Mark Gross. CTS. 11/28 2001.

Affine Manifolds and Mirror Symmetry

Def. A C-Y mfd X is a cpx mfd with $\omega_X \cong \mathcal{O}_X$.

Let S be the unit disk, $\mathcal{X} \to S$ a flat family with $\mathcal{X}_t$ being a C-Y mfd, $t \neq 0$. $\mathcal{X}_0$ may be singular.

There is a monodromy transformation

$T : H^*(\mathcal{X}_t, \mathbb{Q}) \to H^*(\mathcal{X}_t, \mathbb{Q})$

the key data to measure how singular $\mathcal{X}_0$ is.

Thm: If $n = \dim \mathcal{X}_t$, then $(T^p - I)^{n+1} = 0$ for some $p > 0$.

We can always take a $p:1$ cover $S \to S$, $t \mapsto t^p$

$$\begin{array}{ccc} \mathcal{X} & \leftarrow & \mathcal{X} \times_S S \\ \downarrow & & \downarrow \\ S & \leftarrow & S \end{array}$$

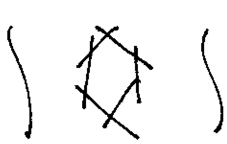
has monodromy T^p

i.e. may always assume that T is unipotent.

i.e. $(T - I)^{n+1} = 0$

Def. We say $\mathcal{X} \to S$ is maximally unipotent if $(T - I)^n \neq 0$ (but $(T - I)^{n+1} = 0$).

Examples: ① A degeneration of an elliptic curve to a cycle of n rational curves



$S^* = S - (0)$

The family of elliptic curves over S^* has periods $1, \frac{n}{2\pi i}\log t$:

$\mathcal{X}_t \cong \mathbb{C}/\langle 1, \frac{n}{2\pi i}\log t \rangle$

$T : H^1(\mathcal{X}_s, \mathbb{Z}) \circlearrowleft$ has matrix $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

(2) In $S \times \mathbb{P}^n$, take the equation

$t f_{n+1} + \prod_{i=0}^{n} x_i = 0$

f_{n+1} is a general equation of degree $n+1$.

Two Questions :

(1) Given an integrally polarized maximally unipotent degeneration $\mathcal{X} \to S$, one should be able to construct a "mirror family" of C-Y's.

(2) what is the combinatorics necessary to describe max. unipotent degenerations?

Rmk : In the elliptic curve case, the only data is the number n alone !

Hoped for answer :

structures appearing in the Strominger-Yau-Zaslow approach to mirror sym. will answer (1) and (2). But today here we do not address the SYZ thing.

(Roughly : $f : X \to B$ s-Lag fibration $\Rightarrow \check{f} : \check{X} \to B$ dual torus fibration.)

The structure on B is "exactly" the data needed ! This leads to alg-geom analogue.

Def. An affine mfd is a mfd B with coordinates charts $\psi_i : U_i \to \mathbb{R}^n$, $\{U_i\}$ open cover of B, st. $\psi_i \circ \psi_j^{-1} \in \mathrm{Aff}(\mathbb{R}^n)$.

Special cases: $\psi_i \circ \psi_j^{-1} \in \mathbb{R}^n \rtimes GL_n(\mathbb{Z})$

or $\psi_i \circ \psi_j^{-1} \in \mathrm{Aff}(\mathbb{Z}^n)$.

In the latter case, we say that the affine structure is integral. In general this is a quite rigid structure.

Examples. ① $\mathbb{R}/n\mathbb{Z} \simeq S^1$ integral affine

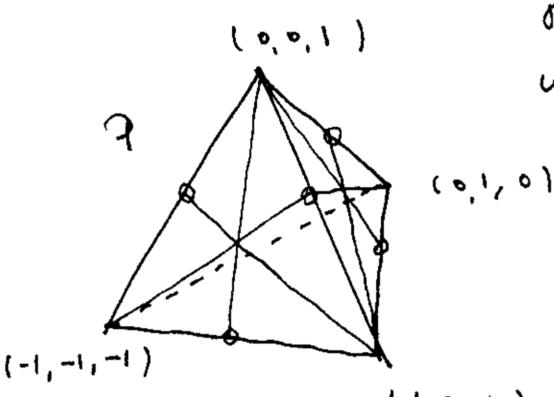
② $\mathbb{R}^n/\Lambda$, Λ a lattice, integral $\Leftrightarrow \Lambda \subseteq \mathbb{Z}^n$.

③ Let $\Xi \subseteq \mathbb{R}^{n+1}$ be a convex integral (lattice) polytope with $0 \in \mathrm{Int}(\Xi)$.

Choose a polyhedral decomposition of Ξ using only integral points as vertices.

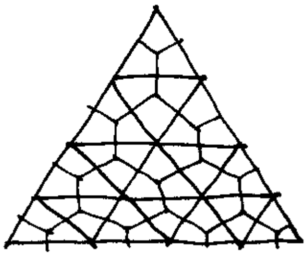
Let $B = \partial \Xi$ (top $\cong$ sphere). Will see a subset of B is an affine mfd:

Take the 1st barycentric subdivision of this decomposition: let Δ = union of simplices in the 1st barycentric subdivision which are disjoint from vertices of P and disjoint from the interior of maximal faces of P.



e.g. in the RHS case, just get $\Delta = \{o\text{'s}\}$.

eg. $\boxed{H}$ = convex hull of $(-1,-1,-1,-1)$,
$(4,-1,-1,-1)$, $(-1,4,-1,-1)$, $(-1,-1,4,-1)$, $(-1,-1,-1,4)$



: a 2-face of $\boxed{H}$
(see Monday's colloquium)

Can construct an affine str on $B_0 = B \setminus \Delta$:
For σ a max. face of P, $\psi_\sigma : \mathrm{Int}(\sigma) \hookrightarrow \mathbb{A}^n \subseteq \mathbb{R}^{n+1}$
In a suitable nbd U_v of each vertex v of P, we can define $\psi_v : U_v \longrightarrow \mathbb{R}^{n+1}/\mathbb{R}v$ by projection. ($U_v \cap U_w = \phi$, $\{U_\sigma, U_v\}$ form an open cover of B_0)
This defines an affine str on B_0. integral if $\boxed{H}$ was a reflexive polytope (in the sense of Batyrev).

Def. Let B be an affine mfd with transition maps in $\mathbb{R}^n \rtimes GL_n(\mathbb{Z})$. Then if $y_1, \cdots, y_n$ are local affine coordinates,
$\frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_n}$ generate a lattice Λ in T_B.
Set $X(B) := T_B/\Lambda$.

$X(B) \to B$ is a T^n bundle. Also $X(B)$ comes with a cpx structure:
if $y_1, \cdots, y_n$, $x_1, \cdots, x_n$ coor on T_B

$$\leftrightarrow \quad \sum x_i \frac{\partial}{\partial y_i} \text{ at } (y_1, \cdots, y_n).$$

Set $z_j = e^{2\pi i (x_j + i y_j)}$,

$z_1, \dots, z_n$ give cpx coordinates on $X(B)$.

(check: change holomorphically).

This defines a cpx str on $X(B)$.

There is also a dual construction:

$dy_1, \dots, dy_n$ generate a lattice $\check{\Lambda}$ of T_B^*,

set $\check{X}(B) := T_B^* / \check{\Lambda}$.

$\check{X}(B)$ has a (canonical) symplectic structure induced by $\check{\omega} = \sum_i dx_i \wedge dy_i$ on T_B^*.

$$
\begin{tikzcd}
\text{complex} & X(B) \arrow[d] \arrow[r, leftrightarrow, "\text{mirror}"'] & \check{X}(B) \arrow[d] & \text{symplectic} \\
& B & B &
\end{tikzcd}
$$

this is the simplest statement in SYZ, we use only part of the data.

Importance of integral affine mfds:

"Prop": $[\check{\omega}] \in H^2(\check{X}(B), \mathbb{Z}) \Leftrightarrow$ the affine structure on B (up to translation) is integral.

(This is on the symplectic side.)

Analogue idea in cpx side: (motivated by Kontsevich and Soibelman).

B integral $\Rightarrow$ in the rigid analytic category.

$X(B)$ is a fiber of a degenerating family.

Let B be any affine mfd, let $B' = B \times \mathbb{R}_{>0}$, with affine str. as follows: if $\psi : U \to \mathbb{R}^n$ is an affine chart for B, $\psi' : U \times \mathbb{R}_{>0} \to \mathbb{R}^{n+1}$ by $\psi'(b,r) = (r\psi(b), r)$. (use the easy trick $\mathrm{Aff}(N) \subset GL(N+1)$ by putting in ht $=1$ slice.)

If B is integral, then B' has transition maps in $GL_{n+1}(\mathbb{Z})$.

Then $X(B')$ is an $(n+1)$-dim'l cpx mfd. Also there is natural projection $B' \to \mathbb{R}_{>0}$ which induces $X(B') \to X(\mathbb{R}_{>0}) = S^*$ punctured disk.

$X(B)$ is a fiber of this map.

Example: $B = \mathbb{R}/n\mathbb{Z}$,

$$X(B') \longrightarrow S^*$$

$X(B)$: elliptic curve

$X(B')$: family of elliptic curves with periods

$$1, \frac{n}{2\pi i}\log t. \quad \text{(Exercise)}$$

Morally speaking, $X(B') \to S^*$ is a maximally unipotent degeneration.

Def'n: An integrally affine mfd with singularities is a mfd B with a discriminant locus $\Delta \subseteq B$ of codim ≥ 2 and an integral affine str on $B_0 := B \setminus \Delta$.

Examples: $B_0 \subseteq B = \partial$ tetrahedron

Thm. $X(B_0)$ has a top. compactification to a K3 surface, and $\check{X}(B_0)$ a symp. compactification to a K3!

2nd example:

$B_0 \subseteq B = \partial$ (convex hull of $(-1,-1,-1,-1)$ ...)

$X(B_0)$ has a top. compactification to the mirror quintic.

$\check{X}(B_0)$ has a top (symplectic?) compactification to the quintic.

Basic Question: Given $B_0 \subseteq B$ with singularities how to construct a cpx mfd $X(B)$?

Idea (Work in progress with Bernd Siebert):

we want to use B to construct a family $\mathcal{X} \to S$ with fibers being (deformations of) the looked-for $X(B)$

we will do this by constructing "$\mathcal{X}_0$" and then smoothing.

(This clearly requires additional data due to birational models problem)

Def. Let B be an affine mfd with singularities. A polyhedral decomposition of B is a cell-decomp of B st. each max. cell σ is given by a map $\nu_\sigma : \underset{\mathbb{R}^n}{\overset{\cap}{\sigma'}} \longrightarrow B$ where σ' is a lattice polytope in $\mathbb{R}^n$.

$\nu_\sigma|_{Int(\sigma')} : Int(\sigma') \longrightarrow B_0 \subseteq B$ is integrally affine linear.

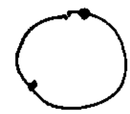
Also, faces of σ' maps to faces of σ, and

$$\begin{array}{ccc} \sigma' \supseteq w' & w'' \subseteq \tau' \\ \nu_\sigma \downarrow & \downarrow \nu_\tau \\ \sigma \leftarrow w & \rightarrow \tau \end{array}$$

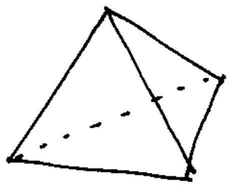
$\nu_\sigma^{-1} \circ \nu_\tau : w'' \longrightarrow w'$ int. affine.

ie. glue only along int.

Example: ① $\mathbb{R}/\mathbb{Z}$,

$[n, n+1] \longrightarrow$  , $\mathbb{R}/3\mathbb{Z}$  , 

remove

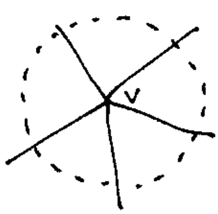
②  (exercise).

We will need for the moment are additional restriction on polyhedral decompositions:
if $\dim B = 2$, all vertices are contained in B_0,
if $\dim B > 2$, slightly more complicated.

Here is the construction:
Given such B with polyhedral decomp. $\mathcal{P}$
construct scheme

$$X_0(B, \mathcal{P})$$

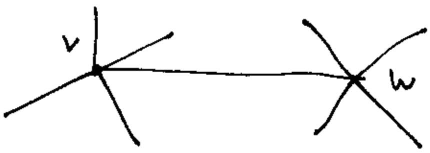
For each vertex v of $\mathcal{P}$



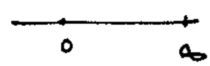
$\subseteq \mathcal{T}_{B,v}$, $\mathcal{P}$ looks like a fan Σ_v near v. (in toric geom sense)
Also, $\mathcal{T}_{B,v}$ has an integral structure
$\Lambda_v \subseteq \mathcal{T}_{B,v}$

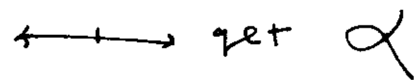
so Σ_v can be thought of as a rational polyhedral fan, defining a toric variety X_v.
(in fact the condition put in $\dim B = 2$ case is to get toric var, if remove this condition, can still do, but get not toric var)

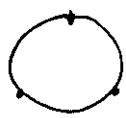
If v and w are joined by an edge, eg.



$\langle v, w \rangle$ define rays in the fans Σ_v, Σ_w
defines toric divisors $D_{v,w} \subseteq X_v$, $D_{w,v} \subseteq X_w$.
which are isomorphic (this is exactly equiv. to the condition to be put to get this!)
Glue X_v and X_w along these divisors.
If we glue all the X_v's along such divisors, we get a scheme $X_0(B, \mathcal{P})$.

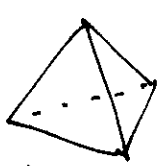
examples: ① $\mathbb{R}/\mathbb{Z}$ $\quad \mathbb{P}^1$ 

$[n, n+1] \to$  get 

$\mathbb{R}/3\mathbb{Z}$,  get ,  get 

(get normal crossing varieties).

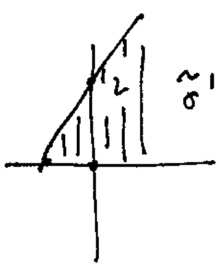
Smoothing on Normal Crossing Varieties:
Kawamata - Namikawa: log-deformation theory

② , $X_0(B, \mathcal{P}) = \{ x_0 x_1 x_2 x_3 = 0 \} \subseteq \mathbb{P}^3$

in dim $B = 1$ case, no difficulty in this smoothing theory. but start with dim $B \geq 2$, the deformation space is very singular (eg. M. Gross)
Basic idea: only allow certain sorts of local deformations.

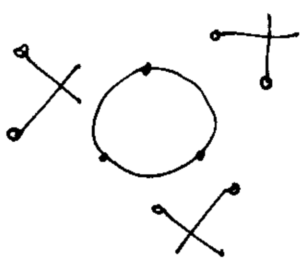
eg. locally X_0 looks like $\prod x_i = 0$,
only deform like $\prod x_i = \epsilon$.
K+N proves: log-deformation space of s.n.c C-Y is smooth.

But K-N's result is not good enough !
since we may also get other thing : (no simple)
Schroer + Siebert : Toroidal crossing varieties
the idea is, locally $X_0(B,P)$ is defined by
a monomial equation $z^m = 0$ in an affine
toric variety. we only allow deformation of the
form $z^m = t$.
These affine toric var comes from faces of P,
$\sigma \in P$ maximal, $\sigma' \subseteq \mathbb{R}^n$, (maximal)
$\tilde{\sigma}' = \{(r, ry) \mid r \geq 0,\ y \in \sigma'\} \subseteq \mathbb{R} \oplus \mathbb{R}^n$



$\partial X(\sigma) :=$ The union of smaller dim'l
torus orbits of the corresponding
affine toric var $X(\sigma)$.
$\{\partial X(\sigma)\}$ form an (étale) open
covering of $X_0(B,P)$.

$xy=0$



We deform locally
$\partial X(\sigma)$ inside $X(\sigma)$.

Everything works well for K3, but a lot need to
be done for dim $B \geq 3$. hopefully at end of 2001?

MIRROR SYMMETRY:

$$\begin{array}{ll} \mathcal{X} & \text{All components of } \mathcal{X}_0 \text{ toric} \\ \downarrow & \quad \mathcal{X} \text{ has toric sing. "generically".} \\ \Delta & \quad \text{(this includes all Batyrev examples)} \end{array}$$

$\rightsquigarrow B, \mathcal{P}$, $H \rightsquigarrow$ multi-valued pl fcn on B,
discrete Legendre transform $\check{B}, \check{\mathcal{P}} \rightsquigarrow \check{X} \to \Delta$.
this procedure will get rid of diff geom
and hopefully get alg. geom. version of SYZ.