_0}.\end{aligned}$$

□

Corollary 2.2.2.1. *Let (X, J, g, Ω) be a Calabi-Yau manifold and let $f : L \to X$ be a spcial Lagrangian submanifold. If $\xi \in C^\infty(NL)$, then*

$$f^*(\iota(\xi)\omega) = (J(\xi))^\flat,$$
$$f^*(\iota(\xi)\mathrm{Im}\ \Omega_0) = \star(J(\xi))^\flat,$$

where $\star$ is the Hodge star operator on L.

Next we review the tubular neighborhood theorem, for the deformation occurs in a star-shaped tubular neighborhood of the initial submanifold.

Proposition 2.2.3 (Tubular neighborhood theorem)**.** *Let (M, g) be a Riemannian manifold, and let X be a submanifold of M, Then there exists an open neighborhood $\tilde{U} \subset NX$ containing the zero section such that*

$$\exp|_{\tilde{U}} : \tilde{U} \to M$$

is a diffeomorphism.

Shrinking $\tilde{U}$ if necessary, we may assume that $\tilde{U} \cap N_x \subset N_x$ is star-shaped.

For the rest of this section, we always assume that (X, J, ω, Ω) is a Calabi-Yau manifold and that $f : L \to X$ is a compact special Lagrangian submanifold of X. Following the notations of the last proposition, we define

$$U = (J\tilde{U})^\flat \subset T^*L,$$
$$\tilde{U}^\infty = \{\xi \in C^\infty(NL) \,|\, \xi_x \in \tilde{U} \ \forall x \in L\},$$
$$U^\infty = \{\eta \in C^\infty(T^*L) \,|\, \eta_x \in U \ \forall x \in L\}.$$

Let $\xi \in C^\infty(NL)$. Then there exists $\epsilon > 0$ small enough such that $t\xi \in \tilde{U}^\infty$ for all $|t| < \epsilon$. This normal vector field ξ defines a deformation of X given by

$$f_{t\xi} : L \to X, \ t \in (-\epsilon, \epsilon),$$
$$f_{t\xi}(x) = \exp_{f(x)} t\xi_x.$$

Note that $f_{t\xi} : L \to X$ is a special Lagrangian submanifold if and only if $f^*_{t\xi}\omega = 0$ and $f^*_{t\xi}\mathrm{Im}\,\Omega = 0$.

Lemma 2.2.4. *Let $\eta = (J\xi)^\flat$. Then*

$$\frac{\partial}{\partial t}(f^*_{t\xi}\omega)\big|_{t=0} = d\eta$$

and

$$\frac{\partial}{\partial t}(f_{t\xi}^*\mathrm{Im}\,\Omega)\big|_{t=0} = d(\star\eta).$$

Proof. Let $\tilde{\xi}$ be an extension of ξ on X.

$$\begin{aligned}\frac{\partial}{\partial t}(f_{t\xi}^*\omega)\big|_{t=0} &= f^*(L_{\tilde{\xi}}\omega)\\ &= f^*(\iota(\tilde{\xi})d\omega + d\iota(\tilde{\xi})\omega)\\ &= d(f^*\iota(\tilde{\xi})\omega)\\ &= d\eta,\end{aligned}$$

where the last equality follows from the previous corollary. The other equation follows similarly. □

From the above lemma it follows that, if $\xi \in C^\infty(NL)$ is a special Lagrangian deformation vector field, then $(J\xi)^\flat$ is necessarily a harmonic 1-form. Our goal is to show that this condition is unobstructed.

Fix $k \geq 2$. Define

$$\begin{aligned}\tilde{U}^{k+1,a} &= \{\xi \in C^{k+1,a}(NL)\,|\,\xi_x \in \tilde{U}\ \forall x \in L\},\\ U^{k+1,a} &= \{\eta \in C^{k+1,a}(T^*L)\,|\,\eta_x \in U\ \forall x \in L\},\end{aligned}$$

and

$$\begin{aligned}\tilde{F} : \tilde{U}^{k+1,a} &\to C^{k,a}(\textstyle\bigwedge T^*L)\\ \xi &\mapsto -\star f_\xi^*\mathrm{Im}\,\Omega + f_\xi^*\omega.\end{aligned}$$

Then we have $\tilde{F}(0) = 0$, since $f_0 = f : L \to X$ is a Special Lagrangian submanifold. More generally, if $\xi \in \tilde{F}^{-1}(0)$, then $f_\xi : L \to X$ is a special Lagrangian submanifold. By composing with the isomorphism $NL \simeq T^*L$, we define $F : U^{k+1,a} \to C^{k,a}(\bigwedge T^*L)$ by

$$F((J\xi)^\flat) = -\star f_\xi^*\mathrm{Im}\,\Omega + f_\xi^*\omega.$$

The idea is to apply the implicit function theorem for Banach spaces to show that $F^{-1}(0)$ is a manifold, parametrized by an open set of the finite dimensional vector space $\mathcal{H}^1$ of the harmonic 1-forms.

Theorem 2.2.5 (Implicit function theorem for Banach spaces)**.** *Let $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{Y}$ be Banach spaces and $\mathcal{U}_1 \subset \mathcal{X}_1$, $\mathcal{U}_2 \subset \mathcal{X}_2$ be open subsets both containing* 0*. Let*

$$\begin{aligned} F : \mathcal{U}_1 \times \mathcal{U}_2 &\to \mathcal{Y} \\ (0,0) &\mapsto 0 \end{aligned}$$

be a map of class C^k such that the partial derivative $F_2'(0,0) : \mathcal{X}_2 \to \mathcal{Y}$ is a topological linear isomorphism. Then there exists open sets $\mathcal{W}_1 \subset \mathcal{X}_1$ and $\mathcal{W}_2 \subset \mathcal{X}_2$, both containing 0*, and a unique C^k map $\mathcal{X} : \mathcal{W}_1 \to \mathcal{W}_2$ such that*

$$F^{-1}(0) \cap (\mathcal{W}_1 \times \mathcal{W}_2) = \{(x_1, \mathcal{X}(x_1)) \,|\, x_1 \in \mathcal{W}_1\}.$$

By Hodge decomposition,

$$C^{k+1,a}(T^*L) = \mathcal{H}^1 \oplus d^*(C^{k+2,a}(\bigwedge^2 T^*L)) \oplus d(C^{k+2,a}(L)). \tag{2.1}$$

Define $\mathcal{X}_1 = \mathcal{H}^1$ and $\mathcal{X}_2$ be the rest of the direct sum. Let $\mathcal{U}_1 \subset \mathcal{X}_1$ and $\mathcal{U}_2 \subset \mathcal{X}_2$ be open sets such that $(0,0) \in \mathcal{U}_1 \times \mathcal{U}_2 \subset U^{k+1,a}$. We restrict F to $\mathcal{U}_1 \times \mathcal{U}_2$.

Lemma 2.2.6. *$F : \mathcal{U}_1 \times \mathcal{U}_2 \to C^{k,a}(\bigwedge T^*L)$ has partial derivative $F_2'(0,0) : \mathcal{X}_2 \to C^{k,a}(\bigwedge T^*L)$ at $(0,0)$ in the $\mathcal{X}_2$-direction which acts as $d^* + d$.*

Proof. Let $\eta = (J\xi)^\flat \in \mathcal{X}_2$. Then

$$\begin{aligned} F_2'(0,0)\eta &= \tfrac{\partial}{\partial t}(F(t\eta))\big|_{t=0} \\ &= \tfrac{\partial}{\partial t}(-\star f_{t\xi}^* \mathrm{Im}\,\Omega + f_{t\xi}^*\omega)\big|_{t=0} \\ &= -\star(d\star\eta) + d\eta \\ &= (d^* + d)\eta. \end{aligned}$$

□

Theorem 2.2.7 (Open mapping theorem)**.** *Let $T : \mathcal{X} \to \mathcal{Y}$ be a bounded linear map between Banach spaces. If T is surjective, then T is an open mapping. If T is bijective, then T is a toplinear isomorphism.*

Theorem 2.2.8. *$F_2'(0,0) : \mathcal{X}_2 \to C^{k,a}(\bigwedge T^*L)$ is a topological linear isomorphism onto the closed subspace*

$$\mathcal{Y} = d^*(C^{k+1,a}(T^*L)) \oplus d(C^{k+1,a}(T^*L)). \tag{2.2}$$

Proof. We know that $F_2'(0,0) = d^* + d$. It's clear that $F_2'(0,0)$ maps $\mathcal{X}_2$ into $\mathcal{Y}$ injectively. To see the reverse inclusion, let $\theta_1, \theta_2 \in C^{k+1,a}(T^*L)$. Then

$$d^*\theta_1 \in \Delta C^{k+3,a}(L)$$

and

$$d\theta_2 \in \Delta C^{k+3,a}(\bigwedge^2 T^*L)$$

by Hodge decomposition. Therefore there exists $f \in C^{k+3,a}$ such that $d^*(df) = d^*\theta_1$. simiarly, there exists $\omega \in C^{k+3,a}(\bigwedge^2 T^*L)$ such that $dd^*\omega + d^*d\omega = d\theta_2$. Since $d^*d\omega$ is orthogonal to $d\theta_2$, $d^*d\omega = 0$. We conclude that

$$(d^* + d)(df + \omega) = d^*\theta_1 + d\theta_2,$$

where $df + \omega \in C^{k+2,a}(T^*L)$. Finally, by open mapping theorem, $F_2'(0,0)$ is a topological linear isomorphism between Banach spaces $\mathcal{X}_2$ and $\mathcal{Y}$. □

To apply implicit function theorem, we show that image$(F) \subset \mathcal{Y}$.

Lemma 2.2.9. image$(F) \subset \mathcal{Y}$.

Proof. Note that there is a homotopy $f_{t\xi}$ between f and f_ξ. It follows that $[f_\xi^* \mathrm{Im}\ \Omega] =$

$[f^*\mathrm{Im}\ \Omega] = 0$ and $[f_\xi^*\omega] = [f^*\omega] = 0$. Consequently, $\star f_\xi^*\mathrm{Im}\ \Omega \in d^*(C^{k+1,\alpha}(T^*L))$ and $f_\xi^*\omega \in d(C^{k+1,\alpha}(T^*L))$. □

It turns out that $F : \mathcal{U}_1 \times \mathcal{U}_2 \to \mathcal{Y}$. By implicit function theorem, there exists open subests $0 \in \mathcal{W}_1 \subset \mathcal{U}_1$ and $0 \in \mathcal{W}_2 \subset \mathcal{U}_2$ and a unique map $\mathcal{X} : \mathcal{W}_1 \to \mathcal{W}_2$ such that

$$F^{-1}(0) \cap \mathcal{W}_1 \times \mathcal{W}_2 = \{(\eta_1, \mathcal{X}(\eta_1)) \mid \eta_1 \in \mathcal{W}_1\}. \tag{2.3}$$

This gives a smooth chart from the set of harmonic forms on L to the moduli space of special Lagrangian submanifolds near L. The dimension of this moduli space $M = F^{-1}(0) \cap (\mathcal{W}_1 \times \mathcal{W}_2)$, $\mathrm{dim}M$, is equal to the first Betti number $b^1(L)$ of L. This completes the proof of McLean's theorem.

2.3 Geometric Structures on the Local Moduli Spaces

Let M be the local moduli space obtained in the last subsection. On M there is a natural Riemannian metric given by the L^2 inner product of harmonic 1-forms.

In [Hit97], Hitchin asked what is the natural geometrical structure on the moduli space of special Lagrangian submanifolds in a Calabi-Yau manifold.

By McLean's theorem, we have an embedding $M \to H^1(L)$. But the construction of this embedding uses implicit funcition theorem, thereby not canoncial. Following [Hit97], we first show that there is a canoncial one.

Redefine $f : \mathcal{M} \to X$ as the full local family of special Lagrangian deformations of L, where $\mathcal{M} \simeq L \times M$. Let $p : \mathcal{M} \to M$ be the projection map. For each $t \in M$, $L_t = p^{-1}(t)$.

Let $t_1, \ldots, t_m$ be alocal coordinate system of M. Of course, $m = b^1(L) = \mathrm{dim}H^1(L, \mathbb{R})$. The tangent vectors $\frac{\partial}{\partial t_j}$ on L_t define harmonic forms

$$\theta_j = (\iota(\frac{\partial}{\partial t_j}) f^*\omega)\big|_{L_t}.$$

These 1-forms can also be obtained as follows. Since $f^*\omega = 0$ on the fibre L_t (L_t is Lagrangian), there are 1-forms $\tilde{\theta}_j$ on $\mathcal{M}$ such that $f^*\omega = \sum dt_j \wedge \tilde{\theta}_j$. Then $\theta_j = \tilde{\theta}_j|_{L_t}$, which coincides the previous definition.

Choose a basis $A_1, \dots A_m \in H_1(L, \mathbb{Z})$ modulo torsion. Let $\alpha_1, \dots \alpha_m \in H^1(L, \mathbb{R})$ be the dual basis of $A_1, \dots A_m$. Then we obtain a period matrix

$$\lambda_{ij} = \int_{A_i} \theta_j.$$

The matrix (λ_{ij}) is invertible, since θ_j are linearly independent.

Proposition 2.3.1. *The 1-forms $\xi_i = \sum \lambda_{ij} dt_j$ on M are closed.*

Proof. Choose smoothly in each fibre L_t a circle representing A_i. Then we have a fibration

$$\begin{tikzcd} S^1 \arrow[r] & \mathcal{M}_i \arrow[d, "p"] \\ & M \end{tikzcd}$$

Define a 1-form ξ on M by

$$\xi = p_* f^*\omega.$$

The push-down map p_* is the integration over the fibre. Since p_* sends closed forms to closed forms, ξ is closed. Now,

$$\begin{aligned} p_* f^*\omega &= p_*(\sum dt_j \wedge \tilde{\theta}_j) \\ &= \sum dt_j \int_{A_i} \theta_j \\ &= \xi_i. \end{aligned}$$

□

Since the 1-forms ξ_i are closed, locally we can find functions $u_i : M \to \mathbb{R}$, welldefined up to a constant, such that $du_i = \xi_i = \sum \lambda_{ij} dt_j$. Since (λ_{ij}) is invertible, $u_1, \ldots, u_m$ define a coordinate chart. More invariantly,

Proposition 2.3.2. *We have a coordinate chart $u : M \to H^1(L, \mathbb{R})$ defined by*

$$u(t) = \sum u_i(t)\alpha_i.$$

Moreover, this embedding is independent of the choice of basis, and is unique up to a translation.

Proof. Let the prime version denote another choice of basis. There exists an invertible matrix (T_{ij}) such that $A_i = T_{ij}A'_j$ and $\alpha_i = T_{ji}\alpha'_j$. The period matrix transforms as $\lambda_{ij} = T_{ik}\lambda'_{kj}$. Now,

$$\begin{aligned}
\sum u_i\alpha_i &= \sum \int^t du_i \wedge \alpha_i \\
&= \sum \int^t \lambda_{ij} dt_j \wedge \alpha_i \\
&= \sum \int^t T_{ik}\lambda'_{kj} dt_j \wedge \alpha_i \\
&= \sum \int^t du'_k \wedge \alpha'_k \\
&= \sum u'_k \alpha'_k.
\end{aligned}$$

□

Paralleling the procedure above, we have a similar result for Im Ω. Since L_t is special Lagrangian, $f^*\text{Im }\Omega = 0$ on the fibre L_t. Therefore there exist $(n-1)$-forms $\tilde{\phi}_j$ on $\mathcal{M}$ such that $f^*\text{Im }\Omega = \sum dt_j \wedge \tilde{\phi}_j$. The restriction $\phi_j = \phi_j|_{L_t}$ is independent of the choice of $\tilde{\phi}_j$. In fact, $\phi_j = (\iota(\frac{\partial}{\partial t_j}) f^*\text{Im }\Omega)|_{L_t}$ and $\phi_j = \star\theta_j$ by Corollary 2.2.2.1.

Let $B_1, \ldots, B_m \in H_{n-1}(L, \mathbb{Z})$ be the Poincaré duals of $\alpha_1, \ldots, \alpha_m$. We form a period matrix

$$\mu_{ij} = \int_{B_i} \phi_j.$$

Then there exist coordinates functions $v_1, \ldots, v_m$ such that $dv_i = \mu_{ij} dt_j$ and an embedding

$$v : M \to H^{n-1}(L, \mathbb{R})$$

given by

$$v(t) = \sum v_i \beta_i,$$

where $\beta_1, \ldots, \beta_m \in H^{n-1}(L, \mathbb{R})$ are the dual basis of $B_1, \ldots, B_m$. The matrices (λ_{ij}) and (μ_{ij}) are related as follows:

Lemma 2.3.3.

$$\sum_i \lambda_{ij} \mu_{ik} = \sum_i \lambda_{ik} \mu_{ij}.$$

Proof.

$$\begin{aligned} \int_L \theta_j \wedge \phi_k &= \int_L \theta_j \wedge \star\theta_k \\ &= \int_L \theta_k \wedge \star\theta_j \\ &= \int_L \theta_k \wedge \phi_j. \end{aligned}$$

Using $\theta_j = \sum \lambda_{ij} \alpha_i$, $\phi_k = \sum \mu_{lk} \beta_l$ and the fact that α_i and β_k are Poincaré duals, we get the identity. $\square$

u and v together give an embedding

$$F : M \to H^1(L, \mathbb{R}) \times H^{n-1}(L, \mathbb{R})$$

by

$$F(t) = (u(t), v(t)).$$

The main result of [Hit97] is to show that the geometry of M is inherited from $H^1(L,\mathbb{R})\times H^{n-1}(L,\mathbb{R})$. We first recall the linear geometry on $V\oplus V^*$ where V is a finite dimensional vector space. There is naturally a symplectic structure on $V\oplus V^*$ given by

$$W((v,\alpha),(v',\alpha')) = \alpha'(v) - \alpha(v').$$

There is also a symmetric bilinear form G on $V\oplus V^*$ given by

$$G((v,\alpha),(v',\alpha')) = \frac{1}{2}(\alpha'(v) + \alpha(v')).$$

It follows that manifold $H^1(L,\mathbb{R})\times H^{n-1}(L,\mathbb{R})$ has these two structures on the tangent space $H^1(L,\mathbb{R})\oplus H^{n-1}(L,\mathbb{R})$; in other words, $H^1(L,\mathbb{R})\times H^{n-1}(L,\mathbb{R})$ is a symplectic manifold together with a symplectic form W and an indefinite metric G. This indefinite metric restricted M is the L^2 metric:

Proposition 2.3.4. *The L^2 metric g on M is F^*G.*

Proof. We have $dF(\frac{\partial}{\partial t_j}) = (\sum \lambda_{ij}\alpha_i, \sum \mu_{lk}\beta_l)$. Thus for two tangent vectors,

$$F^*G(a_j\frac{\partial}{\partial t_j}, b_k\frac{\partial}{\partial t_k}) = \frac{1}{2}(a_jb_k\lambda_{ij}\mu_{ik} + a_kb_j\lambda_{ik}\mu_{ij}) = a_jb_k\lambda_{ij}\mu_{ik},$$

where we sum over repeated indices. On the other hand,

$$g(a_j\frac{\partial}{\partial t_j}, b_k\frac{\partial}{\partial t_k}) = a_jb_k\int_L \theta_j\wedge\star\theta_k = a_jb_k\int_L \theta_j\wedge\phi_k.$$

(Again we are summing over repeated indices.) Plugging $\theta_j = \sum \lambda_{ij}\alpha_i$ and $\phi_k = \sum \lambda_{ik}\beta_i$ into the above equation gives the equality. $\square$

Theorem 2.3.5. *The map F embeds M in $H^1(L)\times H^{n-1}(L)$ as a Lagrangian submanifold.*

Proof. In local coordinates (u,v), the symplectic form W reads

$$W = \sum du_i\wedge dv_i.$$

Pulling back to M, we get

$$
\begin{aligned}
F^*W &= F^* \sum du_i \wedge dv_i \\
&= \sum \tfrac{\partial u_i}{\partial t_j} \tfrac{\partial v_i}{\partial t_k} dt_j \wedge dt_k \\
&= \sum \lambda_{ij}\mu_{ik} dt_j \wedge dt_k.
\end{aligned}
$$

But $\sum_i \lambda_{ij}\mu_{ik} = \sum_i \lambda_{ik}\mu_{ij}$. So W pulls back to 0. $\square$

Any Lagrangian submanifold of $V \times V^* \simeq T^*V$ can be written locally as the image of the exact differential $d\phi : V \to T^*V$, where ϕ is a (locally defined) function. In our case, $H^1(L) \times H^{n-1}(L) \simeq T^*H^1(L)$, and the fibre coordinates v_j can be written as

$$v_j = d\phi(u_j) = \tfrac{\partial \phi}{\partial u_j}.$$

Exchanging the roles of $H^1(L)$ and $H^{n-1}(L)$, we can also write $u_j = \frac{\partial \psi}{\partial v_j}$ for some locally defined function ψ on $H^{n-1}(L)$. In these coordinates, the metric G can be written as

$$G = \sum du_i dv_i = \sum \frac{\partial^2 \phi}{\partial u_i \partial u_j} du_i du_j = \sum \frac{\partial^2 \psi}{\partial v_i \partial v_j} dv_i dv_j.$$

As in the case of deformations of complex manifolds, one can ask when is M itself a special Lagrangian submanifold. The manifold $V \times V^*$ has two constant m-forms, which are the generators of $\wedge^m V^*$ and $\wedge^m V$. Following the previous notion, we can define a submanifold of $V \times V^*$ as a special Lagrangian if it is Lagrangian and a linear combination of these constant m-forms vanishes. For further discussion we refer to [Hit97].

Chapter 3

Ricci-flat metrics on T^*S^n

In this section, we introduce a local model of special Lagrangian spheres, namely the cotangent bundle of the standard n-sphere, T^*S^n. We equip T^*S^n with a Ricci-flat metric, called the Stenzel metric, and show that the zero section S^n is a special Lagrangian submanifold.

3.1 Existence of the Metric

First we fix coordinates by $T^*S^n = \{(x, \xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid |x| = 1, x \cdot \xi = 0\}$. The group SO$(n+1, \mathbb{R})$ acts transitively on T^*S^n by $g \cdot (x, \xi) = (gx, g\xi)$, $g \in$ SO$(n+1, \mathbb{R})$. The general orbit of this action is the sphere bundle SO$(n+1)/$SO$(n-1) \simeq S^n \times S^{n-1}$, while the exceptional orbit being the zero section S^n. It is well-known that there exists a diffeomorphism from T^*S^n to the affine quadric

$$Q^n = \{z = (z_1 + \ldots + z_{n+1}) \in \mathbb{C}^{n+1} \mid \sum_{j=1}^{n+1} z_j^2 = 1\},$$

given by

$$(x, \xi) \mapsto z = x \cosh(|\xi|) + i \frac{\sinh(|\xi|)}{|\xi|} \xi.$$

This diffeomorphism is equivariant with respect to the action of SO$(n+1, \mathbb{R})$ on T^*S^n and the action of SO$(n+1, \mathbb{C})$ on Q^n. Thus we can view T^*S^n as the complexification of the homogeneous space S^n. We make T^*S^n into a complex manifold simply by pulling back the complex structure of Q^n. Our next goal is to describe a method to construct a family of Ricci-flat metrics on Q^n.

Let τ be the restriction of the function $\|z\|^2 = \sum_{j=1}^{n+1} |z_j|^2$ to Q^n. By a simple calculation we see that $\tau : Q^n \to [1, \infty)$ is a strictly plurisubharmonic exhaustion on Q^n. We seek [Ste93] Ricci-flat Kähler potentials of the form $\rho = f \circ \tau$, where $f : [1, \infty) \to \mathbb{R}$ is a smooth function. For later calculations, we fix a local frame on Q^n as follows:

$$\begin{cases} v_1 &= -z_{n+1}\frac{\partial}{\partial z_1} + z_1\frac{\partial}{\partial z_{n+1}}, \\ v_2 &= -z_{n+1}\frac{\partial}{\partial z_2} + z_2\frac{\partial}{\partial z_{n+1}}, \\ &\vdots \\ v_n &= -z_{n+1}\frac{\partial}{\partial z_n} + z_n\frac{\partial}{\partial z_{n+1}}. \end{cases}$$

Let $u^1, \ldots, u^n$ be the dual frame. The Kähler form of ρ is given by

$$\begin{aligned} \omega &= i\partial\bar{\partial}\rho \\ &= i(f''\partial\tau \wedge \bar{\partial}\tau + f'\partial\bar{\partial}\tau) \\ &= i(f''\tau_j\tau_{\bar{k}} + f'\tau_{j\bar{k}})u^j \wedge u^{\bar{k}}, \end{aligned}$$

where $\tau_j, \tau_{\bar{k}}$ denote the differentiations in the directions of v_j and $\bar{v}_k$, respectively:

$$\begin{aligned} \tau_j &= -z_{n+1}\bar{z}_j + z_j\bar{z}_{n+1}, \\ \tau_{\bar{k}} &= -\bar{z}_{n+1}z_k + \bar{z}_k z_{n+1}, \\ \tau_{j\bar{k}} &= |z_{n+1}|^2\,\delta_{jk} + z_j\bar{z}_k. \end{aligned}$$

The Ricci-form of ρ is given by

$$\begin{aligned} \mathrm{Ric}(\rho) &= -i\partial\bar{\partial}\log\det\rho_{j\bar{k}} \\ &= -i\partial\bar{\partial}\log\det(f''\tau_j\tau_{\bar{k}} + f'\tau_{j\bar{k}}). \end{aligned}$$

We can investigate further:

$$\begin{aligned}\det(f''\tau_j\tau_{\bar{k}} + f'\tau_{j\bar{k}}) &= (f')^n \det(\frac{f''}{f'}\tau_j\tau_{\bar{k}} + \tau_{j\bar{k}}) \\ &= (f')^n(1 + \frac{f''}{f'}\tau^{j\bar{k}}\tau_j\tau_{\bar{k}}) \det\tau_{j\bar{k}} \\ &= ((f')^n + (f')^{n-1}f''\tau^{j\bar{k}}\tau_j\tau_{\bar{k}}) \det\tau_{j\bar{k}},\end{aligned}$$

where the second equality follows from the matrix determinant lemma and $\tau^{j\bar{k}}$ is the matrix inverse of $\tau_{j\bar{k}}$, i.e. $\tau^{j\bar{i}}\tau_{j\bar{k}} = \delta_{ik}$.

Lemma 3.1.1.

$$\tau^{j\bar{k}}\tau_j\tau_{\bar{k}} = \frac{\tau^2 - 1}{\tau}.$$

Proof. By a simple calculation, we have

$$\tau^{j\bar{k}} = \frac{1}{|z_{n+1}|^2}(\delta_{jk} - \frac{1}{\tau}z_k\bar{z}_j).$$

After a lenghty but straight-forward computation, we get the equality. □

Lemma 3.1.2. *If the function f solves satisfies the ODE*

$$x(f'(x))^n + (f'(x))^{n-1}f''(x)(x^2 - 1) = c > 0, \tag{3.1}$$

where $c \in \mathbb{R}$ is a constant, then $(f \circ \tau)_{i\bar{j}}$ solves the Ricci-flat equation

$$\partial\bar{\partial} \log \det(f \circ \tau)_{i\bar{j}} = 0.$$

Proof. The eigenvalues of the matrix $(\tau_{j\bar{k}}) = (\delta_{jk}|z_{n+1}|^2 + z_j\bar{z}_k)$ are $|z_{n+1}|^2$ and τ, with multiplicity $n-1$ and 1, respectively. Therefore, $\det\tau_{j\bar{k}} = \tau|z_{n+1}|^{2n-2}$.

If f solves the ODE, then

$$\begin{aligned}\log\det(f\circ\tau)_{j\bar{k}} &= \log(\frac{c}{\tau}\det\tau_{j\bar{k}})\\ &= \log(c|z_{n+1}|^{2n-2}).\end{aligned}$$

It is then straight-forward to check that

$$\frac{\partial^2}{\partial z^j\partial\bar{z}^k}\log|z_{n+1}| = 0$$

to conclude the result. □

The ODE (3.1) can be solved directly. In fact, multiplying both sides of (3.1) with the integral factor $n(x^2-1)^{\frac{n-2}{2}}$, we have

$$((f')^n(x^2-1)^{\frac{n}{2}})' = nc(x^2-1)^{\frac{n-2}{2}}.$$

Hence

$$f'(x) = (\frac{nc\int_1^x(t^2-1)^{\frac{n-2}{2}}dt}{(x^2-1)^{\frac{n}{2}}})^{\frac{1}{n}}.$$

We still need to check that the potential $\rho = f\circ\tau$ defines a metric; in other words, ρ is a strictly plurisubharmonic function. But this can be seen easily as follows.

Lemma 3.1.3. *The function $\rho = f\circ\tau$ defined as above is a strictly plurisubharmonic exhaustion.*

Proof. That ρ is strictly plurisubharmonic translates to that the hermitian matrix $\rho_{j\bar{k}}f''\tau_j\tau_{\bar{k}}+f'\tau_{j\bar{k}}$ is positive definite. On the $(n-1)$-dimensional subspace ker $\partial\tau$, the matrix restricts to $f'\tau_{j\bar{k}}$. Since $f'>0$, this restricted hermitian form is positive definite. It follows that $\rho_{j\bar{k}}$ has at least $n-1$ positive eigenvalues. Since $\det\rho_{j\bar{k}} = \tau|z_{n+1}|^{2n-2} > 0$, the eigenvalues are all positive. This completes the proof. □

We record the results so far as the following theorem.

Theorem 3.1.4 (Stenzel[Ste93])**.** *There exists a Ricc-flat Kähler metric on T^*S^n.*

In [Ste93], Stenzel proved in general that any cotangent bundle of a rank one symmetric space admits such a Ricci-flat Kähler metric.

3.2 Completeness of the Stenzel Metric

In [PW91], Patrizio and Wong give a thorough study of the geometry in Q^n. They first constructed a pluriharmonic exhaustion of the form $g \circ \tau$ on Q^n, which satisfies the homogeous Monge-Ampère equation. In fact, the method that Stenzel constructed the Ricci-flat Kähler metric in the previous section parallels the construction of Patrizio and Wong:

Theorem 3.2.1 ([PW91])**.** *The function $u = \cosh^{-1} \circ \tau$ on Q^n is a plurisubharmonic exhaustion function of Q^n which satisfies the homogeneous Monge-Ampère equation*

$$(\partial\bar{\partial}u)^n = 0.$$

Proof. The function u can be derived in the following way. Suppose there is a smooth function $h : [1, \infty) \to \mathbb{R}$ such that $u = h \circ \tau$. The homogeneous Monge-Ampère equation translates to

$$\begin{aligned}
\det u_{j\bar{k}} &= \det h''\tau_j\tau_{\bar{k}} + f'\tau_{j\bar{k}} \\
&= ((h')^n + (h')^{n-1}h''\tau^{j\bar{k}}\tau_j\tau_{\bar{k}}) \det \tau_{j\bar{k}} \\
&= 0.
\end{aligned}$$

By Lemma 3.1.1, it suffices to find a function h satisfying the following ODE:

$$\frac{h''(x)}{h'(x)} = -\frac{x}{x^2 - 1}.$$

Integrating twice gives the general solution

$$h(x) = c_1 \cosh^{-1} x + c_2.$$

The condition $h(1) = 0$ forces c_2 to be 0. Plurisubharmnoicity implies that $c_1 > 0$. That the resulting function $u = \cosh^{-1} \circ \tau$ is plurisubharmonic can be seen by the same argument as in Lemma 3.1.3. □

The following theorem is the key to the completeness of the metric.

Theorem 3.2.2 ([PW91])**.** *Let $\rho = r^{-1} \circ u$ be a strictly plurisubharmonic function on a complex manifold M, where u satisfies the homogeneous Monge-Ampère equation. Then ρ defines a Kähler metric. The distance minimizing geodesics between level sets of ρ are given by the integral curves of $\nabla\rho$. Consequently the geodesic distance between level sets $\{\rho = a\}$ and $\{\rho = b\}$ is given by*

$$\mathrm{dist}(\{\rho = a\}, \{\rho = b\}) = \left| \frac{1}{\sqrt{2}} \int_a^b \sqrt{\frac{-r''}{r'}} dt \right|.$$

Theorem 3.2.3. *The Ricci-flat Kähler pontential $\rho = f \circ \tau$ constructed in the last subsection gives a complete metric.*

Proof. Let $u = \cosh^{-1} \circ \tau$. Then we write $\rho = g \circ u$, where $g = f \circ \cosh$. g satisfies the ODE

$$[g'(w)^n]' = nc(\sinh(w))^{n-1}.$$

By theorem 3.2.2, the distance between a point $z \in Q^n$ and the zero section $\{\rho = 0\}$ is given by

$$\begin{aligned} \mathrm{dist}(z, \{\rho = 0\}) &= \frac{1}{\sqrt{2}} \int_0^{\rho(z)} \sqrt{\frac{-r''}{r'}} dt \\ &= \frac{1}{\sqrt{2}} \int_0^{u(z)} \sqrt{g''(w)} dw. \end{aligned}$$

We estimate $g'(u)^n$ from above.

$$\begin{aligned} g'(u)^n &= \int_0^u nc(\sinh(w))^{n-1}dw \\ &\leq ncu \sinh(u)^n \end{aligned}$$

for u large enough, say $u > u_0$, since $\sinh(w)$ is strictly increasing. It follows that

$$\int_0^u \sqrt{g''(w)}dw \geq c_0 \int_{u_0}^u (\frac{1}{u})^{\frac{n-1}{2n}} du,$$

where c_0 is a constant. Note that the right hand side is clearly unbounded as $u \to \infty$. Since the function u is a exhaustion, this completes the proof. $\square$

3.3 Special Lagrangian Structures

This Ricci-flat metric, together with a holomorphic nonvanishing $(n, 0)$-form on Q^n, makes Q^n a noncompact Calabi-Yau manifold. The sphere $\{x_1^2 + \ldots x_{n+1}^2 = 1\}$ in Q^n can be shown to be special Lagrangian.

Let $f \circ \tau$ denote our Ricci-flat potential. Let $\omega = i\partial\bar{\partial}f(\tau)$ be the Kähler form. Then $\omega = d\alpha$ is exact, where $\alpha = -\text{Im}\ \bar{\partial}f(\tau)$. $\alpha = f'(\tau)\alpha_0$, where $(\alpha_0)_z(v) = \langle Jz, v\rangle$ is the standard Liouville form on $\mathbb{C}^{n+1}$. The holomorphic volume form Ω on Q^n is given by the Poincaré residue map. Let $h = z_0^2 + \ldots + z_{n+1}^2 - 1$ be the defining equation of Q^n and $\Omega_0 = dz_1 \wedge \ldots \wedge dz_{n+1}$ be the standard holomorphic volume form on $\mathbb{C}^{n+1}$. The adjunction formula

$$K_{Q^n} \simeq K_{\mathbb{C}^{n+1}}|_{Q^n} \otimes NQ^n$$

yields

$$\Omega = \iota(\sum_i \tfrac{\partial h}{\partial z_i} \tfrac{\partial}{\partial z_i})\Omega_0,$$

or more explicitly

$$\Omega(v_1, \ldots, v_n) = 2\Omega_0(z, v_1, \ldots, v_n)$$

for all $z \in Q^n$ and $v_1, \ldots, v_n \in T_z Q^n$.

Since SO$(n+1, \mathbb{C})$ preserves J, ω and Ω, there exists $\lambda \in \mathbb{R}$ such that

$$\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} (\frac{i}{2})^n \lambda^2 \Omega \wedge \bar{\Omega}. \tag{3.2}$$

Therefore $(T^*S^n = Q^n, J, \omega, \lambda\Omega)$ is a Calabi-Yau manifold.

Since $\alpha|_{S^n} = 0$, $\omega|_{S^n} = 0$. Also, since Im Ω contains dy^i terms and the tangent space of S^n is spanned by $\frac{\partial}{\partial x^i}$, we have Im $\Omega|_{S^n} = 0$. We conclude that S^n is a special Lagrangian submanifold of T^*S^n.

Chapter 4

Existence of Lagrangian Spheres

Definition 4.0.1. *Let $f : X \to \mathbb{C}$ be a holomorphic map of a complex manifold X. We say that a critical point $x \in X$ of f is nondegenerate if the hessian of p is a nondegenerate bilinear form.*

Let D denote the unit open disk in $\mathbb{C}$ centered at 0.

Definition 4.0.2. *Let X be a compact Kähler manifold. An ordinary degeneration of X is a Kähler manifold E with a proper holomorphic map $\pi : E \to D$ with the following properties:*

1. *π has a critical point.*
2. *Any critical point p of π is nondegenerate, and is contained in the central fibre $E_0 = \pi^{-1}(0)$.*
3. *There exists $z \in D$ such that X is isomorphic to $E_z = \pi^{-1}(z)$.*

The regular fibres of a degeneration, thanks to Ehresmann's fibration theorem and Moser's theorem on deformation of symplectic forms, are in fact symplectomorphic:

Proposition 4.0.3. *Let $p : E \to \mathbb{C}$ be a proper holomorphic map from a Kähler manifold (E, Ω). Then all regular fibres $E_t = p^{-1}(t)$ are symplectomorphic.*

Proof. May assume that E_0 is a regular fibre. By Ehresmann's fibration theorem, $p : E \to D$ is a fibration. May also assume that E_1 is another regular fibre in the local trivialization near 0. Then we have diffeomorphisms $\phi_t : X_0 \to X_t$, $t \in [0, 1]$. Let $\iota_t : E_t \to E$, $t \in [0, 1]$ be the inclusion maps. The symplectic form Ω restricts to $\omega_t = \iota_t^* \Omega$ on E_t. We have

$$[\phi_t^*\omega_t] = (\iota_t \circ \phi_t)^*[\Omega] = [\omega_0].$$

Thus there is a family of symplectic forms $\phi_t^*\omega_t$ on E_t lying in the same cohomology class. By Moser's theorem, ω_0 is symplectomorphic to $\phi_1^*\omega_1$, and the proposition follows. □

Morse lemma helps us choose a coordinate system shaping the projection π into a quadratic form.

Proposition 4.0.4 (Holomorphic Morse Lemma)**.** *Let $f : X^n \to \mathbb{C}$ be a holomorphic map. Suppose that $x \in X$ is a nondegenerate critical point of f. Then there exists a coordinate system $(z_1, \ldots, z_n)$ centered at x such that under this coordinate system, $f(z_1, \ldots, z_n) = f(0) + z_1^2 + \ldots + z_n^2$.*

The following is the main statement of this section.

Proposition 4.0.5. *Any compact Kähler manifold which admits an ordinary degeneration contains a Lagrangian sphere.*

In [Sei97], two constructions are given. We devote the subsequent subsections to record these results.

4.1 Seidel's Proof

We begin with a linear version.

Lemma 4.1.1. *Let β be a (complex) bilinear form on $\mathbb{C}^n$ which is symmetric and nondegenerate. Then there exists an n-dimensional real subspace $L \subset \mathbb{C}^n$ which is Lagrangian for the standard symplectic form ω_0, and such that β restricted to L is real and positive definite.*

Proof. Since β is nondegenerate, the real part $b = \mathrm{Re}\ \beta$ is a real nondegenerate bilinear form on $\mathbb{R}^{2n} = \mathbb{C}^n$, with complex structure J being the scalar multiplication with i. Let

B be the $\mathbb{R}$-linear map given by $b(v, w) = \langle Bv, w\rangle_{\mathbb{R}}$. Then B is invertible and self-adjoint. Note that B is conjugate-linear: $B(Jv) = -J(Bv)$ for all $v \in \mathbb{R}^{2n}$. We can decompose $\mathbb{R}^{2n}$ into $L^+ \oplus L^-$, where $L^\pm$ are the $\pm$-eigenspaces of B, respectively. Note that $JL^\pm = L^\mp$.

Let $L = L^+$. Then L is Lagrangian with respect to ω_0, since the eigenspaces are orthogonal to each other. The fact β restricted to L is real and positive definite follows from direct computation. □

Corollary 4.1.1.1. *(Notations as the above lemma.) Let q be the quadratic form associated to β. Then for $t > 0$, $V_t = q^{-1}(t) \cap L$ is a Lagrangian $(n-1)$-sphere in the symplectic manifold $(q^{-1}(t), \omega_0|_{q^{-1}(t)})$.*

Proof. Since β is nondegenerate, by the complex version of Regular Value Theorem, $q^{-1}(t)$ is a complex submanifold of $\mathbb{C}^n$, thus also a symplectic submanifold. Since β restricts to a real, positive definite form on L, by Regular Value Theorem, $q^{-1}(t) \cap L$ is a $(n-1)$-dimensional real submanifold. Since the tangent space lies entirely in the Lagrangian subspace L, this is a Lagrangian submanifold. Finally, the quadratic form q is nondegenerate. It follows that $q^{-1}(t) \cap L$ is a $(n-1)$-sphere. □

Lemma 4.1.2. *Let ω be a Kähler form in some open ball $B_\epsilon(0) \subset \mathbb{C}^n$ centered at 0 with radius $\epsilon > 0$, which agrees with the standard form ω_0 at 0. Thene there is a Kähler form ω' on $B_\epsilon(0)$ such that $\omega' = \omega$ on $B_\epsilon(0) \setminus B_{\frac{\epsilon}{2}}(0)$ and $\omega' = \omega_0$ in some neighborhood of 0. Moreover, ω' and ω are connected by a 1-parameter family of symplectic forms, all of them lying in the same cohomology class.*

Proof. By Poincaré lemma, there exists $0 < \delta < \epsilon$ and $f \in C^\infty(B_\delta(0), \mathbb{R})$ such that $(\omega_0 - \omega)|_{B_\delta(0)} = i\partial\bar{\partial} f$. Since $\omega_0 = \omega$ at 0, we may assume that f vanishes up to second order, i.e. $f(0) = \mathrm{D}f(0) = \mathrm{D}^2 f(0) = 0$. Therefore there exists $C > 0$ such that

$$\left|f(x)\right| \leq C|x|^3, \ |\mathrm{d}f_x| \leq C|x|^2 \ \textit{ and } \left|\mathrm{D}^2 f_x\right| \leq C|x| \tag{4.1}$$

for all $x \in B_\delta(0)$.

Let $\psi \in C^\infty$ be a bump function such that $\psi \equiv 1$ on $B_1(0)$ and $\mathrm{supp}(\psi) \subset B_2(0)$. For each $r > 0$, let $\psi_r(x) = \psi(\frac{x}{r})$. For each $0 < r < \frac{\delta}{2}$, let $\eta_r = i\partial\bar{\partial}(\psi_r f)$, which is a real $(1,1)$-form on $\mathbb{C}^n$. We compute:

$$\begin{aligned}\eta_r &= i((\partial\bar{\partial}\psi_r)f - \bar{\partial}\psi_r \wedge \partial f + \partial\psi_r \wedge \bar{\partial} f + \psi_r \partial\bar{\partial} f)\\ &= i(r^{-2}(\partial\bar{\partial}\psi)f + r^{-1}\bar{\partial}\psi \wedge \partial f - r^{-1}\partial\psi \wedge \bar{\partial} f + \psi_r \partial\bar{\partial} f)\end{aligned}$$

It follows from the estimates (4.1) that for $x \in B_{2r}$, $r < \frac{\delta}{2}$, we have

$$\left|(\eta_r)_x\right| \le C' r, \tag{4.2}$$

where C' is a constant independent of r. On the other hand, $(\eta_r)_x = 0$ for $x \notin B_{2r}(0)$. This together shows that $\eta_r \to 0$ uniformly as $r \to 0$. If we choose $r > 0$ small, then $\omega' = \omega + \eta_r$ is also a Kähler form. If$|x| < \frac{r}{2}$, then $\psi_r(x) = 1$ and $\omega' = \omega_0$ at x. Finally, the 1-parameter family is easily got by interpolating between ω and ω'.

□

Proof of the proposition. Let $\pi : E \to D$ be an ordinary degeneration of a compact Kähler manifold X, with Kähler form Ω. Let $x_0 \in \pi^{-1}(0)$ be a critical point of π. By holomorphic Morse lemma, there exists a holomorphic chart $c : B_\epsilon(0) \subset \mathbb{C}^n \to E$ around x_0 such that $q(z) = \pi(c(z))$ is a nondegenerate quadratic form. By a linear change of coordinates, we may assume that $c^*\Omega = \omega_0$ at 0. By the previous lemma, there is a Kähler form Ω' on E such that $c^*\Omega' = \omega_0$ near 0 and $\Omega' = \Omega$ outside a neighborhood of 0. By the previous corollary, there exists a Lagrangian sphere in the symplectic manifold $(E_t, \Omega'|_{E_t})$ whenever $t > 0$ is small enough. By Moser's theorem, $(E_t, \Omega'|_{E_t})$ and $(E_t, \Omega|_{E_t})$ are symplectomorphic. Finally, since every regular fibre is symplectomorphic to each other, we conclude that X contains a Lagrangian sphere.

□

4.2 Donaldson's Proof

Let $\pi : (E, \Omega) \to D$ be an ordinary degeneration of a compact Kähler manifold X, and let $x_0 \in \pi^{-1}(0)$ be a critical point of π. By holomorphic Morse lemma, there exists a coordinate system $z_1, \ldots, z_n$ centered at x_0 such that in this coordinate system, we have $\pi(z) = z_1^2 + \ldots + z_n^2$. Let $f = \text{Re}\ \pi, h = \text{Im}\ \pi$. Then $f(z) = x_1^2 - y_1^2 + \ldots + x_n^2 - y_n^2$, where $x_k = \text{Re}\ z_k, y_k = \text{Im}\ z_k, k = 1, \ldots, n$.

Let ϕ_r be the negative gradient flow of f, i.e.

$$\frac{\partial \phi_r}{\partial r}(p) = -\nabla f \circ \phi_r(p). \tag{4.3}$$

Let W^s denote the stable submanifold of x_0, i.e.

$$W^s = \{p \in E \mid \lim_{r \to \infty}(p) = x_0\}. \tag{4.4}$$

By Morse theory, for small $t > 0$, $V_t = W^s \cap f^{-1}(t)$ is an embedded $(n-1)$-sphere.

Lemma 4.2.1. *$\nabla f = X_h$, the Hamiltonian vector field of h.*

Proof. Let (g, J) denote the metric and the complex structure of E, respectively. For any tangent vector X of E, we compute

$$\begin{aligned} g(X_h, X) &= g(JX_h, JX) \\ &= \Omega(X_h, JX) \\ &= (JX)(h) \\ &= X(f), \end{aligned}$$

where the third equality follows from the definition of X_h and the last equality follows from the Cauchy-Riemann equation. $\square$

By the lemma, we see that ϕ_r is the integral flow of the Hamiltonian vector field X_h. It follows that ϕ_r is a 1-parameter subgroup of symplectomorphisms preserving h. Therefore $W^s \subset h^{-1}(0)$, and hence $V_t \subset (h^{-1}(0) \cap f^{-1}(t)) = E_t$.

For $p \in W^s$, since $\phi_r(p) \to x_0$ as $r \to \infty$, the differential $(d\phi_r)_p : T_pW^s \to T_{\phi_r(p)}W^s$ tends to 0 as $r \to \infty$. But $\phi_r^*\Omega = \Omega$. It turns out that W^s is a Lagrangian submanifold of E.

Since $V_t \subset W^s$, V_t is a Lagrangian sphere. Since all the regular fibres are symplectomorphic to each other, we conclude that X contains a Lagrangian sphere.

Some observations can be given when comparing these two proofs. The approach that Seidel follows is to study the local case first, and then deform the Kähler form given to meet the local case. The lagrangian sphere is obtained as the symplectomorphic image of the local one. The information about the sphere lies in Moser's trick. On the other hand, Donaldson's proof uses the gradient flow. The resulting Lagrangian sphere thus seems to have some kinde of uniqueness. But in both approaches, the sphere still depends on the coordinate choice, which lies in the complex Morse lemma.

Chapter 5

Discussion on the Main Problem

5.1 Formulation of the Main Problem

In this section, we investigate the existence of special Lagrangian spheres in a compact Calabi-Yau manifold M which admits an ordinary degeneration. The case of dimension $n = 3$ is of particular interest.

We formulate the problem as follows. Let M be a compact Calabi-Yau manifold. Suppose that M admits an ordinary degeneration $\pi : E \to D$ such that each nonsingular fibre is also a Calabi-Yau manifold. Let p be a singular point of π. By holomorphic Morse lemma, $\pi(z_1, \ldots, z_{n+1}) = z_1^2 + \ldots + z_{n+1}^2$ in a suitable coordinate system z_i. We consider the fibres E_t, where $t > 0$. Inside a neighborhood $|z| < R$, the fibre looks locally the same as our local model. As $t \to 0$, the tubular neighborhood $U_t = E_t \cap \{|z| < R\}$ of the sphere $\{x_1^2 + \ldots x_{n+1}^2 = t\}$ can be arbitrarily large compared to the size of the sphere. The naive approach is to use partition of unity to combine the metric on E_t and that of our local model, so that near the sphere, the metric is exactly the Stenzel metric. But the resulting metric can hardly be Kähler.

The Ricci-flat metric on T^*S^n has a kind of uniqueness. Therefore, we conjecture that as $t \to 0$, the metric on E_t will be more similar to the local one. The first step is to construct a Kähler metric on E_t such that near the sphere, this metric is exactly the Stenzel metric. Then we can apply the Calabi-Yau theorem to find a Ricci-flat metric in the same Kähler class. We may then compare this metric to the original one on E_t. This is still in progress.

The other approach is to use the mean curvature flow. According to [Smo96], the Lagrangian condition is preserved under the mean curvature flow, if the ambient manifold is Kähler-Einstein. If the long time existence is guaranteed, the flow would converge to a minimal Lagrangian submanifold. A possibly simpler question can be asked: does the Lagrangian mean curvature flow in T^*S^n have long existence? More precisely,

Question 1. *Does a Lagrangian section $\sigma : S^n \to (T^*S^n, \omega_{\mathrm{Stz}})$ converges to a special Lagrangian section under the mean curvature flow?*

For $n \geq 2$, the difficulties come from the fact that the codimension is greater than 1. For $n = 1$, the long time existence is guaranteed.

5.2 Results in $n = 1$

Let M be a compact 1-dimensional Calabi-Yau manifold. By Cartan-Hadamard theorem, the universal cover of M is R^2. It follows that M is just a flat torus. The Stenzel metric on $T^*S^1 \simeq S^1 \times \mathbb{R}$ is also flat for the same reason.

We first study the case of the local model $T^*S^1 = S^1 \times \mathbb{R}$. Let $\sigma : M = S^1 \to S^1 \times \mathbb{R}$ be a section. Recall the mean curvature flow $F(p, t)$ of σ is defined as the following equations:

$$\frac{\partial}{\partial t} F(p,t) = \Delta F(p,t) = H(p,t) \tag{5.1}$$

$$F(p,0) = \sigma(p) \tag{5.2}$$

where $p \in S^1$ and $H(p, t)$ denotes the mean curvature vector of the hypersurface $M_t = F(M, t)$.

Since any 1-dimensional submanifold is Lagrangian in a 2-dimensional symplectic manifold, the Lagrangian condition is empty. We are concerned about the long time existence and the behavior of this flow. Intuitively, the flow should minimize the length of a cross section of the cylinder $S^1 \times \mathbb{R}$, "flattening out" the initial one into a flat one $S^1 \times \{c\}$ for some $c \in \mathbb{R}$. If we lift the flow to the cover $\mathbb{R}^2$, then $\sigma : \mathbb{R} \to \mathbb{R}^2$ is just a graph of a periodic function. The mean curvature flow of entire graphs is well-studied in [EH89], and our situation is just a special case.

For completeness, we reproduce a proof here. The flow (5.1) is now considered in $\mathbb{R}^2$. Let ν be the unit outer normal of σ. The graph condition of the initial data can be captured by the existence of a vector $\omega \in \mathbb{R}^2$ such that $|\omega| = 1$ and $\langle \nu, \omega \rangle > 0$. From now on, we always assume implicitly that $x = F(p, t)$.

The fundamental indgredient is a monotonicity formula due to Huisken.

Definition 5.2.1. *Fix $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^{n+1}$. We define the backward heat kernel ρ_{x_0,t_0} to be*

$$\rho_{x_0,t_0}(x,t) = (4\pi(t_0 - t))^{-\frac{n}{2}} \exp(-\frac{|x_0 - x|^2}{4(t_0 - t)})$$

for $t < t_0$.

We will suppress the notions of x_0 and t_0 when they can be read out from the context. Differentiating $\rho_{x_0,t_0}(x,t)$ with respect to t yields

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\rho(F(p,t),t) = \left(\frac{n}{2(t_0 - t)} - \frac{|x_0 - x|^2}{4(t_0 - t)^2} + \frac{\langle H, x_0 - x \rangle}{2(t_0 - t)} \right) \rho(F(p,t),t).$$

On the other hand,

$$\Delta_t \rho(F(p,t),t) = \sum_i \left(\frac{-n}{2(t_0 - t)} + \frac{\langle H, x_0 - x \rangle}{2(t_0 - t)} + \frac{\left|(x_0 - x)^T\right|^2}{4(t_0 - t)^2} \right) \rho(F(p,t),t),$$

where T denotes the tangent component. Hence the evolution equation of ρ is

$$(\tfrac{\mathrm{d}}{\mathrm{d}t} + \Delta_t)\rho = \left(\frac{\langle x_0 - x, H \rangle}{t_0 - t} - \frac{\left|(x_0 - x)^\perp\right|^2}{4(t_0 - t)^2} \right) \rho,$$

where $\perp$ denotes the normal component. Let $d\mu_t$ denote the volume form of the flow M_t. The first variation of $d\mu_t$ yields

$$\tfrac{\mathrm{d}}{\mathrm{d}t} d\mu_t = -H^2 d\mu_t.$$

The above calculations together give the monotonicity formula:

Lemma 5.2.2.

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\int_{M_t}\rho d\mu_t = -\int_{M_t}\rho\left|H+\frac{1}{2(t_0-t)}(x-x_0)^{\perp}\right|^2 d\mu_t.$$

More generally, suppose $f(x,t) = f(F(p,t),t)$ satisfies some polynomial growth condition. Then we have

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\int_{M_t} f\rho d\mu_t = \int_{M_t}(\tfrac{\mathrm{d}}{\mathrm{d}t}f-\Delta f)\rho d\mu_t - \int_{M_t} f\rho\left|H+\frac{1}{2(t_0-t)}(x-x_0)^{\perp}\right|^2 d\mu_t. \quad (5.3)$$

This monotonicity formula gives the following maximum principle.

Proposition 5.2.3. *Suppose the function $f = f(x,t)$ satisfies the inequality*

$$(\frac{d}{dt}-\Delta)f \le a\cdot\nabla f$$

for some vector a, where ∇ denotes the gradient on M. If $a_0 = \sup_{M\times[0,t_1]}|a| < \infty$, then

$$\sup_{M_t} f \le \sup_{M_0} f.$$

Proof. Put $k = \sup_{M_0} f$ and set $f_k = \max(f-k,0)$. Then

$$\begin{aligned}(\tfrac{\mathrm{d}}{\mathrm{d}t}-\Delta)f_k^2 &\le 2f_k a\cdot\nabla f_k - 2|\nabla f_k|^2\\ &= -2\left|\nabla f_k - \frac{f_k}{2}a\right|^2 + \frac{f_k^2|a|^2}{2}\\ &\le \frac{1}{2}a_0^2 f_k^2.\end{aligned}$$

We then apply the monotonicity formula (5.3) to f_k^2. Pick $t_0 > t$ and $x_0 \in \mathbb{R}^{n+1}$ arbitrary. Then

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\int_{M_t} f_k^2\rho d\mu_t \le \frac{1}{2}a_0^2\int_{M_t} f_k^2\rho d\mu_t.$$

It turns out that $f_k \equiv 0$ and the result follows. $\square$

Let $u = \langle x, \omega \rangle$ denote the *height* of the M_t with respect to the hyperplane defined by ω. It follows from (5.1) that

$$(\frac{d}{dt} - \Delta)u = 0.$$

Since our graph is periodic, u is uniformly bounded at $t = 0$. It follows from proposition 5.2.3 that

$$\inf_{M_0} u \le \inf_{M_t} u \le \sup_{M_t} u \le \sup_{M_0} u$$

whenever M_t exists.

Next we show that the flow stays as a graph. It amounts to show that the quantity $\langle \nu, \omega \rangle$ is bounded from below for all time, or equivalently

$$v = \langle \nu, \omega \rangle^{-1}$$

is bounded from above. For each local frame $\{e_i\}$, let $A = (h_{ij})$ be the second fundamental form, i.e.

$$\langle F_{ij}, \nu \rangle = -h_{ij},$$

where $F_{ij} = e_i e_j F$.

Lemma 5.2.4.

$$\tfrac{\partial}{\partial t}\nu = \nabla H.$$

Proof. In local coordinates,

$$\tfrac{\partial}{\partial t}\nu = g^{ij}\langle \tfrac{\partial}{\partial t}\nu, \tfrac{\partial}{\partial x_i}\rangle \tfrac{\partial}{\partial x_j}.$$

Also,

$$\langle \tfrac{\partial}{\partial t}\nu, \tfrac{\partial}{\partial x_i}\rangle = -\langle \nu, \tfrac{\partial^2 F}{\partial t \partial x_i}\rangle = -\tfrac{\partial}{\partial x_i}\langle \nu, H\rangle = \tfrac{\partial H}{\partial x_i}.$$

This completes the proof. □

Lemma 5.2.5. *v satisfies the evolution equation*

$$(\frac{d}{dt} - \Delta)v = -|A|^2 v - 2v^{-1}|\nabla v|^2 .$$

Proof. From the previous lemma we know that

$$\tfrac{\partial}{\partial t}v = -v^2\langle \nabla H, \omega\rangle.$$

We may assume that $\{x_i\}$ is a normal coordinate system centered at a point we want, and set $e_i = \frac{\partial}{\partial x_i}$. Then,

$$\langle \nabla_{e_i}\nu, e_j\rangle = -\langle \nu, F_{ij}\rangle = h_{ij}.$$

Therefore $\nabla_{e_i}\nu = h_{ij}e_j$. The mean curvature H is given by

$$H = \langle \nabla_{e_j}e_j, -\nu\rangle,$$

and hence

$$e_i(H) = \langle F_{ijj}, -\nu\rangle = e_j\langle F_{ij}, -\nu\rangle = e_j(h_{ij}).$$

Therefore

$$\nabla H = e_i(h_{ij})e_j.$$

Now,

$$
\begin{aligned}
\Delta v &= e_i(-v^2 e_i\langle \nu, \omega\rangle) = 2v^{-1} e_i\langle \nu, \omega\rangle e_i\langle \nu, \omega\rangle + v^2 \Delta\langle \nu, \omega\rangle \\
&= 2v^{-1}|\nabla v|^2 - v^2\langle e_i(h_{ij})e_j, \omega\rangle - v^2\langle h_{ij}\nabla_{e_i} e_j, \omega\rangle \\
&= 2v^{-1}|\nabla v|^2 - v^2\langle \nabla H, \omega\rangle + v^2\langle \sum_{ij} h_{ij}^2 \nu, \omega\rangle \\
&= 2v^{-1}|\nabla v|^2 - v^2\langle \nabla H, \omega\rangle + v|A|^2 .
\end{aligned}
$$

□

From the above lemma and proposition 5.2.3, we deduce that v is uniformly bounded for all t, for v is certainly uniformly bounded for $t = 0$. It follows that our flow remains a Lagrangian section for all time.

The following estimates interior in time for the curvature and all its derivatives will establish the long time existence.

Proposition 5.2.6. *Assume that there is $c_1 > 1$ such that $v < c_1$ at $t = 0$. Let M_t be a smooth solution of* (5.1)*. Then for each $m \geq 0$, there is a constant $C(m)$ depending only on c_1 and m such that*

$$t^{m+1}|\nabla^m A|^2 \leq C(m)$$

holds uniformly on M_t.

From the above proposition we see that the flow keeps "flattening out" the section. Since in our case, the flow does not diverge to ∞, the flow actually converges to a hyperplane, that is a section $S^1 \times \{c\}$ of the cylinder $S^1 \times \mathbb{R}$.

In the torus case, we have the exactly same result, since the neighborhood of a cross section in the torus is a neighborhood in cylinder.

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