Y.P. Lee Gross-Siebert program
lect 1, 6/17, 2013 at TIMS

overview: Greene - Plesser
studied Fermat quintic 3-fold X and its mirror partner (orbifold) $\check{X}$, st Hodge diamond of X & X'

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{22} & & 0 & \\ 1 & & h^{21} & & h^{12} & & 1 \\ & 0 & & h^{11} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

2 non-trivial entries

$h^{1,1}(X) = h^{2,1}(\check{X}) = 1$

$h^{1,2}(X) = h^{1,1}(\check{X}) = 101$

$\check{X} = [X/G]$

$\bar{G} = \{(a_0, \cdots, a_4) \in (\mathbb{Z}/5)^5,\ \prod a_i = 1\} / \{(a, \ldots a)\}$

SYZ conjecture: (1996)
$\exists\ X \to B$ st. fibers $\simeq (S^1)^3$
real 3-dim "mfd"
$\Rightarrow \check{X} \to B$ with fibers $\simeq$ dual torus

Problems: ① No rigorous def'n, no "very non-trivial" examples so far, hence no "proof"
② even if ① is accomplished, still a long way from SYZ to finer str of MS. eg. Candelas formula.

Take this as an inspiration
since fibers $\simeq (S^1)^3$, study the base: what properties B is supposed to have?
what can be gained from these?

$\exists$ ① B = "affine mfd" transition functions $\in Aff(\mathbb{R}^3) = \mathbb{R}^3 \rtimes GL_3(\mathbb{R})$

$$\text{G-S} \downarrow$$

tropical mfd
with singularities

Very roughly, $\Delta \subset B$ sing/disc loci in dim ≥ 2

$B \setminus \Delta$ = mfd with transition fun $\in \mathbb{R}^3 \rtimes GL_3(\mathbb{Z})$

why? B = trop mfd w/o sing, locally on a chart ($\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}$ well-defined up to $GL_3(\mathbb{Z})$)

$\Lambda \subseteq T_B$ 6-dim'l mfd, $T_B/\Lambda \to B$ $(S^1)^3$-fibration

on the other hand, dy_1, dy_2, dy_3 $\check{\Lambda} \subseteq T^*B$

$\check{X}(B) = T^*B/\check{\Lambda} \to B$, $(S^1)^3$-fibration, canonical symplectic str

- These 2 fibrations are canonically dual
- $\check{X}(B)$ can sym str.
  $X(B)$ can complex str. } mirror pair

But in Kähler case the only example $= \mathbb{R}^n/\Gamma$ (Gross)
Hence we must allow singularities.

Kontsevich ($\sim$2000) holo curve in $X(B)$
$\longleftrightarrow$ tropical curve in B
real 1-dim obj — real 3-dim (w/ sing)
= graphs with vertexes, edges.

Mikhalkin (2002) proved it for 2-dim case for toric surfaces.

- Gross-Siebert program

$$\begin{array}{ccccc} \text{A-model} & \leftrightarrow & \text{Log geometry} & \leftrightarrow & \text{B-model} \\ & \searrow & \updownarrow & \swarrow & \\ & & \text{tropical geometry} & & \end{array}$$

On both A & B sides: 3 worlds

tropical world $\leftrightarrow$ log world $\leftrightarrow$ classical world (or holo world)

Very roughly, $\mathcal{X}$ $\quad \mathcal{X}_t$ cpx mfd
$\downarrow$ $\quad \mathcal{X}_0$ = SNC in $\mathcal{X}$
A^1

expect $GW(\mathcal{X}_t) = \text{Log } GW(\mathcal{X}_0) \longleftrightarrow$ tropical count
degeneration formula

Tropical Geometry.

$\mathbb{R}^{trop} = (\mathbb{R}, \oplus, \odot)$ tropical semi-ring

$a \oplus b = \min(a,b)$
$a \odot b = a+b$ } will omit $\odot$

affine tropical geom. $A = \mathbb{R}^{trop}[x_1, \cdots, x_n] \ni f$

trop fcn

$$f = \sum_{(i_1, \cdots, i_n) \in S} a_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$$

$$= \min_{(i_1 \cdots i_n) \in S} \left(a_{i_1 \cdots i_n} + i_1 x_1 + \cdots + i_n x_n \right)$$

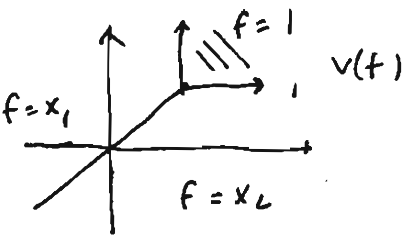
① Σ: $0x$ can not be removed since $0 \cdot x = 0 + x$

② all fcns are piece-wise (affine) linear

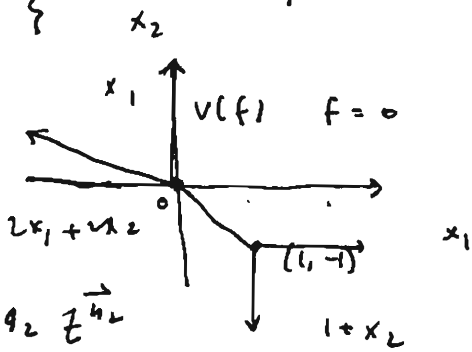
As an exercise, tropical hypersurface $V(f) \subset \mathbb{R}^n$
= the locus where f is not linear.

Examples ① $f = 1 + 0x_1 + 0x_2$ $(n=2)$,
$= \min\{1, x_1, x_2\}$

(again $0 \cdot x_1$ can't be removed!)



② $f = 0 + 0x_1 + 1x_1 + 0x_1^2 x_2^2$
$= \min\{0, x_1, 1+x_1, 2x_1 + 2x_2\}$



Define weights of edges (for $n=2$)
(in general for codim 1)

An edge e separates 2 regions
defined by $f = a_1 z^{\vec{n}_1}$ & $f = a_2 z^{\vec{n}_2}$

$w(e) :=$ index $(\vec{n}_1 - \vec{n}_2)$ via index $(2,2) = \underline{2}$ $(1,1)$

i.e. multiple of primitive vector

o more convenient to use toric notations:

$M = \mathbb{Z}^2$, $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$

$f \in A$ can be considered as $f: M_{\mathbb{R}} \to \mathbb{R}$

$S \subseteq N$ a finite set, $f = f(z) = \sum_{n \in S} a_n z^n$

1.4 Discrete Legendre transformation

for now, it's a good way to draw "$V(f)$"

$\Delta_S :=$ Newton polyhedron of S (now polytope since S is finite)

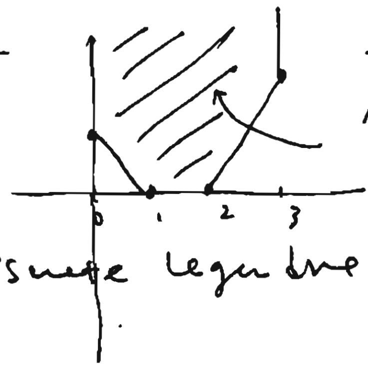
$\hat{S} := \{ (n, a_n) \in S \times \mathbb{R} \mid a_n z^n$ is a monomial in $f \}$

$\tilde{\Delta}_S \subset \tilde{S} \times \mathbb{R}$ upper convex hull of $\hat{S}$

$\varphi(n) := \min \{ a \in \mathbb{R} \mid (n, a) \in \tilde{\Delta}_S \}$

eg $n = 1$ $\quad f = 1 + 0x + 0 \cdot x^2 + 2 \cdot x^3$, $\quad S = \{0, 1, 2, 3\}$

$\Delta_S =$ 

$\mathbb{R}^2$ 

the lower boundary = graph of $\varphi(n)$

$\tilde{\Delta}_S$ gives a polyhedral decomposition

$N_\mathbb{R}$

discrete Legendre transformation of $(\check{\Delta}_S, \mathcal{P}, \varphi)$

polyhedral decomposition

$\tilde{\Delta}_S$

decompose wrt where it bends.

together with all faces.

$\rightsquigarrow (M_\mathbb{R}, \check{\mathcal{P}}, \check{\varphi})$

just dual space $\quad \check{\mathcal{P}}, \check{\varphi}$ defined next time.

Discrete Legendre transf

Input $(\Delta_S, \mathcal{P}, \varphi)$ — convex PL

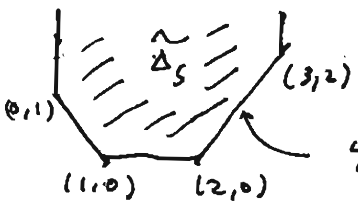
polyhedron $\subset N_{\mathbb{R}}$; polyhedral decomp

Output $(M_{\mathbb{R}}, \check{\mathcal{P}}, \check{\varphi})$

$\check{\mathcal{P}} := \{\check{\tau} \mid \tau \in \mathcal{P}\}$, $\check{\tau} := \{m \in M_{\mathbb{R}} \mid \forall n \in \Delta_S,\ \langle -m, n\rangle + a \le \varphi(n)$, $\exists a$ $\forall n \in \tau,\ \langle -m, n\rangle + a = \varphi(n)\}$

$\check{\varphi}(m) := \max\{a \mid \langle -m, n\rangle + a \le \varphi(n),\ \forall n \in \Delta_S\}$

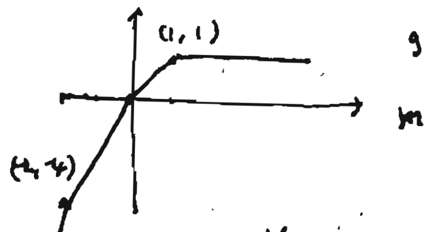
in example.



$\Delta_S = \bigcup_{\sigma \in \mathcal{P}} \sigma$

eg for $\tau = [0,1] \Rightarrow \check{\tau} = \{1\}$

$\tau = \{0\} \Rightarrow \check{\tau} = [1, \infty)$



graph of $\hat{\varphi}$

Rmk: May find $\check{\varphi}$ first, then get $\check{\mathcal{P}}$

Rmk: Usual Legendre transform: $f(x)$ convex diff fcn

$$\check{f}(p) := \sup_x (xp - f(x))$$

M. Gross use "+"

Sup happens: $\frac{\partial}{\partial x}(px - f(x)) = 0$ i.e. $p = f'(x)$

$\check{f}(f'(x)) = x f'(x) - f(x)$ i.e. $xp = \check{f} + f$

(the usual sign is more symmetric).

$V(f) = \bigcup_{\tau \in \mathcal{P}^{[1]}} \check{\tau}$; $\mathcal{P}^{[k]}$ the subset of k-cells

Recall: GWT: virtual counting of holo curves in n-folds (cpx)

$\longleftrightarrow$ (virtual) counting of tropical curves in $n_{\mathbb{R}}$-folds (eg. here $\mathbb{R}^2$)

p. 2 M. Gross' Setup :

$\bar{\Gamma}$ = connected graph w/o bi-valent vertices allowing univalent vertex eg

$\bar{\Gamma}^{(0)}_{\infty}$ = the set of all univalent vertices

$\bar{\Gamma}^{(0)}$ = all vertices

$\bar{\Gamma}^{[1]}$ = all edges , $\bar{\Gamma}^{[1]}_{\infty}$ etc.

Conventions All graphs are weighted

$w : \bar{\Gamma}^{[1]} \to \mathbb{N}_{\geq 0}$

A marked graph $(\Gamma, x_1, \cdots, x_k)$ $\{x_1, \cdots, x_k\} \hookrightarrow \Gamma^{[1]}_{\infty}$

$w(E_{x_i}) = 0 \iff E_{x_i}$ = a marked edge $x_i \mapsto E_{x_i}$.

(usually called a tail in GW th.)

Def'n. A marked parametrized trop curve (MPTC)

$h : (\Gamma, x_1, \cdots, x_k) \to M_{\mathbb{R}} \cong \mathbb{R}^2$ is a conti map

(1) $h|_{E_{x_i}}$ = const , otherwise for E not an marked edge

$h|_E$ = proper embedding of E into a line in $M_{\mathbb{R}}$ with rat'l slope

(2) The balancing condition :

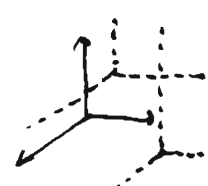
$\forall v \in \Gamma^{[0]}$, let $[E_1, \cdots, E_\ell]$ edges adjacent to v ,

$m_i \in M$ are the primitive tangent vectors to $h(E_i)$, pointing away from $h(v)$

$$\sum_{i=1}^{\ell} w(E_i)\, m_i = 0$$

Rmk : For $V(f) \subset \mathbb{R}^2$ with wt defined in lect 1 , then the balancing condi is satisfied (Exercise)

Fix a fan $\Sigma \subset M_{\mathbb{R}}$ eg



T_Σ := free ab gp gen by $\Sigma^{[1]}$

$\forall \rho \in \Sigma^{[1]}$, $T_\Sigma \xrightarrow{r} M$, $t_\rho \mapsto \rho$

Def. h = MPTC in X_Σ in X_Σ if all unmarked $E \in \Gamma^{[1]}$

$h(E)$ = a translate of $\rho \in \Sigma^{[1]}$

degree of h

$= \Delta(h) := \sum_{\rho \in \Sigma^{[1]}} d_\rho t_\rho \in T_\Sigma$

where $d_\rho = \#$ of unbounded edges $// \rho$

$|\Delta(h)| = \sum d_\rho$.

Exercise: Use balancing condi to show $r(\Delta(h)) = 0$

($\Rightarrow \Delta(h) \in H_2(X)$)

Def'n ① Call 2 MPTC's h, h' equivalent if $\exists$ homeo

$$\begin{tikzcd} \Gamma \arrow[rr, "\sim"] \arrow[rr, "\varphi"'] \arrow[dr, "h"'] & & \Gamma' \arrow[dl, "h'"] \\ & M_{\mathbb{R}} \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

② MTC = an equivalence class of MPTCs.

Rmk "Tropical curve in $n=2$", rather the image $h(\Gamma)$ can always be realized as tropical hypersurfaces

Def'n. $h : \Gamma \to M_{\mathbb{R}} \simeq \mathbb{R}^2$ is called simple if

- Γ is tri-valent
- $h|_{\Gamma^{[0]}}$ is injective
-  Assume $h|_{E_i} \not\equiv 0$, $i=1,2$, $\Rightarrow h(E_1) \not\subset h(E_2)$
- every unbdd unmarked edge E $wt = 1$

Rmk. A general tropical curve is simple

$h : (\Gamma, x_1, \dots, x_n) \to M_{\mathbb{R}} \simeq \mathbb{R}^2$

Def'n: Number of tropical curves

suppose $h : \Gamma \to M_{\mathbb{R}}$ simple

- $\forall v \in \Gamma^{[0]}$ with adj edges E_1, E_2, E_3

$$\mathrm{mult}_v(h) = \begin{cases} 1 & \text{if } E_i \text{ marked for some } i \\ |w(E_1) m_1 \wedge w(E_2) m_2| & \end{cases}$$

m_1 = prim tang vect along $h(E_1)$ pointing out of $h(v)$

S_3 sym due to balancing condi

$w(E_1) m_1 + w(E_2) m_2 + w(E_3) m_3 = 0$

$\mathrm{mult}(h) = \prod_{v \in \Gamma^{[0]}} \mathrm{mult}_v(h)$,

P. 4

$$N^{0,\,trop}_{\Delta,\Sigma} := \sum_h \text{mult}(h)$$

where h (simple) runs over $h \in M_{0,|\Delta|-1}(\Sigma,\Delta)$ = "moduli" of $g=0$ trop curve w/ $|\Delta|-1$ general pts in $M_{\mathbb{R}}$ in X_Σ of deg $=\Delta$. $\Delta(h)=\Delta$

$$\dim_{\mathbb{R}} M_{0,s}(\Sigma,\Delta\,;P_1,\dots,P_s) = |\Delta| + s - 1 - 2(s) \underset{\substack{\uparrow \\ \text{want}}}{=} 0$$

$\Rightarrow s = |\Delta| - 1$

• Mikhalkin's Theorem:

$$N^{0,\,trop}_{\Delta,\Sigma} = N^{0,\,holo}_{\Delta,\Sigma} \underset{\substack{\uparrow \\ \text{in general} \\ \text{have 2} \\ \text{possible def'n}}}{=} \left\{ \begin{array}{l} N^{GW} \\ N^{enum} \end{array} \right.$$

N^{enum} ← Now

in $\mathbb{P}^2$ case these 2 agree but in general NOT

$X = X_\Sigma$ toric surface

$\Delta \in H_2(X_\Sigma)$.

eg. $M_{\mathbb{R}} = \mathbb{R}^2$

$\Delta = d_1 t^{e_1} + d_2 t^{e_2} + d_3 t^{e_3}$

$|\Delta| = d_1 + d_2 + d_3$

Def'n: $Q_1, \dots, Q_{|\Delta|-1}$ general pts in $X_\Sigma \cong \mathbb{P}^2$, $C \subseteq X_\Sigma$ is called toroidally transverse if it's disjoint from all toric strata of codim ≥ 2 ($\Rightarrow C \not\subseteq$ toric div)

Say, $f: C \to X_\Sigma$ tor. transv if $f(C) \subset X_\Sigma$ is torically transv. + no irred comp of C maps to toric bdy $\partial X_\Sigma = \bigcup$ toric div's.

$$N^{0,\,holo}_{\Delta,\Sigma} := \#\{ f \in \overline{M}_{0,|\Delta|-1}(X_\Sigma,\Delta) \mid f(x_i) = Q_i, \text{ \& } f \text{ is toric transverse} \}$$

$(= N^{enum})$

sketch of the pf (based M. Gross' interpretation, YP: not be able to understand Mikhalkin's original pf.)

Will construct 1-parameter degen ⓪

$\mathcal{X} \to \mathbb{A}^1$, $\circlearrowleft Q_1 \dots Q_{s_0}$ st $X_t = \mathcal{X}_{t\neq 0} \simeq X_\Sigma$ (sm)

· $\mathcal{X}_0 = X_0 \subset \mathcal{X}$ div SNC

log sm, but not sm

idea: holo count for $X_\Sigma = X_t$

① $\|$ ⟵ degeneration formula

log hol count for X_0

② $\|$ more than simple double pt

tropical count

NB "deg. formula" is non-existent for genl log, but M. Gross proves a enumerical version of it, without the need of virtual classes, so can prove by bare hand (So will not explain this part)

Will indeed explain ②, and ⓪

Def^{n}: $P_1, \dots, P_s \in M_\mathbb{Q} \simeq \mathbb{Q}^2$ (refine lattice will not change the count.)

$\uparrow M$

A finite polyhedral decomp $\mathcal{P}$ of $M_\mathbb{R}$ is said to be good if

(1) $\forall \sigma \in \mathcal{P}$, σ has vertices in $M_\mathbb{Q}$ & faces of rat'l slope; σ has at least one vertex

(2) $\forall \sigma \in \mathcal{P}$, $\mathrm{Asymp}(\sigma) =$ asymp cone $\in \Sigma$

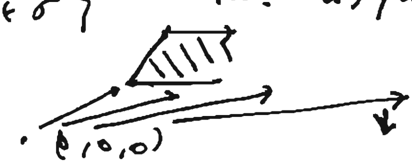
Moreover, $\forall \tau \in \Sigma$ appears as $\mathrm{asymp}(\sigma)$ for some $\sigma \in \mathcal{P}$

$\mathrm{Asym}(\sigma) := \overline{C(\sigma)} \cap (M_\mathbb{R} \oplus \{0\}) \subset M_\mathbb{R}$

$\overline{C(\sigma)} \subset M_\mathbb{R} \oplus \mathbb{R}$; so only unbound part survives

$C(\sigma) = \{(rm, r) \mid r \geq 0, m \in \sigma\}$ i.e. asymptotic cone

$\subset M_\mathbb{R} \oplus \mathbb{R}$



→ the direct survives in closure

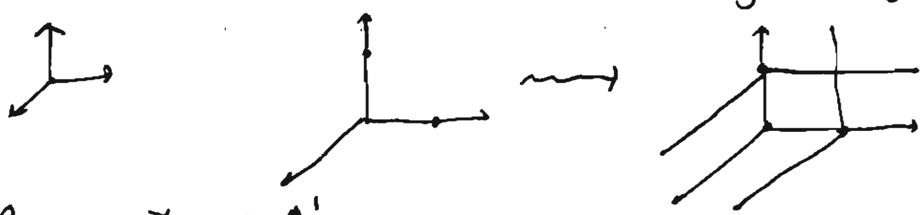
p. 2 (3) $\forall\, h \in M^{trop}_{0,s}(\Sigma, \Delta, P_1, \cdots, P_s)$, $h(\Gamma)$ lies in the 1-dim'l skeleton of $\mathcal{P}$.

(4) $P_i \in \mathcal{P}[0]$

Prop $\exists$ of good polyhedral decomp

eg $\Sigma = \Sigma_{\mathbb{P}^2}$, $|\Delta| = 3$, $s = 2$

good $\mathcal{P}$



$$\left.\begin{array}{l} \mathcal{P} \rightsquigarrow * \to \mathbb{A}^1 \\ \sigma \in \mathcal{P} \mapsto C(\sigma) \end{array}\right\} \mapsto \Sigma_{\mathcal{P}} = \{C(\sigma)\}.$$

claim (Exercise)

① $\Sigma = \{\tau \in \Sigma_{\mathcal{P}} \mid \tau \subseteq M_{\mathbb{R}} \oplus \{0\}\}$

② $X_{\Sigma_{\mathcal{P}}} \longrightarrow \mathbb{A}^1$ induced by the projection $M \oplus \mathbb{N} \longrightarrow \mathbb{N}$.

③ $\pi^{-1}(0) = X_0 = \bigcup_{v \in \mathcal{P}[0]} D_{C(v)}$

④ $X_{\Sigma_{\mathcal{P}}} \setminus X_0 \cong X_\Sigma \times \mathbb{G}_m$.

Rmk: This decomp/degeneration will force the count in log GW level only has mult $= 1$ which is NOT true for holo count or tropical count

Gross-Siebert's program on $\mathbb{P}^2$

Counting curves in 3 worlds.

Tropical world — # of graphs in $M_\mathbb{R} \cong \mathbb{R}^2$ w/ s pt constraints

$\updownarrow$ Thm: Mikhalkin: $\|$ worked for toric surface

log world

$\updownarrow$

holo world — # of $\mathbb{P}^1$ to $\mathbb{P}^2$ with s pt constraints $s = |\Delta| - 1$

GS via degeneration formula

if use enumerative invariants

Last time: start with $\Sigma = \Sigma_{\mathbb{P}^2}$ toric fan (for $\mathbb{P}^2$)

$\rightsquigarrow$ $\mathcal{P}$ = a good polyhedral decomp $\subset \mathbb{R}^2 \cong M_\mathbb{R}$

$\rightsquigarrow$ $\Sigma_\mathcal{P}$ = a fan in $\tilde{M}_\mathbb{R} \cong \mathbb{R}^3$

$\rightsquigarrow$ $X_{\Sigma_\mathcal{P}} \xrightarrow{\pi} \mathbb{A}^1$ with the properties:

1). $\Sigma = \{\tau \in \Sigma_\mathcal{P} \mid \tau \subseteq M_\mathbb{R} \oplus \{0\}\}$

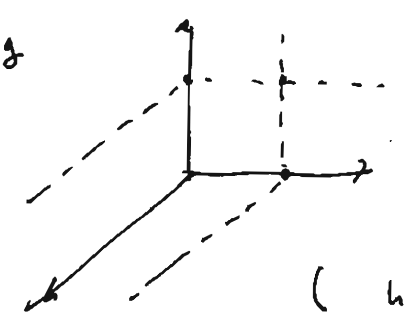
2) $X_{\Sigma_\mathcal{P}} \xrightarrow{\pi} \mathbb{A}^1$ given by proj $p_2 : M \oplus \mathbb{N} \to \mathbb{N}$

3) $\underset{X_0}{\pi^{-1}(0)} = \bigcup_{v \in \mathcal{P}^{[0]}} D_{c(v)}$

4) $X_{\Sigma_\mathcal{P}} \setminus X_0 \cong \underset{\mathbb{P}^2}{X_\Sigma} \times \mathbb{G}_m$

5) the "points" $Q_1, \dots, Q_s$ on $X_\Sigma \cong$ general fiber $X_{t \neq 0}$ extends to X_0.

e.g.



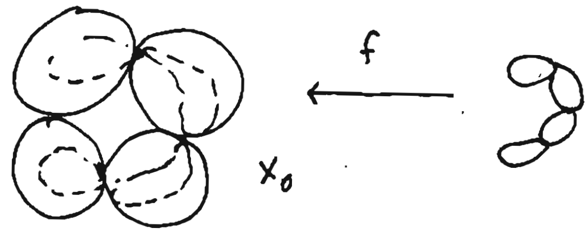
Log $\Rightarrow$ tropical

$$\begin{array}{ccc} C^\dagger & \xrightarrow{f} & X_0^\dagger \\ \downarrow & & \downarrow \\ \operatorname{Spec} k^\dagger & = & \operatorname{Spec} k^\dagger \end{array}$$

$\leftarrow$ standard log pt

(here is easier than Steffen's lecture just need some combinatorial game)

want: f to be tropically transverse + other conditions p 2



construct trop curve

$h : (\Gamma, x_1, \cdots, x_s) \longrightarrow (M_{\mathbb{R}}, P_1, \cdots, P_s)$

for simplicity we omit the discussion on pt constraints

$\{\text{vertices}\} = \{\text{irreducible comp of } C\}/\sim$

$C_i \sim C_j$ if $\exists$ a chain of $C_i, C_{i+1}, \ldots, C_j$ st $C_k \cap C_{k+1} = \text{a pt} =: \{p_k\}$

$f(p_k) \notin \operatorname{Sing} C_0$ (eg. above 4 vertices)

- bdd edges $\longleftrightarrow$ nodes of C mapping to $\operatorname{Sing}(X_0)$
- unbdd edges $\longleftrightarrow$ $f^{-1}(\partial X_0)$: finite number of pts $\;\nwarrow \overline{(\partial X_{\Sigma_p} \setminus X)} \cap X_0$
- weights etc

Thm the converse can also be done

though more difficult (since tropical have multiplicities, but log world does not)

Now we move on to other parts of G-S problem on $\mathbb{P}^2$

B model & mirror for toric varieties

$\Sigma \subset M_{\mathbb{R}}$ a complete fan, X_Σ = complete ns toric var.

$0 \to K_\Sigma \to T_\Sigma \xrightarrow{r} M \to 0$, dualize

$0 \to N \to T_\Sigma^\vee \to \operatorname{Pic}(X_\Sigma) \to 0$

$\otimes_{\mathbb{Z}} \mathbb{C}^\times$ gets $\quad 0 \to N \otimes \mathbb{C}^\times \to \operatorname{Hom}(T_\Sigma, \mathbb{C}^\times) \xrightarrow{\kappa} \operatorname{Pic} X_\Sigma \otimes \mathbb{C}^\times \to 0$

$\mathcal{M}_\Sigma = \operatorname{Spec} \mathbb{C}[K_\Sigma]$ $\qquad \operatorname{Pic} X_\Sigma \otimes \mathbb{C}^\times = \mathcal{M}_\Sigma$ (complexified) Kähler moduli fiber of $K = N \otimes \mathbb{C}^\times = \mathrm{Spec}\, \mathbb{C}[M]$ cpx torus

$\tilde{M}_\Sigma$ = universal cover (induced from $\mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times$)

Mirror family $\check{X}_\Sigma := \mathrm{Hom}(T_\Sigma, \mathbb{C}^\times) \times_{M_\Sigma} \tilde{M}_\Sigma$

defined by

Landau - Ginzburg potential (M. Gross) : (primitive version)

$$W_0 := \sum_{\rho \in \Sigma^{[1]}} z^{t_\rho}$$

In sophisticated version, have to thicken this

Our way to formulate mirror sym's consequence

J-fcn for $\mathbb{P}^2$ (A-model)

= generating fcn of $\underline{\text{oscillating integrals}}$ for mirror (LG)

periods integral

[Gross] define tropical descendents

$J^{\mathrm{trop}}_{\mathbb{P}^2}$ So far only ad hoc definition of 4 classes, NO good geom def'n

Example: $\Sigma = \Sigma_{\mathbb{P}^2}$

$T_\Sigma \simeq \mathbb{Z}^3$ with basis t_0, t_1, t_2

corresp to ρ_0, ρ_1, ρ_2, $x_i := z^{t_i}$

$M_\Sigma \simeq \mathbb{C}^\times$, $\tilde{M}_\Sigma \simeq \mathbb{C}$

Mirror family is simple:

$$\begin{array}{c} \check{X}_\Sigma \subset \mathbb{C}^3 \times \tilde{M}_\Sigma \\ \kappa \downarrow \\ \mathrm{Spec}\, \mathbb{C}[y_1] = \mathbb{C} \simeq \tilde{M}_\Sigma \end{array}$$

κ = projection

defined by $x_0 x_1 x_2 = e^{y_1}$

and $W_0 = x_0 + x_1 + x_2$

(But in general only have "formal family" so actually split just a function as in LG model)

So far,

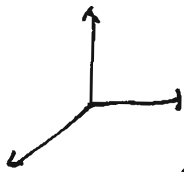
	A model		B model
	3 worlds		"2 worlds"
	$\boxed{\text{trop}}$		$\boxed{\text{trop}}$
	$\updownarrow$	not so sym!	$\Vert$
	$\boxed{\text{Log}}$		
	$\updownarrow$		
	$\boxed{\text{holo}}$		$\boxed{\text{hol = integrals}}$

.4 Baby version of Gross-Siebert program for $\mathbb{P}^2$. Data $(B, \mathscr{P}, \varphi)$ ← strictly convex fcn

tropical mfd with singularity $\updownarrow$ base of SYZ fibration

some sort of <u>good</u> polyhedral decomp $B = \bigcup_{\sigma \in \mathscr{P}} \sigma$

For $\mathbb{P}^2$, $B = M_{\mathbb{R}} \simeq \mathbb{R}^2$ (no sing at all)

$\mathscr{P}$ = 

$\varphi_{\mathbb{P}^2}$ = strictly convex PL fcn associated to $\mathcal{O}_{\mathbb{P}^2}(1)$.

Given this data

G-S gives 2 different constructions.

① Fan picture. $(\Sigma, \mathscr{P}) \rightsquigarrow \Sigma_{\mathscr{P}} \subset M_{\mathbb{R}} \oplus \mathbb{R}$ (here $\mathbb{R}^3$)

$$\Sigma_{\mathscr{P}} = \{ C(\sigma) \mid \sigma \in \mathscr{P} \}$$

$$X_{\Sigma_{\mathscr{P}}} \to \mathbb{A}^1$$ alg family, this is what we just did

② Polyhedral picture (Σ: Called cone picture in Gross' book), modelled on ~~Mumford's degeneration~~

Recall: 2 ways to construct toric varieties

- fan construction
- polytope / polyhedron construction

Δ = convex $\{(0,0), (0,1), (1,0)\}$, $\tilde{\Delta} = \{(m,r) \mid m \in \mathbb{R}^2, r \geq \varphi(m)\}$

$C(\tilde{\Delta}) := \overline{\{(sm, sr, s) \mid (m,r) \in \tilde{\Delta}, s \geq 0\}}$ ← closure

asymptotic cone $(\tilde{\Delta}) = C(\tilde{\Delta}) \cap (N_{\mathbb{R}} \oplus \mathbb{R} \oplus \{0\}) = \{0\} \oplus \mathbb{R}_{\geq 0} \oplus \{0\}$

- define degree by projection to last component

$$k[C(\tilde{\Delta}) \cap (N \oplus \mathbb{Z}) \oplus \mathbb{Z}] \hookleftarrow (\deg = 0 \text{ piece}) = k[\text{Asymp}(\tilde{\Delta}) \cap N \oplus \mathbb{Z}] \simeq k[\mathbb{N}]$$

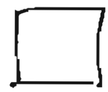
f.g. graded ring $\big\downarrow$ $\mathbb{Z}$

$\Rightarrow$ Proj$[\cdots] \longrightarrow$ Spec $k[\mathbb{N}] = \mathbb{A}^1$ ~~proj~~ family

$P_{\tilde{\Delta}}$:

es then construction is in order preserving fashion

$$\begin{array}{c} \mathbb{P}_{\tilde{\Delta}} \\ \downarrow \\ \mathbb{A}^1 \end{array}$$

say if we subdivide  to 
then get 4 pieces in the fiber (with same dot pattern)

Exercise:

For $\mathbb{P}^2$, Fan picture gives trivial family

$$\begin{array}{c} \mathbb{P}^2 \times \mathbb{A}^1 \\ \downarrow \\ \mathbb{A}^1 \end{array}$$

This picture is A-model of X_Σ itself

polyhedral picture B-model family we just constructed!

$$(B, \rho, \varphi)$$

$$x_0 x_1 x_2 = e^{y_1} \subset \mathbb{C}^3 \times \mathbb{C} \to \mathbb{C}$$

B-model for mirror

Can ask, how can I get reverse?

B for X_Σ? A for $\check{X}_\Sigma$?

Answer: Use discrete Legendre transformation

$$(\check{B}, \check{\rho}, \check{\varphi}) \quad !!!$$

End