The First

NCTS Summer School on Algebraic Geometry

July 19 — 30, 1999

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(Notes by Chin-Lung Wang)

NCTS Summer School in Alg. Geom.
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Lecture I, 7/14 at Academia Sinica

1954. Shafarevich Conjectures:
K number field, F curve over K, $g = g(F)$

1) $\# \{ F/K$ with given set of places S of bad red. $\}$ is finite

2) $K = \mathbb{Q}$, $S = \phi$ then there no such curves.

$\xRightarrow{\text{Parshin}}$ Mordell. (Faltings, Yes).

F/K $\mathcal{O}_K$: ring integers

$\rightsquigarrow$

$X \xrightarrow[f]{} \operatorname{Spec} \mathcal{O}_K$, f smooth over $\operatorname{Spec} \mathcal{O}_K - S$

1) $\operatorname{Spec} \mathcal{O}_K \longleftrightarrow$ Base curve B
$\operatorname{Spec} \mathcal{O}_{\mathbb{Q}} = \operatorname{Spec} \mathbb{Z} \longleftrightarrow \mathbb{A}^1 = B$.

Notations, $k = \bar{k}$, char $k = 0$ ($k = \mathbb{C}$)

$f : X \to B$, X, B non-singular proj $/k$
f : alg. fiber space + f flat
($\Leftrightarrow$ in this case $\dim f^{-1}(b)$ is constant)
f is a "family of algebraic varieties".

$S \subseteq B$, S : NCD
$B_0 = B - S$, $X_0 = f^{-1}(B_0) \xrightarrow[f_0]{} B_0$ smooth

let F = general fiber. $s = \# S$ if B curve.
F curve, $g = g(F)$, B curve, $S \subset B$ finite.

(I). $\#\{$Isom. classes of smooth non-iso-trivial families of curves $/ B_0$ of genus $g\} < \infty$

Def: $f : X \to B$ isotrivial $\Longleftrightarrow$

$$X \times_B \overline{k(B)} \underset{\text{birat'l}}{\sim} F \times_k \overline{k(B)}$$

means all fibers all birat'l. (make sense for higher dim'l also)

(II). If $2g(B) - 2 + s \leq 0$, then there are no non-isotrivial families
$(B, S) \neq (\mathbb{P}^1, \{0, \infty\}) \neq (\text{Ellipt. } \emptyset)$

Parshin (67?) $S = \emptyset$
Arakelov In general //

Problem: what about if $\dim F > 1$.

Assumption 1) F should be a minimal model
2) consider $f : X \to B$ (proj) with a polarization $\mathcal{L}$ ie. $\mathcal{L}$ inv. sheaf, f-ample.

(otherwise, $\exists$ families of K3 surfaces $X \to \mathbb{P}^1$ "twistor spaces". but this is non-algebraic.)
or just F = torus. but still total space no algebraic.

Faltings: (83) (I) is NOT true:

For (B, S) fixed, there exists pos. dim families of abelian varieties $/ B - S$.

Better : $\omega_{X/B} = \mathcal{O}(K_X - f^* K_B)$

$= \omega_X \otimes f^* \omega_B^{-1}$, $\omega_X = \Lambda^{\max} \Omega^1_X$.

$f: X \to B$ families of polarized minimal models of Kodaira dim $\kappa(F) \geq 0$.

i.e. ω_{X_0/B_0} f-nef.

Let $\mathcal{L}$ be the polarization

$h(\nu) := \chi((\mathcal{L}|_F)^\nu)$ the hilbert polynomial

(I_a) Hope : Isom classes of non-isotrivial families $f_0 : X_0 \to B_0$ smooth polarized with given Hilb poly h should be parametrized by a scheme T of <u>finite type</u> / k

(I_b) Search for conditions implying that $\dim T = 0$.

(II) $2g(B) - 2 + s \leq 0 \Rightarrow T = \emptyset$.

Known :

(II) $\kappa(F) = 2$ (dim F), families of surfaces of general type

$(B, S) = (\mathrm{Ell}, \emptyset)$ Migliomni

$(B, S) = (\mathbb{P}^1, \{0, \infty\})$ Kovacs

(II) OK if local Torelli theorem holds true

$\Rightarrow \kappa(F) = 0$, $\dim F = 2$.

For $\dim F = 2$, $\kappa(F) = 1$, & Keiji Oguisio

true except $(B, S) = (\mathbb{P}^1, \{0, \infty\})$ & $\chi(\mathcal{O}_F) = 1$

2 multi fibers of multi m_1, m_2

$(m_1, m_2) = 1$, $m_1, m_2 \in \{2, 3, 4, 5\}$.

(I_a) F surface of general type , true by Bedulev, — .

(Ia) $\kappa(F)=0$ $\dim F = 2$ follows from Hodge theory
(M. H. Saito, Zucker, Peters, Jost - Zuo, $\dim F \geq 2$)

$\kappa(F) = 1$, open.

$\dim F > 2$, $\kappa(F) = \dim F$

ω_F ample,

MMP ($\dim F + 1$) $\Rightarrow$ Ia true. Bedulov, —

Karu (recently) Existence of comp. of moduli

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§1 Moduli (Introduction)

Fix $h(t) \in \mathbb{Q}[t]$ hilb. poly.

$\mathcal{F}(k) = \{ (X, \mathcal{L}) :$ X proj CM scheme + some prop.

$\omega_X^{[r]}$ invertible, nef, $\mathcal{L}$ ample and

$h(v) = \chi(\mathcal{L}^v) \} / \simeq$

$\omega_X^{[r]} := (\omega_X^r)^{\vee\vee}$.

$\mathcal{F}(T) = \{ (f : X \to T, \mathcal{L}) \mid$ f flat, fibers are in $\mathcal{F}(k) \}$

Definition (Mumford):

M coarse Moduli scheme $\overset{\text{def}}{\iff}$

$\exists$ natural transformation $\phi : \mathcal{F} \longrightarrow \mathrm{Hom}(-, M)$
such that

(1). $\phi(\operatorname{spec} k) : \mathcal{F}(k) \xrightarrow{1-1} M(k)$
but in order to fix the structure sheaf, eg. elli case

(2) If $\psi : \mathcal{F} \to \mathrm{Hom}(-, N)$ nat'l transf. $\mathbb{P}^1 \cup$ cusp.

$\Rightarrow \exists ! \; M \to N$ $\quad$ $\mathrm{Hom}(-, M)$.

Ex. Mumford,

$\mathcal{F} = M_g$, $g \geq 2$

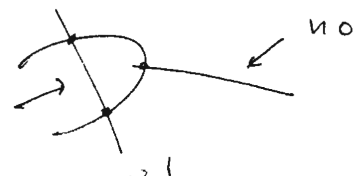
$M_g(k) = \{ C/k \mid C$ proj stable curve of genus $g \}/\cong$

i.e. C reduced 1-dim'l sing. are

s.t. ω_C ample.

$g = h^1(C, \mathcal{O}_C)$.

eg.



Theorem. (Mumford–Knudson)

M_g coarse moduli scheme for stable curves exists and M_g projective.

Moreover, $\exists$ invertible sheaf "λ_ν" such that

$$M_g(B) \ni f \quad \begin{array}{c} X \\ \downarrow \\ B \xrightarrow[\varphi_f]{} M_g \end{array} \qquad \text{"} \varphi_f^* \lambda_\nu = \det (f_* \omega_{X/B}^{\nu}) \text{"}$$

unfortunately this is not true. the truth is

$\exists \ \lambda_\nu^{(p)}$ $p >> 0$ st. $\varphi_f^* \lambda_\nu^{(p)} = \det (f_* \omega_{X/B}^{\nu})^p$.

But for simplicity, we omit the p. (or think the $=$ in sense of $\mathbb{Q}$-divisor)

Addendum: $\lambda_\nu^{-\alpha} \otimes \lambda_{\nu\mu}^{\beta}$ ample on M_g $\alpha, \beta > 0$

$\Rightarrow$ λ_ν ample on M_g for $\nu \geq 2$, $\nu, \mu >> 0$.

Interpretation of (II) $\longleftrightarrow$ M_g° locus of smooth curves is algebraically hyperbolic

i.e. $\nexists \ \mathbb{C}^* \subseteq M_g^\circ$

$\text{Elli} \subseteq M_g^\circ$.

2) [Kollár, Shepherd Barron, Alexeev]
There exists a definition of stable surfaces and a proj moduli space of stable surfaces, compactifying M_h° = moduli of surfaces of general type.

3) [Karec]:
MMP (dim + 1) $\Rightarrow \exists$ "stable canonically polarized n-fold"
Again Addendum true:
i.e. λ_ν ($\longleftrightarrow$ $\det(f_* \omega_{X/B}^\nu)$) is ample on M_h, $\nu \geq 2$.
Very optimistic interpretation of (II).
find universal family after a cover

$$\begin{array}{ccc} Y_0 & \xrightarrow{\text{finite}} & M_h^\circ \\ \cap & & \cap \\ Y & \longrightarrow & M_h \end{array}$$

$D = Y - Y_0$ NCD

$\Omega_Y^1(\log D)$ = sheaf of diff forms with log poles along D.

locally $D = Z(x_1 \cdots x_s)$

$$\Omega_Y^1(\log D) = \langle \frac{dx_1}{x_1}, \cdots \frac{dx_s}{x_s}, dx_{s+1}, \cdots, dx_n \rangle \mathcal{O}_Y$$

$$\Omega_Y^1(\log D) \dashrightarrow \Omega_{\mathbb{P}^1}^1(\log S)$$

may think its the ampleness of $\Omega_Y^1(\log D)$ but this may not be true in general since we take a covering, and is outside somewhere??

Def: $\mathcal{E}$ locally free on Y, $\mathcal{E}$ ample wrt. Y_0
$\Leftrightarrow$ $\exists\, \alpha > 0$, $\mathcal{H}$ ample inv. on Y
& $\exists \bigoplus \mathcal{H} \hookrightarrow S^\alpha(\mathcal{E})$, $\cong$ over Y_0.

Remark: "Hope" true for M_g, $g \geq 2$

In general; open problems: M_h compatified moduli scheme (of can. polarized varieties), M_h° smooth part

$$\begin{array}{ccccc} Y - D & \xrightarrow[\varphi]{\text{finite}} & M_h^\circ & \overset{?}{\Rightarrow} & \det(\Omega^1_Y(\log D)) \\ \cap & & \cap & & = \omega_Y(D) \\ Y & \longrightarrow & M_h & & \text{ample w.r.t. } Y - D \end{array}$$

D: NCD

(I_a), $\kappa(F) = \dim F$, Assume M_h exists

<u>Problem</u> (B) (boundedness) $f : X \to B$, B curve

f: families of minimal models on $f_0 : X_0 \to B_0$.

$h(\nu) = \chi(\omega_F^\nu)$

<u>Show</u>: $\exists$ polynomial P depends only on h st.

$\deg f_* \omega^\eta_{X/B} \leq P(g(B), \# S, \eta)$

Now standard: $(B) \Rightarrow (I_a)$:

pf: remember λ_η ample on M_h.

$$\begin{array}{ccc} \mathbb{H} = \mathrm{Hom}\big((B, B_0), (M_h, M_h^\circ)\big) & \subseteq & \mathrm{Hom}(B, M_h) \\ \cup\!\mathsf{I} & & \cup \\ \mathbb{H}_P\big((B, B_0), (M_h, M_h^\circ)\big) & \subseteq & \mathrm{Hom}_P(B, M_h) \\ \cup\!\mathsf{I} & & \uparrow \end{array}$$

scheme of finite type

$\varphi : B \longrightarrow M_h$

$\varphi^* \lambda_\eta \leq P(\cdots)$

Since the moduli is only coarse, consider

$T = \{ \varphi : (B, B_0) \to (M_h, M_h^\circ)$, induced by "$f : X \to B$" $\in \mathcal{M}_h(B) \} / \simeq$. $\square$

§2. Diff forms and Kodaira-Spencer map.

Ex. X elliptic surface

$$\begin{array}{ccc} f: X \to B & \text{non-isotrivial}, \ \exists\ B \xrightarrow{\text{finite}} & M_1 = \mathbb{P}^1 \\ \quad\ \ \| & & \cup \\ \quad\ \mathbb{P}^1 & & M_1^0 = \mathbb{A}^1 \end{array}$$

$\overset{?}{\Rightarrow}$ f has at least 3 deg. fibers.

$$\omega_{X/B} = f^* \Theta, \quad \Theta \geq 0 \qquad (\text{Kodaira's formula})$$

$\deg(f_* \omega^2_{X/B}) > 0$.

consider: $0 \to f^* \Omega^1_B(\log S) \to \Omega^1_X(\log f^{-1}S) \to \Omega^1_{X/S}(\log f^{-1}(S)) \to 0$

exact sequence of vector bundles.

$$0 \to T_{X/B}\langle -f^{-1}(S)\rangle \to T^1_X\langle -f^{-1}S\rangle \to f^* T^1_B \langle -S \rangle \to 0$$

$$\begin{tikzcd} T^1_B\langle -S\rangle \arrow[r, "*"] \arrow[d, equal] & R^1 f_* T_{X/B}\langle -f^{-1}(S)\rangle \arrow[dd, dash] \\ \mathcal{O}_B \arrow[ur, "\neq 0"] & \\ & (f_* \omega^2_{X/B})^\vee \end{tikzcd}$$

← cheating a little
fit here
(need handle multi fibers)

← deg < 0.

$B = \mathbb{P}^1$, $S = \{0, \infty\}$.

this proof is the "most complicated pf" in the ellipt case. but it holds true for general g, and also to higher dim. #.

<u>To be continued</u>.

$f : X \to B$ families of curves,
surfaces of general type
or. can. polarized manifolds.

Need diff forms & K-S Map:
positivity properties for $f_* \omega^{\nu}_{X/B}$ (not just $\nu = 1$)
$\rightsquigarrow$ moduli schemes.

§3. Construction of moduli

$h \in \mathbb{Q}[t]$, $\deg h = n = \dim(\text{—})$

$$\mathcal{M}_h^{\circ} = \left\{ X \;\middle|\; \begin{array}{l} X \text{ normal reduced scheme with at} \\ \text{most RDP}, \ \omega_X \text{ ample}, \ \chi(\omega_X^{\nu}) = h(\nu) \end{array} \right\} \Big/ \cong$$

Need compactification of moduli Problem.

Ex. $n=1$, $\mathcal{M}_h^{\circ} = \mathcal{M}_g^{\circ}$

$$\mathcal{M}_g^{\circ} = \left\{ X \;\middle|\; \begin{array}{l} X \text{ stable curve}, \ \omega_X \text{ ample} \\ \text{only transversal int. as sing.} \end{array} \right\} \Big/ \cong$$

$n=2$: $\exists$ definition of stable surfaces [Kollár, S-B].
More general definition will be given later.

$\mathcal{M}_h$ bounded $\iff \exists H$ & $\mathfrak{X} \xrightarrow[f]{} H$ fiber space
H scheme of finite type
st. each $X \in \mathcal{M}_h(k)$ occurs as
a fiber of f.

Ex. $\mathcal{M}_h^{\circ}(k)$ bounded.

Def: X $\mathbb{Q}$-Gorenstein, $\omega_X^{[r]}$ invertible some $r >> 0$, X semi-log can. sing.

$\Leftrightarrow$ i) X satisfies Serre's condition S_2

ii) X NCD in codim 1

iii) $\forall$ $f: Y \to X$, Y normal $\mathbb{Q}$-Gorenstein

$$\omega_Y^{[r]} = f^* \omega_X^{[r]} \otimes \mathcal{O}(\Sigma a_i E_i) \quad a_i \geq -r.$$

Def: Stable n-folds:

connected proj n-dim'l X

* semi-log-can. sing.
* smoothable
* $\omega_X^{[r]}$ ample, (invertible)

$M_h(k) = \{ X : X \text{ stable } n\text{-fold, Hilb poly} = h \}$

$M_h(T) = \{ f: \mathfrak{X} \to T$, that fiber $\in M_h(k)$ and $\omega_{\mathfrak{X}/T}^{[r]}$ invertible $\} / \simeq$

Properties:

A). Boundedness: $\exists$ M st. $\forall$ $X \in M_h(k)$ $\omega_X^{[M]}$ very ample (& invertible)

$n=1$ exercise

$n=2$ Alexeev

$n \geq 2$: Thm (Karu): Assume MMP $(n+1)$

$\Rightarrow$ M_h is bounded (in dim n).

Idea of pf: M_h° bounded $\rightsquigarrow$ $\mathfrak{X}_0 \xrightarrow{f_0} H_0$

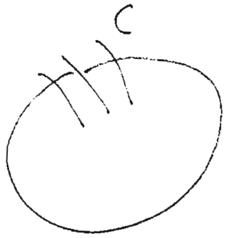
compactify $\mathfrak{X} \xrightarrow{f} H$, Apply weakly s.s. alteration

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "f'"] & H' \arrow[d, "\text{finite}"] \\ \mathcal{X} \arrow[r, "f"] & H \end{tikzcd}$$

Need to know

$$\mathcal{Y} = \mathrm{Proj}\Big(\bigoplus_{\nu} f'_* \omega^{[r]^{\nu}}_{\mathcal{X}'/H'}\Big)$$

need to be finitely generated.



$\bigoplus f'_* \omega^{[r]\nu}_{\mathcal{X}/C} \,/C$

need Siu-Kawamata's thm on invariance of plurigenera. $\square$

Cor. There exists a Hilbert scheme H and universal family $f : \mathcal{X} \to H \in \mathcal{M}_h(H)$ together with

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[dr] & \mathbb{P}(f_* \omega^{[r]}_{\mathcal{X}/H}) \simeq \mathbb{P}^{h(r)-1} \times H \arrow[d] \\ & H \end{tikzcd}$$

Idea (old): $r \gg 0$

$|\omega_X^{[r]}| .\ X \hookrightarrow \mathbb{P}^{h(r)}, \ \forall X \in \mathcal{M}_h(k)$

$\Rightarrow$ subvarieties of bounded degree ($\leftrightarrow h$) $\subseteq \mathbb{P}^{\cdot}$ are parametrized by $\mathrm{Hilb}(\cdot, h)$ & $\mathcal{Y} \to \mathrm{Hilb}$.

$\to$ need locally closedness.

B). H quasi-projective:

$\mathcal{X} \xrightarrow{f} H, \quad f_* \omega^{[r]}_{\mathcal{X}/H} \cong \bigoplus^{h(\nu)} N, \quad N$ inv.

$$\underbrace{N^{-n} \otimes S^{\mu} f_* \omega^{[r]}_{\mathcal{X}/H}}_{\text{just copies of } \mathcal{O}} \twoheadrightarrow f_* \omega^{[\nu r]}_{\mathcal{X}/H} \otimes N^{-\mu}$$

$\Rightarrow X \to \mathrm{Grass}(\cdots)$, which is projective.

& ample sheaf is of the form

$\lambda_\eta := \det(f_* \omega^\eta_{X/H})$

$\lambda_{\nu\mu}^{\alpha} \otimes \lambda_{\nu}^{-\beta}$ $\alpha, \beta \in \mathbb{N}$ will do. □.

c). "stable reduction"

true by definition for $n \geq 2$.
exercise for $n = 1$.

D). $G = SL(h(\nu), k)$ acts on H

$G \times H \xrightarrow{\sigma} H$ action is proper with finite stabilizers.

proper $:=$ $G \times H \xrightarrow[(\sigma, pr_2)]{} H \times H$ is proper

this actually forces birat'l maps of general fiber in family extends to the central fiber.
(or maybe isom. on general fiber ⇒ birat'l)

THEOREM: A, B, D ⇒ ∃ coarse quasi-projective moduli scheme M_h, and
C ⇒ M_h projective.

Ex. MMP $(n+1)$ ⇒ M_h as above
Without ⇒ $M_h^o \longleftrightarrow X$ with normal RDP
similarly X with can. sing. (Kawamata).

Proof: Assume H/G exists

$$\underset{?}{\Longrightarrow} \quad H/G = M_h$$

Def: H q. proj. G red. linear alg. gp

$\pi : H \longrightarrow Z = H/G$ geom. quotient

$\Leftrightarrow$ 1) π compatible with G-action

2) $\mathcal{O}_Z = (\pi_* \mathcal{O}_H)^G$

3) $W_i \subset H$ G-inv closed $\Rightarrow \pi(W_i)$ closed, $i=1,2$

& $W_1 \cap W_2 = \phi \Rightarrow \pi(W_1) \cap \pi(W_2) = \phi$

4) $\pi^{-1}(w)$ one G-orbit.

Cor. If H/G geom. quotient exists $\Rightarrow H/G \cong M_h$.

pf: glueing: $g : \mathcal{Y} \longrightarrow T$

$$g_* \omega_{\mathcal{Y}/T}^{[r]} \big|_H \xrightarrow{\sim} \oplus \mathcal{O}_U \quad \text{on an open set.}$$

⋮

(2nd) universal property follows from 1) and 2).
(easy exercise).

So the important thing is to construct H/G:

Seshadri's elimination of finite isotropies ($\sim$ to)

G: reductive linear alg. gp H quasi-proj

$\sigma : G \times H \to H$ proper gp action with f stable

$\Rightarrow \exists$
$$\begin{array}{ccc} V & \xrightarrow{\pi} & Z \\ \downarrow p & & \\ H & & \end{array}$$
with 1) σ lifts to $\Sigma : G \times V \to V$

2) $\pi : V \to Z$ geom. quotient & principal G-bundle in Zariski top, $\pi^{-1}(U) \cong G \times U$, U open.

3) V/H Galois cover, gp Γ, and Γ commutes with G.

Pf is a simple construction:

$$\begin{tikzcd} G \times Y \arrow[d] \\ V_X \arrow[d, "p_X"', "\text{not proper}"] \\ H \end{tikzcd}$$

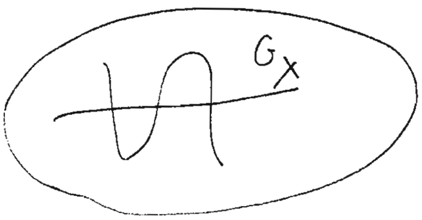
1) true by definition

2) true in nbd of $p_X^{-1}(G_X)$

Now use glueing:

Consider

$$\begin{tikzcd} & V_X \times_H V_Y \arrow[dl] \arrow[d] \arrow[dr] & \\ V_X \arrow[r] & H & V_Y \arrow[l] \end{tikzcd}$$



take normalization $V_{XY} = \widetilde{V_X \times_H V_Y}$.

1), 2) in nbd of G_X and G_Y.

After a finite number of steps: get

$V' \xrightarrow{p'} H$ s.t. 1), 2). surj. not proper

irred.

V = normalization of H in Galois hull of

$K = \langle k(V_i) \rangle \supset k(H)$, $\Rightarrow$ 1), 2), 3).

this is approximately the proof. $\square$

Cor. $\tilde{H}/G = Z/\Gamma$ if the later exists.

2 problems. $\sim$ and Γ.

Remark: Z q.proj $\Rightarrow Z/\Gamma$ exists and q.proj. but ~~our~~ Z is not q.p.

Recall:

$$\begin{tikzcd} \mathcal{X} \times_H V \arrow[dr] & & \\ & V \arrow[r] \arrow[d] & Z \\ \mathcal{X} \arrow[r] & H & \end{tikzcd}$$

locally $E \times U \to U$

$\uparrow$

$\mathcal{X} \times_H V \big|_{e \times U} = \mathcal{X}_U$

$\Rightarrow$ glue to $Y \xrightarrow{g} Z$

$\exists\ g : Y \to Z$ st. $\forall z_0 \in Z$
$\{ z \in Z \mid g^{-1}(z_0) \cong g^{-1}(z) \}$ is finite $(*)$

- Thm: $g : Y \to Z$ family with property $(*)$ & $g \in M_h(Z)$ then Z is quasi-proj

This thm is true, but very hard to prove. But if with Z proper, then the pf is much easier, & ample sheaves $\det(g_* \omega_{Y/Z}^{[r]})$ $v >> 0$.

Cor. Z/P exists $= \tilde{H}/G$.

Prob: $\tilde{H}/G$ Normalization of unknown object M_h.

Two ways out:

A). Show that M_h exists as a <u>proper</u> alg. space — this method only works for proper moduli prob.
$\Rightarrow \tilde{M}_h \xrightarrow{p} M_h$ with λ_v on M_h,
$p^* \lambda_v$ ample $\Rightarrow M_h$ proj. Scheme.

B). Construct H/G by GIT.
use Methods of proof of Thm o (stability).

c). For our problem, $\tilde{M}_h$ is enough

Remarks: for M_h°, B) is the only known method.

For proof of Thm:

Def: a) $\mathcal{E}$ locally free on Z, $\mathcal{E}$ is semi-positive:
if Z proper, $\forall \tau : C \to Z$, C curve,
$\forall \tau^* \mathcal{E} \to \mathcal{Y}$ inv $\Rightarrow \deg_c(\mathcal{Y}) \geq 0$.

For Z arbitrary, $\mathcal{E}$ semi-positive $\Leftrightarrow$

$\forall\ \tau : Z' \hookrightarrow Z$, Z' q. proj

$\forall\ \mathcal{H}'$ inv. ample on Z, $\forall\ \alpha > 0$

$$S^\alpha(\tau^* \mathcal{E}) \otimes \mathcal{H}' \text{ ample}$$

Remark: Z q. proj, $Z' = Z$ enough.

We Need:

A) $g_* \omega^{[r]}_{Y/Z}$ semi positive for all $r \gg 0$ & divisible.

easy (Z proper); hard (Z not proper)

B) A) + $(*)$ $\Rightarrow$ Theorem. $\square$

<u>To be continued</u>.

Positivity of Sheaves

- $f : X \to B$: curve $\qquad \kappa(F) = \dim F$
- $g : Y \to Z \in M_h(Z)$

$f_* \omega^{\nu}_{X/B}$. for $\nu = 1$ already in Kawamata's lecture.

$\mathcal{E}$ loc. free on Y

Y proper, $\mathcal{E}$ num. s.p. (nef) $\overset{def}{\iff}$ $\forall \tau : C \to Y$

$\forall \tau^* \mathcal{E} \to \mathcal{L} \to 0$, $\deg \mathcal{L} \geq 0$.

Y arbitrary, $\mathcal{E}$ s.p. (w.p.) $\overset{def}{\iff}$ $\forall \tau : Y' \to Y$ q. projective

$\forall \mathcal{H}'$ ample inv on Y', $\forall a > 0$

$S^a(\tau^* \mathcal{E}) \otimes \mathcal{H}'$ ample.

Ex. $f : X \to B$: B arbitrary.

$f_* \omega_{X/B}$ n.s.p. if monodromy are unipotent
($B - B_0$ NCD, B smooth)

without monodromy condition, can only say

$\exists \mathcal{E} \subset S^?(f_* \omega_{X/B})$, $\mathcal{E}$ num. s.p.

But for B curve, this is $\iff$ $f_* \omega_{X/B}$ n.s.p.

Main Reference : Mori : Bowdoin '85 (general)

For B curve : Esnault, —, Compositio '76

also : Bedulev, —. (preprint)

Starting Point:

Theorem (Fujita): $f: X \to B^1$ family of manifolds

$\Rightarrow f_* \omega_{X/B}$ n.s.p, X, B nonsingular.

One possible proof using alg. method:

pf: Kollár's vanishing:

X projective mfd, $\mathcal{L}$ semi-ample on X

D ef. st. $H^0(X, \mathcal{L}^\nu(-D)) \neq 0$ for some $\nu > 0$

$\Rightarrow H^i(X, \mathcal{L} \otimes \omega_X(D)) \twoheadrightarrow H^i(D, \mathcal{L} \otimes \omega_D) \ \forall i$.

Take $\mathcal{L} = f^* \mathcal{O}_B(pt)$, $D = F =$ fiber, get

$$\begin{array}{ccc} H^0(X, \mathcal{L} \otimes \omega_X(F)) & \twoheadrightarrow & H^0(F, \mathcal{L} \otimes \omega_F) \\ \| & & \| \\ H^0(B, (f_* \mathcal{L} \otimes \omega_X) \otimes \mathcal{O}(p)) & \twoheadrightarrow & f_*(\mathcal{L} \otimes \omega_X) \otimes k(p) \end{array}$$

ie. $f_* \omega_{X/B} \otimes \omega_B \otimes \mathcal{O}_B(2p)$ is globally generated.

Now

$$\begin{tikzcd} X^r = X \times_B \cdots \times_B X \arrow[r, "f^r"] & B \\ X^{(r)} \arrow[u, "\text{resd.} - \delta"] \arrow[ur, "f^{(r)}"'] & \end{tikzcd}$$

so $f_*^{(r)} \omega_{X^{(r)}/B} \otimes \omega_B(2p)$ is also globally gen.

$\downarrow$ over some open set

$$f_*^\nu \omega_{X^\nu/B} \otimes \omega_B(2p) = \left(\otimes^\nu (f_* \omega_{X/B})\right) \otimes \omega_B(2p)$$

$$\longrightarrow S^\nu(f_* \omega_{X/B}) \otimes \omega_B(2p)$$

Now B curve $\Rightarrow f_* \omega_{X/B}$ n.s.p.

Same proof works except the last step.
and also sing. occurs in the resolution map.

Theorem (Kawamata): $\dim B \geq 1$

$f: X \to B$ family of manifolds, X, B nonsingular
$B - B_0$ NCD, $\Rightarrow f_* \omega_{X/B}$ n.s.p.
for $f: X \to B \in M_n(B)$.

Recall Multiplier ideals:

X mfd, $D \geq 0$ of div, $N \in \mathbb{N}$, $\tau: X' \to X$ st.
$\tau^* D$: NCD. $\quad \omega_X\{-\frac{D}{N}\} = \tau_* \omega_{X'}(-[\frac{\tau^* D}{N}])$

$\left(= \omega_X(\lceil -\frac{D}{N} \rceil) \text{ in Kawamata's notation} \right)$

1). independent of τ
2). "$\mathcal{L}^N = \mathcal{O}(-D)$" $\Rightarrow \mathcal{L} \otimes \omega_X\{\frac{-D}{N}\}$ has similar properties as ω_X (eg. vanishing)

Reason: a) $R^i \tau_* \omega_{X'}(-[\frac{D'}{N}]) = 0$

b) $\exists$ cyclic covering $\gamma: Z' \to X'$ st.

$\gamma_* \omega_{Z'} \rightleftarrows \mathcal{L} \otimes \omega_{X'}(-[\frac{D'}{N}])$ a factor. (split)

Cor. $\mathcal{L}$ inv. on X, $\mathcal{L}^N = \mathcal{O}_X(D)$

$\Rightarrow f_* \left(\omega_{X/B}\{-\frac{D}{N}\} \otimes \mathcal{L} \right) = (*)$ n.s.p.

for B curve.

Reason: $(*) \rightleftarrows f_* \omega_{Z'/B}$.

Properties:

$$e(\mathcal{L}) := \mathrm{Min}\left\{ e \;\middle|\; \omega_{X/B}\left\{-\frac{D}{e}\right\} = \omega_{X/B},\ \forall D \geq 0 \right\}$$

where $\mathcal{L} = \mathcal{O}_X(D)$

i.e. the minimal number to kill all multiply ideals.

1) $e(\mathcal{L}) < \infty$

2) $\mathcal{L}$ very ample : $e(\mathcal{L}) \leq u(\mathcal{L})^{\dim X} + 1$

3) Behave Nicely under product : $X \times \cdots \times X \xrightarrow{pr_i} X$

$$e\left(\otimes pr_i^* \mathcal{L}\right) = e(\mathcal{L})$$

Cor. $f_* \omega_{X/B}^{\nu}$ n.s.p. $\forall \nu > 0$

Pf: $f_* \omega_{X/B}^{\nu} \otimes \mathcal{O}_B(\eta)$ n.s.p. for some $0 < \eta = \nu\rho - 1$ with ρ min.

$\Rightarrow f_* \omega_{X/B}^{\nu} \otimes \mathcal{O}_B(\nu\rho)$ ample

$\Rightarrow \omega_{X/B}^{\underline{\nu\mu}(\nu-1)}(\mu(\nu-1)\cdot\underline{\nu\cdot\rho})$ "gen. by global sections along general fiber".

$\mathcal{O}(D) :=$

in fact not quite correct unless take away some base locus in fiber.

eg. ω_F semi-ample. (let's only do this case)

take $\mathcal{L} = \omega_{X/B}^{\nu-1}((\nu-1)\rho)$, $N = \nu\mu$

$\Rightarrow f_*\left(\omega_{X/B}^{\nu-1}((\nu-1)\rho) \otimes \omega_{X/B}\left\{-\frac{D}{N}\right\}\right)$ n.s.p.

$\downarrow \cong$ over some open set

$f_* \omega_{X/B}^{\nu}((\nu-1)\rho) \Rightarrow$ n.s.p.

so must $\nu(\rho-1)-1 < (\nu-1)\rho$ ie. $\rho < \nu+1$

ie. the twist is only bounded by ν !

Same true if

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow \\ G:\ B' & \longrightarrow & B \end{array} \quad \text{finite cover}$$

$f'_* \omega^{\nu}_{X'/B'} \subset G^* f_* \omega^{\nu}_{X/B}$

$\simeq$ over some open set

Some may take cover to make "$\rho < \nu + 1$" number < 1. Then done. $\square$

Main Prob. for general B: D near singular fiber would be very bad!

Main ingredients for above pf:

- cyclic cover.
- product.

Next cor. (weak stability) true for arbitrary morphism and arb. variety, but again we only deal with semi-ample fiber case: and B curve.

Assume that $f_* \omega^{\nu}_{X/B} \neq 0$ $(\nu \geq 2)$

Equivalently:

a) $f_* \omega^{\nu}_{X/B}$ ample

b) $\det(f_* \omega^{\nu}_{X/B})$ ample

c) $\exists\, \eta, \alpha > 0$: $\mathcal{O}(\eta p) \hookrightarrow \bigotimes^{\alpha} (f_* \omega^{\nu}_{X/B})$

a) $\Rightarrow$ b) $\Rightarrow$ c), $\alpha = rk(f_* \omega^{\nu}_{X/B})$

For c) $\Rightarrow$ a). replace B by some finite cover $\rightsquigarrow \eta \gg 0$.

$$\begin{tikzcd} X^\alpha = X \times_B \cdots \times_B X \arrow[r, "F^\alpha"] & B \\ X^{(\alpha)} \arrow[u, "\delta"] \arrow[ur, dashed, "f^{(\alpha)}"'] & \end{tikzcd}$$

$\mathcal{M} = \delta^* \otimes^\alpha pr_i^* \omega_{X/B}$, 2 morphisms :

$$f^\alpha_* \delta_* \omega_{X^{(\alpha)}/B} \otimes \mathcal{M}^{\nu-1} \to f^\alpha_* \omega^\nu_{X^\alpha/B} = \overset{\alpha}{\otimes} f_* \omega^\nu_{X/B}$$

$$\otimes^\alpha f_* \omega^\nu_{X/B} \to f^\alpha_* \left(\delta_* \mathcal{O}_{X^{(\alpha)}}\right) \otimes \otimes\, pr_i^* \omega^\nu_{X/B} = f^{(\alpha)}_* \mathcal{M}^\nu$$

$\mathcal{M}^\nu$ has section with zero divisor $\Gamma + \eta F$

$(\mathcal{M}^\nu(\overset{\alpha}{F}))^N$ has lots of sections

since $f_* \mathcal{M}^\nu$ is n.s.p.

Remember we assume ω_F semi-ample

let D general section, $D|_{F^\alpha}$ nonsingular

Apply Fujita's positivity thm :

$$f_* \left(\omega^{\nu-1}_{X/B} \otimes \omega_{X/B}\left\{-\frac{D}{N}\right\} \otimes \mathcal{O}((\nu-1)\, \rho F^\alpha)\right) \leftarrow \text{cancel}$$

consider

$$\mathcal{M}^{\nu(N+1)(\nu-1)} = \mathcal{M}^{\nu N}(NF^\alpha) \otimes \mathcal{O}(\Gamma + (\eta - N) F^\alpha)^{\nu-1}$$

$$= \mathcal{O}\Big((\nu-1) D + \underbrace{(\nu-1)\Gamma + (\nu-1)(\eta-N) F}_{\Delta}\Big)$$

$\mathcal{L} = \mathcal{M}^{\nu-1}$

cor. $\Rightarrow$ $f^{(\alpha)}_* \omega_{X^{(\alpha)}/B}\left\{-\frac{\Delta}{\nu(N+1)}\right\} \otimes \mathcal{M}^{\nu-1} = (*)$

$$\Big\downarrow \cong \text{ near open set}$$

$$f^{(\alpha)}_* \omega^\nu_{X^{(\alpha)}/B} \subset \overset{\alpha}{\otimes}\left(f_* \omega^\nu_{X/B}\right).$$

On general fiber:

$$\omega_{F\alpha}\left\{-\frac{D(\nu-1)+P(\nu-1)}{\nu(N+1)}\right\} \simeq \omega_{F\alpha}$$

for $N \gg 0$

$(\eta - N) > 0$, $\left[\frac{(\nu-1)(\eta-N)}{\nu(N+1)}\right] > 1$ (may choose)

this is the place to play. see below

hence $(*) \hookrightarrow f_*^{(\alpha)}\, \omega^{\nu}_{X^{(\alpha)}/B}(-F\alpha)$

$\simeq$ over the general fiber

and we are done. $\square$

More Precise (will be needed later for Shaf. conj (B) and (II)):

<u>Cor.</u> $r(\nu) = \alpha = \operatorname{rank}(f_* \omega^{\nu}_{X/B})$, $e(\omega^{\nu}_F) = e(\nu)$

$$\Rightarrow S^{r(\nu)\, e(\nu)}(f_* \omega^{\nu}_{X/B}) \otimes \det(f_* \omega^{\nu}_{X/B})^{-1} \text{ n.s.p.}$$

(quite precise numbers!)

Mainly because:

$$e(M^{\nu}_{F r(\nu)}) = e(\omega^{\nu}_F). \qquad \text{EXERCISE.}$$

Rem. & Cor:

Thm: Assume $f: X \to B$ is semi-stable, Then

f isotrivial $\Leftrightarrow \deg\left[\det(f_* \omega^{\nu}_{X/B})\right] = 0 \quad \forall\, \nu \geq 0$

If $\kappa(F) \geq 0$, either F general type (Kollár / Viehweg) or F has minimal model F' and $\omega_{F'}$ semi-ample (Kawamata).

Cor. $f : X \to B$, F as in thm.

$$\begin{tikzcd} Y \arrow[r, "\sigma"] \arrow[dr, "g"'] & X \arrow[d, "f"] \\ & B \end{tikzcd}$$ dominant

g isotrivial $\Rightarrow$ f isotrivial.

$f_* \omega^{\vee}_{X/B} < g_* \omega^{\vee}_{Y/B}$.

Recently, Mok-Hwang have similar result of this cor for families of Fano varieties.

To finish the discussion of semi-positivity, we need

§ Strong Positivity:

Recall:

$$\begin{tikzcd} & V \arrow[r, "\pi"] \arrow[d] & Z & \mathcal{Y} \arrow[l, "g"'] \\ \mathcal{X} \arrow[r, "f"] & H & & \end{tikzcd}$$

$g \in \mathcal{M}_h(Z)$ "universal family"

$\mathcal{E}_\nu := g_* \omega^{\vee}_{\mathcal{Y}/Z}$ semi-positive (n.s.p if Z proper)

The proof is <u>very hard</u> (if Z not proper) which does not follow from above discussion.

(Most book in moduli spent 1/2 to do this!)

Proof: If Z proper $\rightsquigarrow$ reduce to curve (since have numerical criterion)

"<u>as above</u>"

$$\begin{array}{l} X \\ \downarrow f \in \mathcal{M}_h(B) \\ B \subseteq Z \end{array}$$

$f_* \omega^{\vee}_{X/B} = g_* \omega^{\vee}_{\mathcal{Y}/Z}|_B$ n.s.p!

(a little cheating: should be $\omega^{[\nu]}$!).

Theorem: $\lambda_\eta := \det(g_* \omega_{Y/Z}^{\eta})$

$\exists\ \alpha, \beta, \mu \gg 0$ st. $\lambda_{\nu\mu}^{\alpha} \otimes \lambda_{\mu}^{\beta}$ ample Z.

If Z proper, $\exists\ \mu \gg 0$ st. $\lambda_{\nu\mu}$ ample.

a) similar to GIT

b) observed by Kollár via Numerical criteria (which is simpler)

Will explain 1st pf a) here:

let $\xi = \xi_\nu = g_* \omega_{Y/Z}^{\nu}$

$\mathbb{P} = \mathbb{P}(\oplus^{\nu} \xi^{\vee}) \xrightarrow{\pi} Z \Rightarrow \pi^* \oplus^{\nu} \xi^{\vee} \twoheadrightarrow \mathcal{O}(1)$

① $\mathcal{O}(1) \otimes \det(\pi^* \xi)$ (n.) s.p.

$$\uparrow\!\!\uparrow$$

$$\oplus^r \pi^* (\underbrace{\xi^{\vee} \otimes \det(\xi)}_{\Lambda^{r-1}\xi})$$

② $\oplus^r \mathcal{O}(-1) \overset{s}{\hookrightarrow} \pi^* \xi$, let $D =$ degenerate divisor $=$ zero$(\det s)$

$\underline{\mathcal{O}(-r)(D) = \pi^* \det \xi}$

on $V = \mathbb{P} - D$, $\underline{\oplus^{\nu} \mathcal{O}(-1)|_V \xrightarrow{\sim} \pi^* \xi|_V}$ $(*)$

Rmk:

$(*) \rightsquigarrow$
$$\begin{array}{ccc} V & \xrightarrow{\pi|_V} & Z \\ \downarrow p & & \\ H & & \end{array}$$
this is somehow reverse the previous construction.

① + ② = ③: $\pi^* \det(\xi)^{r-1} \otimes \mathcal{O}(D)$ is (n.) s.p.

Next time will see ③ + Rmk $\Rightarrow$ Thm.

To be conti

Recall.

$$\begin{tikzcd} & & \mathcal{Y} \arrow[dl, "g"'] & \in \mathcal{M}_h(Z) \\ V \arrow[r] \arrow[d, "p"', "\text{finite}"] & Z & & \\ H \arrow[r, hook] & \mathbb{P}^M & & \end{tikzcd}$$

plucker embedding

Having H, may reconstruct V via shishadri's method

Having Z may also reconstruct V:

$$V \subseteq \mathbb{P} = \mathbb{P}(\oplus^r \mathcal{E}_\nu^\vee), \quad D = \mathbb{P} - V$$

$\mathcal{E}_\nu = g_* \omega^\nu_{\mathcal{Y}/Z}$ semi. pos. $\lambda_\nu = \det \mathcal{E}_\nu$

③ $\mathcal{O}_{\mathbb{P}}(D) \otimes \pi^* \lambda_\nu^{\nu-1}$ semi. pos.

④ $\mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_{\mathbb{P}}(D) = \mathcal{O}_{\mathbb{P}}(r) \otimes \pi^* \lambda_\nu$

$$\left.\begin{array}{l} \mathcal{O}_Z \longrightarrow S^r(\oplus^r \mathcal{E}^\nu) \otimes \lambda_\nu \\ \quad \| \\ \mathcal{O}_Z \rightleftarrows (\otimes^r \mathcal{E}^\nu) \otimes \lambda_\nu \end{array}\right\} \begin{array}{l} \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(D) \\ \text{natural splitting} \end{array}$$

Cor. If $\mathcal{L}$ inv. on Z & $\pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}}(\Delta)$ ample

$\Delta \geq |D|$, $\Rightarrow \mathcal{L}$ ample.

Idea of pf (simple exercise, in fact):

$$\begin{tikzcd} H^0(\mathbb{P}, \pi^*(\mathcal{L}^\eta)(\eta\Delta)) \arrow[r, two heads] \arrow[d, shift left] & H^0(\mathbb{P}_p, \pi^*\mathcal{L}^\eta \otimes \mathcal{O}(\eta\Delta)) \arrow[d, shift left] \\ H^0(Z, \mathcal{L}^\eta) \arrow[r, two heads] \arrow[u, shift left] & H^0(p, \cdots) = k \arrow[u, shift left] \end{tikzcd} \quad \Rightarrow \text{b.p.f.}$$

similarly for sep. 2 pts. tangents etc. //

Blow up $\mathbb{P}' \xrightarrow{\sigma} \mathbb{P}^M$

ample $\rightarrow \pi'^* \lambda_\nu^{-\alpha} \otimes \lambda_{\nu\mu}^{\beta} (\Delta')$

by ③ since s.p + ample = ample. get

$$\pi'^* \lambda_\nu^{\alpha'} \otimes \lambda_{\nu\mu}^{\beta} (\Delta' + \underbrace{(\alpha' - \alpha)}_{> |D'|} D') \; ; \quad \alpha' \gg 0 \quad \text{ample}$$

$\Rightarrow \lambda_\nu^{\alpha'} \otimes \lambda_{\nu\mu}^{\beta}$ ample on Z. done.

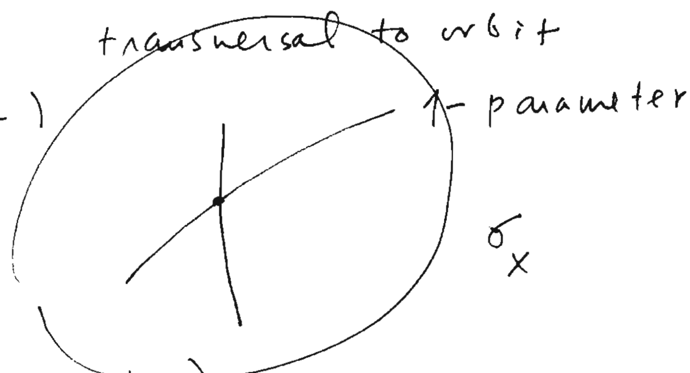
((sometimes) $\Rightarrow \lambda_\nu$ is ample on Z already)

cor. $\widetilde{M}_h$ exists with ample sheaf induced by λ_ν.
 – normalization of M_h

For people experienced in GIT, the above is "Hilbert - Mumford criterion":

H.M. $\rightsquigarrow \Delta' \geq \mathbb{P}' - V$

(but with diff ample sheaf)



Remark:

Similar argument shows that $H = H^s(\lambda_\nu)$ (stable points)

hence $\Rightarrow M_h$ exists.

§ Shafarevich Problem:

Up to now,

A). $\tilde{M}_h$ exists, λ_ν ample

B). $f: X \to B$ non-isotrivial, F surface (curve) of general type, or can. polarized mfd.

$$\Rightarrow \deg f_* \omega^\nu_{X/B} > 0 \quad (\nu \geq 2,\ f_* \omega^\nu_{X/B} \neq 0)$$

(for $n > 2$ use MMP to do boundedness)

c) $S^{r(\nu)e(\nu)}(f_* \omega^\nu_{X/B}) \otimes N^{-1}$ ample for $\deg N < \deg(f_* \omega^\nu_{X/B})$. $\quad r(\nu) = \mathrm{rk}\, f_* \omega^\nu_{X/B}$.

Today, remains to show

Theorem (Bedulev, —): F general type, assume $\Phi_{|\omega^\nu_F|}: F \longrightarrow \mathbb{P}$ has at most 1-dim'l fibers ($\nu \gg 0$) & non-isotrivial

$s = \# S$, f smooth over $B_0 = B - S$.

a) (Migliorini, Kovacs, Qi Zhang) $\longleftrightarrow$ (II)

$$2g(B) - 2 + s = \deg \omega_B(s) > 0$$

b) f semi-stable:

$$n(2g(B) - 2 + s) \cdot \nu \cdot e(\nu) \cdot r(\nu) \geq \deg(f_* \omega^\nu_{X/B})$$

c) f not semi-stable $\longrightarrow$.

Proof: a) If < 0, add points to S

$\deg \omega_B(S) = 0 \rightsquigarrow f$ smooth / $\underset{k^*}{E}$

so may assume f semi-stable (via covering)

hence a) is a special case of b).

b). Assume bound does not hold:

use property c). $A = \omega_B(s)^n$,

$S^{er}(f_* \omega^{\nu}_{X/B}) \otimes A^{-er\nu}$ ample

$A = \omega_B(s)^{n-m}$, $m \geq 0$ is even better.

$\Rightarrow \omega_{X/B} \otimes f^* A$ is 1-ample wrt. $X_0 = f^{-1}(B_0)$

Def: $\mathcal{L}$ inv / X, proj. $X_0 \subseteq X$, $\Gamma = X - X_0$,

i) $\mathcal{L}$ is semi-ample wrt $X_0 \Leftrightarrow$ for some $\eta > 0$

$$\iota_\eta : H^0(X, \mathcal{L}^\eta) \otimes \mathcal{O}_X \longrightarrow \mathcal{L}^\eta \text{ surj on } X_0.$$

ii) $\mathcal{L}$ 1-ample wrt $X_0 \Leftrightarrow$ for some η in i)

the induced map $\phi_\eta : X_0 \twoheadrightarrow V \subseteq \mathbb{P}(H^0(X, \mathcal{L}^\eta))$

(induced by ι_η) is proper, birational, &

$$\dim \phi_\eta^{-1}(v) \leq 1, \quad v \in V$$

this is old notation, new notation is $\Omega^1_X(\log)$

Σ_1, $0 \to f^* \omega_B(s) \to \Omega^1_X \langle \Gamma \rangle \longrightarrow \Omega^1_{X/B} \langle \Gamma \rangle \longrightarrow 0$

$\qquad \qquad \omega_B(s) = \Omega^1_B(\log S)$

$n=2$: $0 \to f^* \omega_B(s) \otimes \Omega^1_{X/B}\langle\Gamma\rangle \to \Omega^2_X \longrightarrow \Omega^2_{X/B}\langle\Gamma\rangle \to 0$

$\qquad \qquad \Omega^2_{X/B}\langle\Gamma\rangle = \omega_{X/B}$

motivation:

$$\begin{tikzcd} \otimes \omega_{X/B} : f_* \omega_{X/B}^2 \arrow[r] \arrow[dr] & R^1 f_* \left(\omega_{X/B} \otimes \Omega^1_{X/B} \langle \Gamma \rangle \right) \otimes \omega_B(S) \arrow[d, "\text{Kodaira - Spencer theory}"] \\ & \left(R^2 f_* \omega_{X/B} \right) \otimes \omega_B(S)^2 \end{tikzcd}$$

want to show $\neq 0$. but even if use Torelli thm + KS theory, still can not conclude it.

Actual way to prove this: via vanishing thm:

$\mathcal{L}$ 1-ample, Assume V in ii) allows projective morphism $V \xrightarrow{\gamma} W$, W affine, then

$\Rightarrow \exists$ blowing up $\tau : X' \to X$ centers in Γ such that $\Gamma' = \tau^* \Gamma$ NCD.

$\exists \; 0 \leq \Sigma \leq m\Gamma'$ with

$$H^q(X', \Omega^p_{X'} \langle \Gamma' \rangle \otimes \tau^* \mathcal{L}^{-1} \otimes \mathcal{O}_{X'}(\Sigma)) = 0, \quad p+q < \dim X.$$

Rmk: If $\mathcal{L}$ ample. this is Nakano-Akizuki-Kodaira.

(See eg. LN H. Esnault, —, DMV-Lecture notes).

$\Sigma^\bullet_m := \Lambda^m \Sigma_1$, tautological sequence

$$0 \to f'^* \omega_B(S) \otimes \Omega^{m-1}_{X'/B} \to \Omega^m_{X'} \langle \Gamma' \rangle \to \Omega^m_{X'/B} \langle \Gamma' \rangle \to 0$$

$\Sigma^\bullet_m \otimes \tau^* \mathcal{L}^{-1}(\Sigma)$: $H^{n-m}(\text{Middle term}) = 0 \Rightarrow$

$$H^{n-m}(\Omega^m_{X'/B} \langle \Gamma' \rangle \otimes \tau^* \mathcal{L}^{-1}(\Sigma)) \hookrightarrow H^{n-m+1}(\Omega^{m-1}_{X'/B} \langle \Gamma' \rangle \otimes \tau^* \mathcal{L}^{-1} \otimes f^* \omega_B(S)).$$

choose $\mathcal{L}_m = \omega_{X/B} \otimes f^*\omega_B(S)^{n-m}$

call $\mathcal{L}_{m-1}^{-1}(\Sigma) = \tau^*\mathcal{L}_m^{-1}(\Sigma) \otimes f^*\omega_B(S)$.

$m=0$: $\quad H^n(\tau^*\mathcal{L}_0^{-1}(\Sigma)) = 0 \quad$ by iteration of inclusion

$$\cup\!|$$

$m=n$: $\quad H^0\left(\Omega^n_{X/B}\langle P'\rangle \otimes \tau^*\omega_{X/B}^{-1} \otimes \mathcal{O}(\Sigma)\right) \neq 0$

$$\underbrace{\Omega^n_{X/B}\langle P'\rangle \otimes \tau^*\omega_{X/B}^{-1}}_{\hookrightarrow\, \mathcal{O}(E) \quad E \geq 0}, \qquad \tau^*\omega_{X/B}^{-1} \supseteq \Omega^n_{X/B}\langle \Gamma\rangle^{-1}$$

since $f: X \to B$ is semi-stable

a contradiction. $\square$.

About Vanishing Theorem : Idea :

X proj. $\mathcal{L}$ inv. $\quad \mathcal{L}^N = \mathcal{O}(D)$, D NCD

$\mathcal{L}^{(1)} := \mathcal{L}\left(-\left[\frac{D}{N}\right]\right)$, Then

i) $\mathcal{L}^{(1)-1}$ has a flat connection

$\nabla : \mathcal{L}^{(1)-1} \longrightarrow \mathcal{L}^{(1)-1} \otimes \Omega^1_X\langle D\rangle \qquad$ (comes from cyclic cover)

ii) $H^p(\Omega^q_X\langle D\rangle \otimes \mathcal{L}^{(1)-1}) \Rightarrow H^{p+q}(\Omega^\bullet_X\langle D\rangle \otimes \mathcal{L}^{(1)-1})$

degenerate at E_1 term.

iii) Residues along components of $D \notin \mathbb{Z}$

($\Leftarrow D = \sum a_i D_i$, $N \nmid a_i$) & $\mathbb{V} = \ker(\nabla|_{X_0 = X - |D|})$

$\Rightarrow H^{p+q}(\Omega^\bullet_X\langle D\rangle \otimes \mathcal{L}^{(1)-1}) = 0$.

iv) So, $H^p(\Omega^q_X\langle D\rangle \otimes \mathcal{L}^{(1)-1}) = 0$ for $p+q < \text{c.d.}(X-D)$.

c.d. $(X-D) = \dim X$ if $X-D$ affine

or if $\exists\ X-D \xrightarrow[\Phi]{} W$, $\Phi^{-1}(w) \leq 1$

birat'l.

Idea of the pf:

l.f. nec.; blow up st. $\mathcal{L}^{\eta}(-\Sigma_1)$ globally generated with $\Sigma_1 > 0$.

May assume that $|\Sigma| = P' = (\tau^* P)_{red.}$ and

η & multiplicities, as in Kawamata's talk, can be achieved by perturbation a little bit.

Induced: $X' \twoheadrightarrow Z \supset V$ over V fiber dim ≤ 1

$\Rightarrow D = H + \Sigma_1$ satisfies the assumption, $X' - D \to V - V \cap H$.

$\to$ get Vanishing for

$$\Omega^q_X \langle H + \Gamma' \rangle \otimes \mathcal{L}^{-1} \otimes \mathcal{O}\left(\left[\frac{\Sigma_1}{\eta}\right]\right)$$

$$\Omega^q_{X'}\langle \Gamma' \rangle \to \Omega^q_{X'}\langle H + \Gamma' \rangle \longrightarrow \Omega^{q-1}_H \langle \Gamma'|_H \rangle \longrightarrow 0$$

+ induction on dimensions. //.

Final Remarks:

let $T^1_{X/B}\langle -\Gamma \rangle := \left(\Omega^1_{X/B}\langle \Gamma \rangle\right)^\vee$.

$$0 \to T^1_{X/B}\langle \Gamma \rangle \longrightarrow T^1_X\langle \Gamma \rangle \longrightarrow f^* T^1_B \langle S \rangle \longrightarrow 0$$

$$T^1_B\langle S \rangle \to R^1 f_* T^1_{X/B}\langle \Gamma \rangle$$

not yet use 1-dim'l fiber.

for moduli scheme $\underset{\text{finite}}{B \to M_h}$, $\underset{\text{univ. family}}{X = \mathfrak{X}}$

should get $T^1_B\langle S\rangle \xrightarrow{\sim} R^1 f_* T^1_{X/B}\langle \Gamma\rangle$

dualize: $R^{n-1} f_*\left(\Omega^1_{X/B}\langle \Gamma\rangle \otimes \omega_{X/B}\right) \xrightarrow{\sim} \Omega^1_B\langle S\rangle$

For $n=1$ (family of curves):

$$\underset{\text{ample}}{f_* \omega^2_{X/B}} \xrightarrow{\sim} \Omega^1_B\langle S\rangle$$

(at least proven for curves)

$f_*\omega^2_{X/B}$ nef & $\underset{\text{ample}}{\mathcal{A}} \subset \underset{\cong \text{ over } B-S}{S^2(f_* \omega^2_{X/B})}$

So may say $f_*\omega^2 X/B$ ample wrt $(B-S)$.

$\Rightarrow \Omega^1_B\langle S\rangle$ ample wrt $(B-S)$

$\Rightarrow B-S \not\supset E$ (elliptic curve) or $k^\times$.

May this be general situation for $n \geq 2$?

Problem: Positivities for $\Omega^1_{M_h}\langle M_h - M_h^\circ\rangle$
for larger class of moduli.

Remark: $\Omega^1_{M_h}\langle M_h - M_h^\circ\rangle$? semi-positive &

$(Y, Y_0) \subseteq (M_h, M_h^\circ) \underset{?}{\rightsquigarrow} \Omega_Y^{\dim Y}\langle Y - Y_0\rangle$

ample wrt Y_0.

<u>END.</u>