LOCALIZATION ON MODULI SPACES AND MULTIPLE COVER FORMULAS

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ABSTRACT. In this report I present new proofs and improvement of results for two kinds of multiple cover formula, which were previously established in [8], [7].

1. INTRODUCTION

Morrison-Aspinwall [10] and [11] computed the genus zero multiple cover formula for Gromov-Witten invariants of $\mathbb{P}^1$. For more general cases or for higher genus, it often happens that the coarse moduli space is not of the correct dimension. And one needs to replace the fundamental class by the *virtual fundamental class* to define the Gromov-Witten invariants. This was done by Li-Tian [9] and Behrend-Fantechi [2], [3]. Graber-Pandharipande [6] used this to develop the virtual localization formula, which may be thought as a generalization version of the Atiyah-Bott localization formula, and applied it to compute the genus 1 multiple cover formula and gived the conjecture for higher genus. This conjecture was later proved by Faber-Pandharipande [5].

In this article we recall Faber-Pandharipande's result and previous techniques, and give a direct-localization proof on two other multiple cover formulas on higher dimensional cases, which was proved by Lee, Lin and Wang [8], [7] using more advanced tools (mirror symmetry, quantizations). In particular, the result of Faber and Pandharipande together with our results determine Gromov-Witten invariant of the following moduli of stable maps into local Calabi-Yau of the form $E = O(-1)^{r+1} \to \mathbb{P}^r$, in the sense that we integral over $e(R^1\pi_* f^* N) \cap [\bar{M}_{g,n}(\mathbb{P}^r, d)]^{vir}$, where $[\bar{M}_{g,n}(\mathbb{P}^r, d)]^{vir}$ is the virtual fundamental class of $\bar{M}_{g,n}(\mathbb{P}^r, d)$ and N is the normal bundle $O(-1)^{r+1}$.

2. EQUIVARIANT COHOMOLOGY AND LOCALIZATION FORMULA

The group action in consideration throughout this article will be $G = {\mathbb{C}^*}^{r+1}$ acting on $\mathbb{P}^r$ and other G-action induced by this action (of course, ${\mathbb{C}^*}^r$ is enough, but this setting make our formula cleaner) , where r is some fixed positive integer. We have

$$H^*_G(pt) = H^*(B_G) = \mathbb{Q}[\lambda_0, ..., \lambda_r],$$

where B_G is the classifying space for G. $\lambda_i \in H^2_G(pt)$ is the Euler class of a trivial line bundle, on which some piece of $\mathbb{C}^*$ acts by $w.z = wz$. Let G acts on $\mathbb{P}^r$ by $(w_0, ..., w_r).(z_0 : ... : z_r) = (w_0^{-1} z_0 : ... : w_r^{-1} z_r)$. We have

$$H^*_G(\mathbb{P}^r) = \mathbb{Q}[h][\lambda_0, ..., \lambda_r]/(\prod(h - \lambda_i)).$$

Under the action of G, $\mathbb{P}^r$ has $r + 1$ fixed points, which we will denote by $p_0, ..., p_r$. The tautological bundle $O(-1)$ has Euler class $-\lambda_i$ when restricted to $\{p_i\}$. When a trivial line bundle (in particular, a 1-dimensional vector space) has Euler class $\mu \in H^2_G(pt)$, we say that the bundle has weight μ.

This is a report which fulfills the course *Special Project for Undergraduate*, supervised by Professor Chin-Lung Wang, at National Taiwan University from September 2009 to January 2000, in which I studied moduli problems and Gromov-Witten theory.

Consider the map $\iota_i : pt \to \mathbb{P}^r$, where the image is p_i. For $\tau \in H^*_G(\mathbb{P}^r)$, $\iota_i^*\tau = \tau(\lambda_i) := \tau|_{h=\lambda_i}$. Moreover $\iota_{i*}1 = \prod_{j\neq i}(h-\lambda_j)$. This can be seen as follows: firstly $\iota_i^*\iota_{i*} = \prod_{j\neq i}(\lambda_i - \lambda_j)$, the Euler class of the tangent space of $\mathbb{P}^r$ at p_i. Also $\lambda_i\iota_{i*}1 = \iota_{i*}(\lambda_i) = \iota_{i*}(\iota_i^*h) = h\iota_{i*}1$ by the projection formula, that is $(h-\lambda_i)\iota_{i*}1 = 0$, and thus the result.

The Atiyah-Bott localization formula [1] is as follows: suppose $\{F_i\}$ are the fixed loci under G of some space M. $\iota_i : F_i \to M$ is the imbedding. Then for every $\mu \in H^*_G(M)$, we have, for some polynomial $f \in H^*_G(pt)$, in the localized module $H^*_G(M)_f$

$$\mu = \sum_i \iota_{i*}\left(\frac{\iota_i^*\mu}{\iota_i^*\iota_{i*}1}\right), \qquad \int_M \mu = \sum_i \int_{F_i} \frac{\iota_i^*\mu}{\iota_i^*\iota_{i*}1}$$

where $\iota_i^*\iota_{i*}1$ may also be seen as the Euler class of the normal bundle of F_i in M. Originally, this result was proven in equivariant cohomology. For our purpose, the equivariant Chow group was constructed and algebraic localization for schemes has been established by [4]. Generalization to Delinge-Mumford stack may be found in [6].

3. Virtual fundamental class and perfect obstruction theory

For the moduli of stable maps, the expected dimension is often different from the dimension of the moduli space. The expected dimension for the moduli of stable maps $\bar{M}_{g,n}(X,d)$ is $rk(H^0(C, f^*T_X)) - rk(H^1(C, f^*T_X)) + (3g-3) + n = d.c_1(T_X) + (\dim X - 3)(1-g) + n$. Now $\bar{M}_{2,0}(\mathbb{P}^1, 2)$ has expected dimension $2[\mathbb{P}]^1.c_1(T\mathbb{P}^1) + (\dim \mathbb{P}^1 - 3)(1-2) = 6$. However, the set of maps of a collapsing genus two curve glued with a double rational cover has dimension 7.

Nevertheless, there is a virtual fundamental class of the moduli space that lies in the correct dimension component of the rational Chow group, constructed via perfect obstruction theory. A perfect obstruction theory on a space X consists of:

(i) A two term complex of vector bundles $E^\bullet = [E^{-1} \to E^0]$ on X.

(ii) A morphism ϕ in the derived category of bounded above complexes of quasi-coherent sheaves from $E^\bullet$ to $L^\bullet X$, the cotangent complex of X, such that ϕ induces an isomorphism in degree 0 cohomology and a surjection in degree -1 surjection.

Here, a morphism in the derived category $\tau : F^\bullet \to G^\bullet$ may be explained as follows: there exists an object $\tilde{F}^\bullet$ in the category such that we have actual morphism of complexes $\tilde{F}^\bullet \to G^\bullet$ and $\tilde{F}^\bullet \to F^\bullet$, such that the later morphism is a quasi-isomorphism, i.e. it induces isomorphism in cohomology.

The virtual fundamental class is of dimension $rk(E^0) - rk(E^{-1})$. In our case, the moduli of stable maps $\bar{M}_{g,n}(\mathbb{P}^r, d)$ can be constructed as a quotient of a locally closed scheme of some Hilbert scheme by PGL. One may embed the Hilbert scheme into a larger Grassmannian, so that its quotient by PGL is still non-singular [6]. Thus $X = \bar{M}_{g,n}(\mathbb{P}^r, d)$ can be embedded into a non-singular Deligne-Mumford stack Y (this still holds if $\mathbb{P}^r$ is replaced by a subscheme V which possess an equivariant embedding $V \hookrightarrow \mathbb{P}^r$, since we have the embedding $\bar{M}_{g,n}(V,\beta) \hookrightarrow \bar{M}_{g,n}(\mathbb{P}^r, d)$). In this case, the cotangent complex can be taken to be its two-term cutoff:

$$L^\bullet X = [I/I^2 \to \Omega_Y]$$

where I is the ideal sheaf of X in Y. We assume for simplicity that $E^\bullet \to [I/I^2 \to \Omega_Y]$ is an actual map of complexes. This is not needed for the construction in [9], [3]. On the other hand, the embedding into quotient of Hilbert scheme shows that $\bar{M}_{g,n}(\mathbb{P}^r, d)$ has enough locally frees, so we may indeed make this assumption. The mapping cone associated to the map gives rise to an exact sequence

$$E^{-1} \to E^0 \oplus I/I^2 \to \Omega_Y \to 0$$

whose exactness is equivalent to the condition imposed on ϕ. Let Q be the kernel of $E^0 \oplus I/I^2 \to \Omega_Y$. The associated exact sequence of abelian cones reads

$$0 \to TY \to C(I/I^2) \times_X E_0 \to C(Q) \to 0$$

The normal cone of X in Y, $C_{X/Y}$ is a closed substack of $C(I/I^2)$. Let $D = C_{X/Y} \times_X E_0$. Then TY is a subcone of D [3], and the quotient $D^{vir} := D/TY$ is a subcone of $C(Q)$ and thus E_1. The virtual fundamental class $[X^{vir}]$ is defined to be the refined intersection of D^{vir} with the zero section of the vector bundle E_1.

Suppose now (and indeed in our case) $X \hookrightarrow Y$ is G-equivariant, together with a G-equivariant lifting to $E^\bullet$.

Then we have an equivariant perfect obstruction theory and an equivariant virtual fundamental class. Let X^f be the G-fixed locus of X. If $X = Spec(A)$, then X^f is defined by the ideal of functions with non-trivial G-characters. Similarly defining Y^f, we have $X^f = Y^f \cap X$. Moreover we have

$$\Omega_Y|^f_{Y^f} = \Omega_{Y^f}, \qquad \Omega_X|^f_{X^f} = \Omega_{X^f}$$

where S^f denote the fixed part of a sheaf S. In [6] it was then proved

Lemma 3.1 The map $\phi^f : (E^\bullet|_{X^f})^f \to L^\bullet X^f$ is a perfect obstruction theory on X^f.

Now, we define the virtual normal bundle N_f^{vir} to X^f to be the moving part of $E_\bullet|_{X^f}$. This is a two-term complex of vector bundles, and for such a complex $[B_0 \to B_1]$ its Euler class is defined to be $e(B_0)/e(B_1)$. Let K be the quotient field of $A^*_G(pt)$. In this setting, [6] proved the following virtual localization formula

$$[X]^{vir} = \iota_* \frac{[X^f]^{vir}}{e(N_f^{vir})}$$

where $\iota_* : A_*^{C^*}(X^f) \otimes_{A^*_G(pt)} K \to A_*^{C^*}(X) \otimes_{A^*_G(pt)} K$ is the push-forward map.

For the moduli of stable maps, there is a canonical perfect obstruction theory. In [2], the relative deformation problem

$$\bar{M}_{g,n}(V, \beta) \to \mathfrak{M}_{g,n}$$

is studied to obtain the canonical obstruction theory. Here $\mathfrak{M}_{g,n}$ is the nonsingular Artin stack of prestable curves. The explicit result we need in the computation will be written down in the next section.

4. LOCALIZATION THE MODULI OF STABLE MAPS

In this section we discuss the combinatorial summation that appear when we apply the virtual localization formula to the moduli of stable maps $\bar{M}_{g,n}(\mathbb{P}^r, d)$.

We discuss the fixed locus of $\bar{M}_{g,n}(\mathbb{P}^r, d)$ by G. Since $G = {\mathbb{C}^*}^{r+1}$ acts on the universal curve $\bar{U}_{g,n}(\mathbb{P}^r, d)$, on each point in the fixed locus of $\bar{M}_{g,n}(\mathbb{P}^r, d)$, we obtain an induced action of G on the fiber curve which is compatible with the stable map and the action of G on $\mathbb{P}^r$, i.e. we obtain an G-equivariant map from the fiber curve to $\mathbb{P}^r$. In particular, for stable maps in the fixed locus, any special points (nodes and marked points, images of contracted components and ramified points) must be mapped to some $p_i \in \mathbb{P}^r$.

Moreover, for any component not contracted, since its image has dimension 1, the image must be a coordinate line connecting some p_i and p_j. Moreover, we obtain a group action on the component, with at least two fixed points, which forces the component to be rational. Since the map must only ramify at p_i and p_j, by Riemann-Hurwitz formula the map must be of the form: $\mathbb{P}^1 \to \mathbb{P}^1$ given by $z \mapsto z^d$, where 0 and ∞ correspond to p_i and p_j.

Let f denote the stable map(s) to $\mathbb{P}^r$. By the above analysis, a closed point on the fixed locus of M (by action of G) is described by the following graph:

(i) A set of vertices, corresponding to each connected component of $f^{-1}(\{p_0, ..., p_r\})$. For each vertex one number is associated, namely its genus. Also there are two labels on it, one being its image (some of the p_i's) and the other being a subset of all the marked points.

(ii) A set of edges, corresponding to each non-contracted irreducible component. A edge has two vertices at its endpoints, which are the connected component of the two points corresponding to some p_i and p_j. Also, there is a positive number associated, the degree of the map from the component to $\mathbb{P}^r$.

The genus of such a graph is defined to be the first Betti number of the graph plus the sum of the genus of all vertices. The degree of the graph is the sum of degrees of all edges. The valence of a vertex, denoted $val(v)$ below, is the sum of the number of neighboring edge and the number of marked points. Let $S_{g,n,d}$ be the set of such graphs with correct genus, degree and marked point set and is connected. For each $\Gamma \in S_{g,n,d}$, define

$$M_\Gamma = \prod_{\text{vertex } v} \bar{M}_{g(v),val(v)}$$

where $\bar{M}_{0,1}$ and $\bar{M}_{0,2}$ may appear, and in that case shall be realized as a point. Then there is a natural morphism $\bar{M}_\Gamma \to \bar{M}_{g,n}(\mathbb{P}^r, d)$ given by the moduli property. There is a "kernel" A of the morphism, which is a group acting on M_Γ as well as the universal curve U_Γ. A can be described by the exact sequence of groups

$$1 \to \prod_{\text{edge } e} \mathbb{Z}/d(e) \to A \to Aut(\Gamma) \to 1$$

where the left term of the sequence arises from the automorphisms of the non-contracted rational component. We thus have a closed immersion $M_\Gamma/A \to \bar{M}_{g,n}(\mathbb{P}^r, d)$, and a component of the fixed locus of $\bar{M}_{g,n}(\mathbb{P}^r, d)$ by G is supported on M_Γ/A for each Γ. It can be shown that the fixed component is non-singular, and thus equal to M_Γ/A.

To apply the virtual localization formula, we must compute the virtual normal bundle of the fixed locus, which by definition is the moving part of the dual of the perfect obstruction theory. Let $E_{\bullet,\Gamma}$ denote the dual canonical perfect obstruction theory restricted to M_Γ/A. There are exact sequences [6]

$$0 \to T^1 \to E_{0,\Gamma} \to E_{1,\Gamma} \to T^2 \to 0 \tag{4.1}$$

$$0 \to Aut(C) \to H^0(C, f^*T\mathbb{P}^r) \to T^1 \to Def(C) \to H^1(C, f^*T\mathbb{P}^r) \to T^2 \to 0 \tag{4.2}$$

where we represent the bundle by its fiber. C is the fiber curve and $Aut(C)$ and $Def(C)$ are the bundle of infinitesimal automorphism and deformation fixing the marked points, respectively. In general, when one applies the virtual localization formula on $\bar{M}_{g,n}(\mathbb{P}^r, d)$, one compute a summation over all possible connected graph of correct genus, degree and number of marked point, and reduces to intersection theory on the moduli of curves via the above exact sequence. Also we may carry out the computation on $\bar{M}_\Gamma$ and divided the result by $|A|$ in the end.

We now claim that the bundle $H^1(C, f^*T\mathbb{P}^r)$ has no fixed part (on M_Γ/A), so is T^2. This shows that the fixed loci is nonsingular, as promised. For a vertex $v \in \Gamma$, let $i(v)$ be given by $f(C_v) = \{p_{i(v)}\}$, where C_v denote the component corresponding to the vertex v. Also for each edge e let C_e denote the corresponding component. Then from the exact sequence of sheaves

$$0 \to \mathcal{O}_C \to \bigoplus_v \mathcal{O}_{C_v} \oplus \bigoplus_e \mathcal{O}_{C_e} \to \bigoplus_{(v,e)} \mathcal{O}x_{(v,e)} \to 0$$

where the summation for the right term are taken over all adjacent pairs of vertices and edges, and $x_{(v,e)}$ is the intersection of C_v and C_e, tensoring $f^*\mathcal{F}$ we have the long exact sequence:

$$\begin{aligned} 0 \to H^0(C, f^*\mathcal{F}) \to \textstyle\bigoplus_v H^0(C_v, f^*\mathcal{F}) \oplus \bigoplus_e H^0(C_e, f^*\mathcal{F}) \to \bigoplus_v (\mathcal{F}|_{p_{i(v)}})^{\oplus deg(v)} \\ \to H^1(C, f^*\mathcal{F}) \to \textstyle\bigoplus_v H^1(C_v, f^*\mathcal{F}) \oplus \bigoplus_e H^1(C_e, f^*\mathcal{F}) \to 0 \end{aligned}$$

(4.3)

where we use the case $\mathcal{F} = T\mathbb{P}^r$. In this case, $H^1(C_e, f^*\mathcal{F})$ all vanish, and $H^1(C_v, f^*\mathcal{F}) = H^1(C_v, \mathcal{O}_{C_v}) \otimes \mathcal{F}|_{p_{i(v)}}$ has no fixed part since $H^1(C_v, \mathcal{O}_{C_v})$ is a fixed bundle, but $\mathcal{F}|_{p_{i(v)}}$ is a trivial bundle with action weight $\lambda_i - \lambda_j$ for all $j \neq i$. Also $\mathcal{F}|_{p_{i(v)}}$ has no fixed part as it is the tangent space of $p_{i(v)}$. Thus $H^1(C, f^*\mathcal{F})$ has no fixed part and the claim is proved.

5. MULTIPLE COVER FORMULAS

In this section we stated the result of Faber and Pandharipande for higher genus and $r = 1$, as well as our results which give different and more direct proofs of Lee, Lin and Wang's results in the higher dimensional case.

Theorem 1 [5] Let $g \geq 1$. Consider the moduli $\bar{M}_{g,0}(\mathbb{P}^1, d)$ and a normal bundle $O(-1) \oplus O(-1)$ on $\mathbb{P}^1$. Let f be the map from the universal curve $\bar{U}_{g,0}(\mathbb{P}^1, d)$ to $\mathbb{P}^1$, π be the projection from the universal curve to the moduli. Then

$$\int_{[\bar{M}_{g,0}(\mathbb{P}^1,d)]^{vir}} e(R^1\pi_* f^*(O(-1) \oplus O(-1)) = \frac{|B_{2g}| d^{2g-3}}{2g \cdot (2g-2)!}$$

where $e()$ denote the Euler class.

Theorem 2 [7] Consider the moduli $\bar{M}_{1,0}(\mathbb{P}^r, d)$ and a normal bundle $O(-1)^{\oplus r+1}$ on $\mathbb{P}^r$. Let f be the map from the universal curve $\bar{U}_{1,0}(\mathbb{P}^r, d)$ to $\mathbb{P}^r$, π be the projection from the universal curve to the moduli. Then

$$\int_{[\bar{M}_{1,0}(\mathbb{P}^1,d)]^{vir}} e(R^1\pi_* f^*(O(-1)^{\oplus r+1})) = (-1)^{(r+1)d} \frac{r+1}{24d}$$

Theorem 3 Consider the moduli $\bar{M}_{0,n}(\mathbb{P}^r, d)$ and a normal bundle $O(-1)^{\oplus r+1}$ on $\mathbb{P}^r$. Let f, π be similar. Let $e_1, ..., e_n : \bar{M}_{0,n}(\mathbb{P}^r, d) \to \mathbb{P}^r$ be the evaluation maps. Let $l_1, ..., l_n$ be such that $l_1 + ... + l_n = 2r + n - 2$. Let $u(l_1, ..., l_n)$ be the coefficient of $\alpha^r\beta^r$ in $\prod_{k=1}^n(\alpha^{l_k} - \beta^{l_k}) \cdot (\alpha-\beta)^{2-n}$. Then

$$\int_{[\bar{M}_{0,n}(\mathbb{P}^r,d)]^{vir}} e(R^1\pi_*f^*(O(-1)^{\oplus r+1}))\prod_{i=1}^n e_i^*(h^{l_i}) = (-1)^{(r+1)(d-1)+1}\frac{u(l_1,...,l_n)}{2}d^{n-3}$$

where h is the hyperplane class in $A^1(\mathbb{P}^r)$.

Theorem 3 was proved in [8] without knowing the exact value of the universal constant. Our extended result gives the first explicit formula of the constant and our proof is much more elementary in nature (without using mirror transformations).

The general idea is that when the bundle is large (has dimension larger than the space $\mathbb{P}^r$) then we may greatly simplify the combinatorics that appears in the summation over graphs for localization. In particular, one may easily reach the result that the number is a monomial in d (possibly negative power).

The reason we use the term "normal bundle" to coin $O(-1)^{\oplus r+1}$ is because we are thinking the problem as counting the cover formula for $\mathbb{P}^r$ the vector bundle $O(-1)^{\oplus r+1}$, which looks like a Calabi-Yau space. Note that $\mathbb{P}^r$ is rigid (i.e. no deformation) in the bundle. Let E denote the bundle. Then the moduli $\bar{M}_{g,0}(E,d)$ has expected dimension $(\dim E-3)(g-1)+K_E\cdot d = (\dim E - 3)(g-1)$.

When we introduce marked points for the Gromov-Witten invariant (as in theorem 3) and give constraints that some marked point must be mapped to some class, because of the divisor equation, we obtain new information only if the class is of codimension two or higher. In that case, $\bar{M}_{g,n}(E,d)\cap e_1^*(h_1)...e_n^*(h_n)$ has dimension $(\dim E-3)(g-1) - \sum(\text{codim } h_i - 1)$, where $h_i > 1$. Since $\dim E = 2r+1$, the only non-trivial case with marked points are $g = 0$, and when $g > 1$ we must have $r = 1$, that's the case for Faber-Pandharipande [5].

6. Direct localization toward the proof

Below we give the proof of our result, namely theorem 2 and 3.

Proof of theorem 2 The bundle $O(-1)$ can be considered as the tautological bundle on $\mathbb{P}^r$. In that case, we have an equivariant G action on the bundle, with the weight of the vector space $O(-1)|_{p_i}$ equal to $-\lambda_i$. We have $r+1$ pieces of $O(-1)$, which we will called the 0-th, ..., r-th piece. We interpret the k-th piece (which we will denote by $O(-1)_k$) as the tensor product of the tautological bundle with a trivial bundle with equivariant weight λ_k. In this way, $O(-1)_k|_{p_i}$ has weight $\lambda_k - \lambda_i$. The key point is that the k-th piece has zero weight at p_k.

Applying the localization formula in this setting, we observe that all graph that has more than one edges gives zero contribution. The reason is that, for such a graph there must be a vertex which has degree larger than or equal to two. Consider the exact sequence (4.3) with $\mathcal{F} = O(-1)^{\oplus r+1}$. According to the last result of the last paragraph, for each vertex the third term in (4.3) always produces $deg(v)$ zeroes, one of them cancels out with $H^0(C_v, f^*\mathcal{F}) = H^0(C_v, \mathcal{O}_{C_v})\otimes\mathcal{F}|_{p_{i(v)}} = \mathcal{F}|_{p_{i(v)}}$. Thus when the degree ≥ 2, the contribution vanishes.

Since we are considering genus 1 case, the graph must be an edge with degree number d, connecting two vertices of genus 1 and 0. Let $\Gamma_{i,j}$ denote the graph for which the genus 1 component are mapped to p_i and the genus zero vertex (in fact a point) are mapped to p_j. The contribution is

$$\int_{M_{\Gamma_{i,j}}} \frac{e([H^1(C, f^*O(-1)^{\oplus r+1})])}{e(N^{vir}_{\Gamma_{i,j}})}$$

Let e and v denote the only edge and the genus 1 component, respectively, then for $\mathcal{F} = O(-1)^{\oplus r+1}$, by (4.3), the facts that $H^0(C, f^*\mathcal{F}) = 0$ and that each vertex has degree 1, one has $e([H^1(C, f^*\mathcal{F})]) = e([H^1(C_v, f^*\mathcal{F})])e([H^1(C_e, f^*\mathcal{F})])$. Here $H^1(C_v, f^*\mathcal{F}) = H^1(C_v, \mathcal{O}_{C_v}) \otimes \mathcal{F}|_{p_i}$ and thus

$$e([H^1(C_v, f^*\mathcal{F})]) = \prod_{k=0}^{r}(-\lambda + \lambda_k - \lambda_i)$$

where λ is the Euler class of the Hodge bundle on $\bar{M}_{1,1}$, the fiber of which is the global sections of the dualizing sheaf on the curve. On the other hand, $H^1(C_e, f^*\mathcal{F}) = H^0(C_e, f^*\mathcal{F}^* \otimes \Omega_{C_e})^*$. Let q_0, q_1 be the points on C_e with $f(q_0) = p_i$ and $f(q_1) = p_j$, respectively. At q_0 and q_1, the k-th piece of $f^*\mathcal{F}^*$ has induces weight $\frac{\lambda_i - \lambda_k}{d}$ and $\frac{\lambda_j - \lambda_k}{d}$, while the space of global sections of T_{C_e} has induced weights $\frac{\lambda_i - \lambda_j}{d}, \frac{\lambda_j - \lambda_i}{d}, 0$. Thus one computes

$$e([H^1(C_e, f^*\mathcal{F})]) = \prod_{k=0}^{r} \eta_{\lambda_k - \lambda_i, \lambda_k - \lambda_j}$$

where $\eta_{a,b} = \prod_{t=1}^{d-1} \frac{ta + (d-t)b}{d}$. For the term $e(N^{vir}_{\Gamma_{i,j}})$, let $\mathcal{F} = T\mathbb{P}^r$ and recall from (4.1) and (4.2) that

$$\frac{1}{e(N^{vir}_{\Gamma_{i,j}})} = \frac{e(Aut(C)^m)e(H^1(C, f^*\mathcal{F})^m)}{e(Def(C)^m)e(H^0(C, f^*\mathcal{F})^m)}$$

where E^m denote the moving part of the bundle E. $Aut(C)$ is a rank two bundle on the fixed locus, which splits into a fixed part consisting of the (infinitesimal) automorphism fixing both q_0 and q_1, and a moving part consisting of the automorphism for which the only fixed point on C_e is q_0. The fixed part is not considered in the computation and the contribution of the moving part is the induced weight of the tangent space of q_1, which is $\frac{\lambda_j - \lambda_i}{d}$.

For $Def(C)$, the deformation of the genus 1 curve is fixed by G and thus give no contribution. Thus $e(Def(C))$ is equal to the contribution of the deformation resolving the node $C_e \cap C_v$, thus

$$e(Def(C)) = e(TC_e|_{q_0}) - \psi = \frac{\lambda_i - \lambda_j}{d} - \psi$$

where ψ is the Euler class of the cotangent bundle of the marked point on $\bar{M}_{1,1}$. For the other two terms, we similarly uses (4.3), obtaining (for $\mathcal{F} = T\mathbb{P}^r$)

$$e([H^1(C_v, f^*\mathcal{F})]) = \prod_{k \neq i}(-\lambda + \lambda_i - \lambda_k)$$

and

$$e([H^0(C_e, f^*\mathcal{F})]) = \tilde{\eta}_{\lambda_i - \lambda_j, \lambda_j - \lambda_i} \cdot \prod_{k \neq i,j} \tilde{\eta}_{\lambda_i - \lambda_k, \lambda_j - \lambda_k}$$

where $\tilde{\eta}_{a,b} = \prod_{t=0}^{d} \frac{ta+(d-t)b}{d}$. Note that there is a zero in $\tilde{\eta}_{\lambda_i-\lambda_j,\lambda_j-\lambda_i}$, which should be discarded (as it is not in the moving part). Now we have

$$\int_{[\bar{M}_{1,0}(\mathbb{P}^1,d)]^{vir}} e(R^1\pi_* f^*(O(-1)^{\oplus r+1})) = \sum_{i\neq j} \frac{1}{|A|} \int_{M_{\Gamma_{i,j}}} \frac{e([H^1(C, f^*O(-1)^{\oplus r+1})])}{e(N^{vir}_{\Gamma_{i,j}})}$$

$$= \sum_{i\neq j} d^{-1} \frac{\lambda_j-\lambda_i}{d} \int_{\bar{M}_{1,1}} \frac{\prod_{k=0}^{r}(-\lambda+\lambda_k-\lambda_i)\prod_{k\neq i}(-\lambda+\lambda_i-\lambda_k)}{(\lambda_i-\lambda_j-\psi)/d} \cdot \frac{-1^{(r+1)(d-1)}}{\sigma_i\sigma_j}$$

where $\sigma_i := \prod_{k\neq i}(\lambda_i-\lambda_k)$ comes from the cancellation of $e([H^1(C_e, f^*O(-1)^{\oplus r+1})])$ and $e([H^0(C_e, f^*T\mathbb{P}^r)])$. We have $\lambda^2 = 0$ since $\bar{M}_{1,1}$ has dimension only 1. Also since after cancellation, there is still a λ in the numerator, the ψ in the denominator has no effect due to dimension reason. Thus the above summation is equal to

$$\sum_{i\neq j} \frac{-1^{r+(r+1)(d-1)}}{d} \frac{\sigma_i^2}{\sigma_i\sigma_j} \int_{\bar{M}_{1,1}} \lambda = \frac{-1^{r+(r+1)(d-1)} \int_{\bar{M}_{1,1}} \lambda}{d} \sum_{i\neq j} \frac{\sigma_i}{\sigma_j}$$

Now $\lambda = \frac{\delta_0}{12}$, where δ_0 is the boundary divisor of $\bar{M}_{1,1}$, since δ_0 corresponds to a weight 12 modular function while λ corresponds to a weight 1 modular function. And $\int_{\bar{M}_{1,1}} \delta_0 = 1/2$ because of the $\mathbb{Z}/2$-automorphism for genus 1 curve. Moreover $\sum_{i=0}^{r} \frac{1}{\sigma_i} = 0$, which is the equality for degree r coefficient of Lagrange interpolation for the constant function 1 at $\{\lambda_0, ..., \lambda_r\}$. Thus $\sum_{i\neq j} \frac{\sigma_i}{\sigma_j} = -(r+1)$ and we obtain the answer $-1^{(r+1)d}\frac{r+1}{24d}$.

Proof of theorem 3 We use the same equivariant setting for $O(-1)^{r+1}$, e.g. $O(-1)_k|_{p_i}$ has weight $\lambda_k - \lambda_i$. For the same reason, this equivariant setting makes the contribution of all graphs zero except for those having only one edge. For those graphs, we have a choice of two points $\{p_i, p_j\}$, such that the image of C_e is the line connecting p_i and p_j. Here C_e denotes the non-contracted component as before.

Since the vertices have genus zero, $H^1(C_v, \mathcal{O}_{C_v}) = 0$ and consequently the term $e([H^1(C, f^*O(-1)^{\oplus r+1})])$ is simply equal to $e([H^1(C_e, f^*O(-1)^{\oplus r+1})]) = \prod_{k=0}^{r} \eta_{\lambda_k-\lambda_i,\lambda_k-\lambda_j}$.

Let $\{1, ..., n\}$ denotes the set of marked points. We have $\{1, ..., n\} = N_1 \sqcup N_2$, where N_1 and N_2 denote the set of marked points mapped to p_i and p_j, respectively. We temporarily assumes $|N_1|, |N_2| \geq 2$, the corresponding moduli $M_\Gamma = \bar{M}_{0,|N_1|+1} \times \bar{M}_{0,|N_2|+1}$. In such case, $Aut(C)$ contains only one fixed part, and $e(Def(C)^m) = (\frac{\lambda_i-\lambda_j}{d} - \psi_1)(\frac{\lambda_j-\lambda_i}{d} - \psi_2)$, where ψ_1 and ψ_2 are the pull-back of the ψ class from $\bar{M}_{0,|N_1|+1}$ and $\bar{M}_{0,|N_2|+1}$, respectively, corresponding to the extra marked point.

And $H^1(C_v, f^*T\mathbb{P}^r) = 0$ as above. Also as before,

$$e([H^0(C_e, f^*\mathcal{F})]) = \tilde{\eta}_{\lambda_i-\lambda_j,\lambda_j-\lambda_i} \cdot \prod_{k\neq i,j} \tilde{\eta}_{\lambda_i-\lambda_k,\lambda_j-\lambda_k}$$

one zero that appears in the $\tilde{\eta}$ term in the above formula is to be ignored, since we are computing the moving part. Finally, if the k-th marked point is mapped to p_i, then $e_k^*(h) = \lambda_i$. Thus the contribution of the graph is

$$\frac{(-1)^{(r+1)(d-1)}}{d} \int_{\bar{M}_{0,|N_1|+1}} \frac{1}{\frac{\lambda_i-\lambda_j}{d} - \psi} \cdot \int_{\bar{M}_{0,|N_2|+1}} \frac{1}{\frac{\lambda_j-\lambda_i}{d} - \psi} \cdot \frac{\prod_{k\in N_1} \lambda_i^{l_k} \prod_{k\in N_2} \lambda_j^{l_k}}{\sigma_i\sigma_j}.$$

And we have

$$\int_{\bar{M}_{0,|N_1|+1}} \frac{1}{\frac{\lambda_i-\lambda_j}{d}-\psi} = \left(\frac{\lambda_i-\lambda_j}{d}\right)^{1-|N_1|} \int_{\bar{M}_{0,|N_1|+1}} \psi^{|N_1|-2} = \left(\frac{\lambda_i-\lambda_j}{d}\right)^{1-|N_1|}$$

Thus the contribution is

$$(6.0) \qquad (-1)^{(r+1)(d-1)+1-|N_2|} d^{n-3} (\lambda_i-\lambda_j)^{2-n} \frac{\prod_{k\in N_1} \lambda_i^{l_k} \prod_{k\in N_2} \lambda_j^{l_k}}{\sigma_i\sigma_j}$$

Now, an interesting fact is that (6) still holds when $|N_1| \leq 1$ (similar for N_2). When $|N_1| = 1$, then the integral about N_1 does not appear since the corresponding vertex must be contracted to a point. Replacing it by 1, the formula is unchanged. When $|N_1| = 0$, not only that the integral does not appear, but there is an extra term $\frac{\lambda_i-\lambda_j}{d}$ from $e(Aut(C)^m)$. Thus the result is unchanged. Thus (6) always holds.

Summing over all partition $N_1 \sqcup N_2$ and then over all pairs $\{i, j\}$, we have

$$\int_{[\bar{M}_{0,n}(\mathbb{P}^r,d)]^{vir}} e(R^1\pi_* f^*(O(-1)^{\oplus r+1})) \prod_{i=1}^{n} e_i^*(h^{l_i}) = (-1)^{(r+1)(d-1)} \sum_{0\leq i<j\leq r} -d^{n-3} \frac{\prod_{k=1}^n (\lambda_i^{l_k} - \lambda_j^{l_k})}{\sigma_i\sigma_j(\lambda_i-\lambda_j)^{n-2}}$$

Now let $p(\alpha,\beta) = \prod_{k=1}^n (\alpha^{l_k} - \beta^{l_k}) \cdot (\alpha-\beta)^{2-n}$, which is a polynomial of degree $2r$ since $l_1 + ... + l_n = 2r + n - 2$. There is a polynomial $\tilde{p}(\alpha,\beta)$, such that $\deg_\alpha \tilde{p} \leq r, \deg_\beta \tilde{p} \leq r$ and

$$\tilde{p} = p - q_1 \prod_{k=0}^{r} (\alpha-\lambda_k) - q_2 \prod_{k=0}^{r} (\beta-\lambda_k)$$

for some polynomial q_1, q_2 in α, β of degree $r-1$. We may now apply Lagrange interpolation on $\tilde{p}$, obtaining

$$\tilde{p} = \sum_{i=0}^{r}\sum_{j=0}^{r} \frac{\tilde{p}(\lambda_i,\lambda_j)\prod_{k\neq i}(\alpha-\lambda_k)\prod_{l\neq j}(\alpha-\lambda_l)}{\sigma_i\sigma_j}.$$

Moreover, one observes that $\tilde{p}(\lambda_i,\lambda_j) = p(\lambda_i,\lambda_j)$, $p(\lambda_i,\lambda_i) = 0$ and the coefficient of $\alpha_r\beta_r$ in $\tilde{p}$ is equal to that in p. Thus the coefficient of $\alpha^r\beta^r$ in $p(\alpha,\beta)$ is equal to

$$2 \sum_{0\leq i<j\leq r} \frac{\prod_{k=1}^n (\lambda_i^{l_k} - \lambda_j^{l_k})}{\sigma_i\sigma_j(\lambda_i-\lambda_j)^{n-2}}$$

and the theorem is obviously obtained.

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