Yi Hu at TIMS: 7/9 2013

Modular Blow-ups

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathbb{P}^n \leftarrow \mathcal{O}_{\mathbb{P}^n}(k) \\ \pi \downarrow & & \\ \overline{M}_g(\mathbb{P}^n, d) & \ni & [u: C \to \mathbb{P}^n] \end{array}$$

$R^\cdot \pi_* f^* \mathcal{O}_{\mathbb{P}^n}(k)$

Goal To diagonalize $R^\cdot \pi_* f^* \mathcal{O}_{\mathbb{P}^n}(k) = [E \xrightarrow{\varphi} F]$.

by birational base change — e.g. $\begin{pmatrix} p_1 I_* & & \\ & p_2 I_2 & \\ & & \ddots \end{pmatrix}$ $p_i \mid p_j$

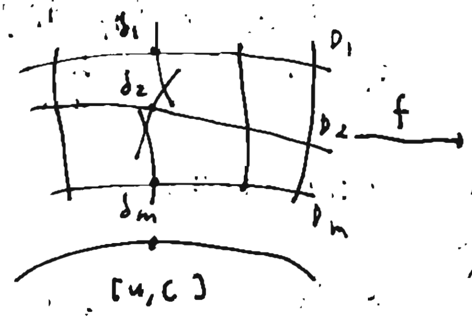
smooth

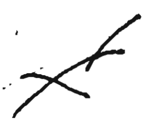
- pt $[u, C] \rightsquigarrow (C, [s_0, \cdots, s_n])$, $s_i \in H^0(u^* \mathcal{O}_{\mathbb{P}^n}(1))$

- $\overline{M}_g(\mathbb{P}^n, d) \longrightarrow \mathcal{P}_g \ni (C, L)$ — moduli stack of genus g curves with a line bundle L (Artin stack).

take nbd of $[u, C]$ — local morphism — will see

$\mathcal{D}_g \ni (C, D) \rightsquigarrow (C, \mathcal{O}(D))$ in $\mathcal{P}_g$ — larger Artin stack using divisors





by pulling back general hyperplane to get D_i

$f^* \mathcal{O}_{\mathbb{P}^n}(k)$ has degree $m = dk$

$\shortparallel \mathcal{O}_C(D)$

shrinking nbd if necessary, can write $D = \sum D_i$ s.t. D_i are disjoint sections. (can do this only locally)

Let $(\mathcal{C}, \mathcal{D})$ be the univ. curve and univ divisor over $\mathcal{D}_g$

So the pt is $(C, \delta_1 + \cdots + \delta_m)$

p.2:

$$\begin{array}{c} \mathcal{C} \\ {\scriptstyle \rho}\downarrow \\ \mathcal{D}_g \end{array}$$ be the projection

Need to diagonalize $R^1\rho_* \mathcal{O}_\mathcal{C}(D)$ ……

$$\begin{array}{c} \mathcal{C} \\ {\scriptstyle \rho}\downarrow \\ \mathcal{V} \end{array}$$

choose sections $A_1, \dots, A_g$ in general position in an open set. let $A = A_1 + \dots + A_g$

Lemma: $R^1\rho_* \mathcal{O}_\mathcal{C}(D) = [\, \rho_* \mathcal{O}_\mathcal{C}(D+A) \xrightarrow[\text{restriction}]{\psi} \rho_* \mathcal{O}_A(A) \,]$

Pf: $0 \to \mathcal{O}_\mathcal{C}(D) \to \mathcal{O}_\mathcal{C}(D+A) \to \mathcal{O}_A(A) \to 0$

$\quad \to 0 \to \rho_* \mathcal{O}_C(D) \to \rho_* \mathcal{O}_C(D+A) \xrightarrow{\psi} \rho_* \mathcal{O}_A(A) \to R^1\pi_* \mathcal{O}_C(D) \to 0$

Notice that $\rho_* \mathcal{O}_A(A) = \bigoplus \rho_* \mathcal{O}_{A_i}(A_i) \simeq \mathcal{O}_\mathcal{V}^{\oplus g}$ $\quad rk = g$

and by R-R, for $\rho_* \mathcal{O}_\mathcal{C}(D+A)$ it is $m+g+1-g = m+\underline{1}$

the $\underline{1}$ can really be split out… by a general section B.

Lemma: 1) $\rho_* \mathcal{O}_\mathcal{C}(D) = \mathcal{O}_\mathcal{V} \oplus \rho_* \mathcal{O}_C(D-B)$

2) $\rho_* \mathcal{O}_C(D-B) = \ker \varphi$, $\varphi : \rho_* \mathcal{O}_\mathcal{C}(D+A-B) \to \rho_* \mathcal{O}_A(A)$

3) $\varphi = \oplus \varphi_i : \bigoplus_{i=1}^{m} \rho_* \mathcal{O}_C(D_i+A-B) \longrightarrow \bigoplus_{i=1}^{g} \rho_* \mathcal{O}_{A_i}(A_i)$

$$\begin{array}{ccc} \rho_* \mathcal{O}_C(D_i+A-B) & \xrightarrow{\varphi_i} & \rho_* \mathcal{O}_{A_i}(A_i) \\ \wr\!\! \mid & & \wr\!\! \mid \\ \mathcal{O}_\mathcal{V} & & \mathcal{O}_\mathcal{V} \end{array}$$

Key: Describe $\varphi_i : \rho_* \mathcal{O}_C(D_i+A-B) \to \rho_* \mathcal{O}_A(A)$.

$$\begin{array}{ccc} \rho_* \mathcal{O}_C(D_i+A-B) & & \\ \wr\!\! \mid & \overset{\varphi_{ij}}{\searrow} & \\ \mathcal{O}_\mathcal{V} & & \rho_* \mathcal{O}_{A_j}(A_j) \simeq \mathcal{O}_\mathcal{V} \end{array}$$

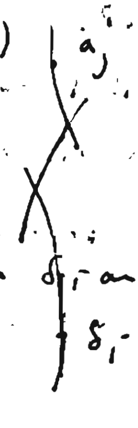
φ_{ij} is just a function

Lemma $\quad \varphi_{ij} = c_{ij}\, \xi_{[\delta_i, a_j]}$, $c_{ij} \in \Gamma(\mathcal{O}_g^*)$

where $\xi_{[\delta_i, a_j]} = \prod_{p \in N[\delta_i, a_j]} \xi_p$

1) $N[\delta_i, a_j]$ = Set of separable nodes between δ_i and a_j

2) ξ_p is a node smoothing parameter



This way, $\varphi = \left(c_{ij}\, \xi_{[\delta_i, a_j]} \right) = M_\varphi$

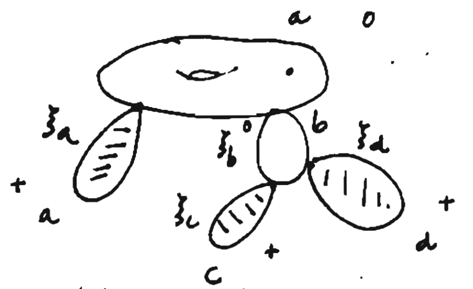
Theorem Local eq'n of $\overline{M}_g(\mathbb{P}^n, d)$ at $[u, c]$ is
(Set $k = 1$)

$$M_\varphi \begin{bmatrix} w_1^j \\ \vdots \\ w_d^j \end{bmatrix} = 0, \quad j = 1, \dots, n$$

where $w_i^j \in \mathbb{A}^1$ are coordinates on fibers.
(for $g = 1$, this reduces to a single eq'n just one row.)
This allows us to determine singularities and read off how to desingularize for $g = 1, 2$

Now, consider $g = 1$
$\varphi = (c_1 \xi_{[\delta_1, a]}, \dots, c_d \xi_{[\delta_d, a]})$ WLOG set $c_i = 1$
in this case, we want to do blow up st one factor divides all the others, and hence can reduce all other terms to 0

Ex.



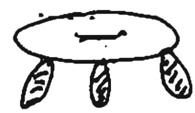
Local eq'n:

$$\xi_a w_a + \xi_b (\xi_c w_c + \xi_d w_d) = 0$$

1st blow-up $\xi_a = \xi_b = 0$
(locus of 2 tails)

ie $\dfrac{\xi_a}{u_a} = \dfrac{\xi_b}{u_b}$ $\Rightarrow$ At $(u_a = 1)$ smooth
At $(u_b = 1)$, the equation becomes

$$u_a w_a + \xi_c w_c + \xi_d w_d = 0$$



2nd blow-up along $u_a = \xi_c = \xi_d = 0$
(proper transf. of 3 tail locus)
it becomes smooth

So indeed no complication in the genus 1 case

$g = 2$ case:

$$\varphi = \begin{pmatrix} c_{11} \xi_{[\delta_1, a_1]}, & c_{12} [\delta_2, a_1], & \dots, & c_{1d} \xi_{[\delta_d, a_1]} \\ c_{21} \xi_{[\delta_1, a_2]}, & c_{22} \xi_{[\delta_2, a_2]}, & \dots, & c_{2d} \xi_{[\delta_d, a_2]} \end{pmatrix}$$

now the coeff plays role, can't be ignored. eg. det of minor has meaning!

Need to further "simplify" the matrix φ

there are several cases:

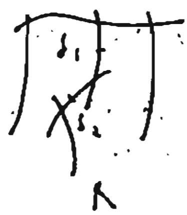
Let F be the minimal sub curve of C of genus 2, called the core curve of C,

1) $\deg F \cap D \geq 2$

let $\varphi F = \bigoplus_{\delta_i \in D \cap F} \varphi_i$, $\varphi F = \begin{pmatrix} c_{11} & \cdots & c_{1r} \\ c_{21} & & c_{2r} \end{pmatrix}$.

$\det (c_{ij})(0) = 0 \iff \delta_i, \delta_j$ are conjugate points for i and j columns

2) Assume $\deg F \cap D = 0$,



Let $\varphi_R = \bigoplus_{\delta_i \in D \cap R} \varphi_i$. Then after changing basis (so different c_{ij}'s)

$$\varphi R = \begin{bmatrix} c_{11} \xi[\delta_1, q_1] & c_{12} \xi[\delta_1, q_1] \xi[\delta_2] & c_{13} \xi[\delta_1, q_1] \xi[\delta_2] \xi[\delta_3] \\ c_{21} \xi[\delta_1, q_2], & c_{22} \xi[\delta_1, q_2] \xi[\delta_2] & c_{23} \xi[\delta_1, q_2] \xi[\delta_2] \xi[\delta_3] \end{bmatrix}$$

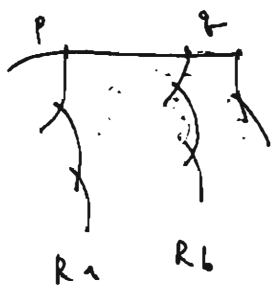
where $\xi[\delta_i] = \prod_{p \in N[\delta_i, F]} \xi_p$, further

(1) $\operatorname{rank} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}(0) = 2$

(2) $\det \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}(0) = 0 \iff p$ is a Weierstrass point.

Still another kind of geometry needs to be dealt with:

Prop:



$$\varphi R_a = \begin{pmatrix} a_{11} \xi[\xi_a, q_1], & \cdots \\ a_{12} \xi[\xi_a, q_2], & \cdots \end{pmatrix}$$

$$\varphi R_b = \begin{pmatrix} b_{11} \xi[\xi_b, q_1], & \cdots \\ b_{12} \xi[\xi_b, q_2], & \cdots \end{pmatrix}$$

$\varphi = (\varphi_{R_a}, \varphi_{R_b}, \quad)$

$$\det \begin{pmatrix} a_{11} & b_{11} \\ a_{12} & b_{12} \end{pmatrix}(0) = 0$$

$\iff p, q$ are conjugate

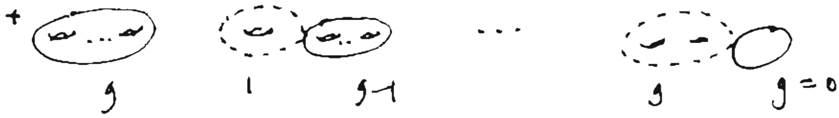
<u>Theorem</u>: $\exists$ 4 rounds blow-ups, 1st, 2nd are just like $g=1$ case (topological). 3rd is geometric (dealing with Weierstrass & conj loci). 4th round on hyperelliptic locus

Pf of $N_{1,d} = N'_{1,d} + \frac{1}{12} N_{0,d}$

$N_{g,d} = \deg [\overline{M}_g(Q,d)]^{vir}$, $Q \subset \mathbb{P}^4$ quintic

conj : $N_{g,d} = N'_{g,d} + c_1 N'_{g-1,d} + \cdots + c_g N'_{0,d}$

heuristic



other components eg.

can be get rid off via Clemens's conj

piece leading to 2 $\mathbb{P}^1$'s which is excluded

$g=0$:

$[\overline{M}_0(Q,d)]^{vir} = \int_{[\overline{M}_g(\mathbb{P}^4,d)]} e(\pi_* f^* \mathcal{O}_{\mathbb{P}^4}(5))$

higher genus g : $[\overline{M}_g(Q,d)]^{vir} = \int_{[\overline{M}_g(\mathbb{P}^4,d)]^{vir}} e(\ ?\ )$

As a stack : $\overline{M}_g(Q,d) \subset \pi_* f^* \mathcal{O}_{\mathbb{P}^4}(5) = R^0$

$$\downarrow \qquad R^1 \pi_* \mathcal{O}_{\mathbb{P}^4}(5)$$

$$\overline{M}_g(\mathbb{P}^4,d)$$

get $\overline{M}_g(Q,d) \subset [R^0/R^1]$

$$\downarrow$$

$$\overline{M}_g(\mathbb{P}^4,d)$$

but in literature, not much had been said in this way (virtual cycle, normal cone ...)

Moduli space of stable maps with p-fields :

$\overline{M}_g(\mathbb{P}^4,d)^p = \{ ([u,C],\psi) \mid [u,C] \in \overline{M}_g(\mathbb{P}^4,d),\ \psi \in \Gamma(C, u^*\mathcal{O}(-5) \otimes \omega_C) \} \longrightarrow \mathbb{P}^4$

ψ can be non-zero exactly over stable maps where $\pi_* f^* \mathcal{O}_{\mathbb{P}^4}(5)$ is NOT locally free (get a cone)

S.D. $H^1(C, u^*\mathcal{O}(5))$

$\searrow \overline{M}_g(\mathbb{P}^4,d)$

still has perfect obst. th. hence virtual cycle .

§2 Some obstruction theory (then)

$\mathcal{X} = \overline{M}_1(\mathbb{P}^4, d)$, $(\pi_{\mathcal{X}}, f_{\mathcal{X}}) : \mathcal{C}_{\mathcal{X}} \to \mathcal{X} \times \mathbb{P}^4$

$Y = \overline{M}_1(\mathbb{P}^4, d)^p$, $(\pi_Y, f_Y) : \mathcal{C}_Y \to Y \times \mathbb{P}^4$

$\psi_Y \in P(\mathcal{C}_Y, \mathcal{P}_Y)$, $\mathcal{P}_Y = f_Y^* \mathcal{O}(-5) \otimes \omega_{\mathcal{C}_Y/Y}$

Now its more natural to get theory rel to Picard scheme

$D = \{(C, L)\}$

A perfect obstruction th of $\mathcal{X}/D$ is

$$E_{\mathcal{X}/D} = R\, \pi_{\mathcal{X}*} f_{\mathcal{X}}^* \mathcal{O}(1)^{\oplus 5}$$

A perfect obs th of Y/D is

$$E_{Y/D} = R^\cdot\, \pi_{Y*} \left(f_Y^* \mathcal{O}_Y(1)^{\oplus 5} + \mathcal{P}_Y \right)$$

- Cosection, localized virtual class

$Q = (W=0)$, $W = x_1^5 + \cdots + x_5^5$

it induce $\sigma : Ob_Y \to \mathcal{O}_Y$ (in general $\sigma : E \to F$)

Non-surjective locus $D(\sigma) = \overline{M}_g(Q, d) \subset \overline{M}_g(\mathbb{P}^4, d) \subset \overline{M}_g(\mathbb{P}^4, d)^p$

Let $C_{Y/D}$ be the intrinsic normal cone

$$\Rightarrow [C_{Y/D}] \in Z_* \, h^1/h^0 (E_{Y/D})$$

Kiem-Li, "localized virtual class"

$$\star \quad [Y]^{vir}_{loc} = 0^!_{\sigma, loc} [C_{Y/D}] \in A_0(D(\sigma)) = \overline{M}_g(Q, d)$$

Def: $N^p_{1,d}(\mathbb{P}^4) = \deg [Y]^{vir}_{loc}$

Theorem (Chang-Li) $\quad N^p_{1,d}(\mathbb{P}^4) = (-1)^{5d} N_{1,d}(Q)$

$g=1$ To get "structure" of $C_{Y/D}$, we need "resolution"

Recall we have $\widetilde{\mathfrak{M}}^w \to \mathfrak{M}^w$ - weighted by deg $\in \mathbb{N}$

$\widetilde{\mathcal{X}} = \mathcal{X} \times_{\mathfrak{M}^w} \widetilde{\mathfrak{M}}^w \qquad \widetilde{D} = D \times_{\mathfrak{M}^w} \widetilde{\mathfrak{M}}^w$

$\widetilde{Y} = Y \times_{\mathfrak{M}^w} \widetilde{\mathfrak{M}}^w$

$\mathcal{C}_{\widetilde{Y}} = \mathcal{C}_Y \times_Y \widetilde{Y} \quad \to$

$$\begin{array}{ccc} \mathcal{C}_{\widetilde{Y}} & \xrightarrow{f_{\widetilde{Y}}} & \mathbb{P}^4 \\ \downarrow{\scriptstyle \pi_{\widetilde{Y}}} & & \\ \widetilde{Y} & & \end{array}$$

Let $\xi : \tilde Y \to Y$

obstruction theory is simply the pull back:

$E_{\tilde Y/\tilde D} = \xi^* E_{Y/D}$, $\sigma : ob_{Y/D} \to \mathcal{O}_Y$, $\tilde\sigma = \xi^*\sigma : ob_{\tilde Y/\tilde D} \to \mathcal{O}_{\tilde Y}$

$D(\tilde\sigma) = D(\sigma) \times_Y \tilde Y = \bar{M}_g(Q,d) \times_{\mathfrak{m}^w} \tilde{\mathfrak{m}}^w$

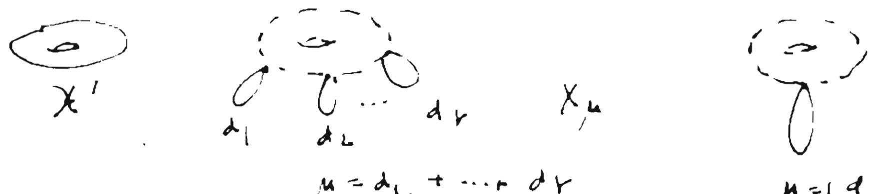
$\Rightarrow \deg [\tilde Y]^{vir}_{loc} = \deg [Y]^{vir}_{loc} = (-1)^{5d} N_{1,d}$

the point is that then the virtual normal cone ★' is almost just the normal cone since now everything is Sm in this pull back:

★' : $[\tilde Y]^{vir}_{loc} = 0^!_{\tilde\sigma, loc}[C_{\tilde Y/\tilde D}] \in A_0(D(\tilde\sigma))$.

Stratification of $\mathcal{X}, Y, \tilde{\mathcal{X}}, \tilde Y, C_{\tilde Y/\tilde D}$

$\mathcal{X}$



$\mathcal{X} = \mathcal{X}' \cup \bigcup_\mu \mathcal{X}_\mu$, $Y = Y' \cup \bigcup_\mu Y_\mu$

$\tilde{\mathcal{X}} = \tilde{\mathcal{X}}' \cup \bigcup_\mu \tilde{\mathcal{X}}_\mu$, $\tilde Y = \tilde Y' \cup \bigcup_\mu \tilde Y_\mu$

1) It is the zero section of $h^1/h^0(E_{\tilde Y/\tilde D})\Big|_{\tilde Y \setminus \bigcup_{\mu \vdash d} \tilde Y_\mu}$

2) it is the rk 2 subbundle of

$h^1/h^0(E_{\tilde Y/\tilde D})|_{\tilde Y \setminus \tilde Y'}$

ie μ is a partition of d

Now we may write down the cone as pieces:

$$[C_{\tilde Y/\tilde D}] = [C] + \sum_{\mu \vdash d} [C_\mu] + Z_* h^1/h^0(E_{\tilde Y/\tilde D})$$

$$[\tilde Y]^{vir}_{loc} = 0^!_{\tilde\sigma, loc}[C'] + \sum_{\mu \neq (d)} 0^!_{\tilde\sigma, loc}[C_\mu] + \sum_{\mu = (d)} 0^!_{\tilde\sigma, loc}[C_{(d)}]$$

claim: 1) should be the reduced MV, ie 2) $\downarrow$ no contribution, 3) should be the genus 0 part

Prop: $\deg 0^!_{\tilde\sigma, loc}[C'] = (-1)^{5d} N_{1,d}$.

the pf is a formal argument

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Prop 2 If $\mu \neq (d)$, then $\deg 0^!_{\tilde{\sigma}, loc}[C_\mu] = 0$

Prop 3 $\deg 0^!_{\tilde{\sigma}, loc}[C_{(d)}] = \frac{(-1)^{5d}}{12} N_{0,d}$.

These amounts to get the cone explicitly:

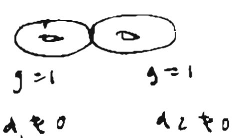
$$H_1 = h^1/h^0 \left(R^\cdot \pi_{\tilde{Y} *} f^*_{\tilde{Y}} \mathcal{O}(1)^{\oplus 5}\right)$$

$$H_2 = h^1/h^0 \left(R^0 \pi_{\tilde{Y} *} P_{\tilde{Y}}\right)$$

Show that the cone sits in some very "small" space, hence sum is wrong get 0 (dimension reason) → Prop 2

Prop 3 is hard, and not yet followed by the speaker //

<u>Rmk</u> For $g=2$ the contribution



is already counted in the main component $N_{2,d}$ hence not contradict to BCOV.

<u>End</u>