Orbifold GW th

Tom Coates lect 1 6/17 TIMS

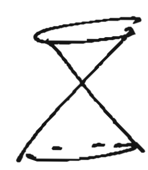
plan * orbifold
* orbifold GW inv
* crepant resol conj
* prove CRC in some examples
  some new joint w Bryan-Iritani

What is an orbifold ?
cpx mfds are locally modelled on $\mathbb{C}^n$.
str of a $\mathbb{C}$ mfd on X, I given
$\{U_\alpha \mid \alpha \in I\}$ open cover of X
$\{\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}^n \mid \alpha \in I\}$ homeo for open sets
st $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ analytic

First point of view. (Chen-Ruan)
intuitive, but slightly dangerous
X to have a covering by open sets U_α
Open sets $V_\alpha \subset \mathbb{C}^n$, finite gps G_α acting on V_α
homeomorphisms φ_α $V_\alpha/G_\alpha \longrightarrow U_\alpha$ st
the cover is closed under taking finite intersections
and whenever $U_\alpha \subset U_\beta$, $\exists$ inj gp homo
$\rho_{\alpha\beta}$ $G_\alpha \to G_\beta$, and a G_α-equiv. hol. embedding
$V_\alpha \to V_\beta$ st

$$\begin{tikzcd} V_\alpha \arrow[r, hook] \arrow[d] & V_\beta \arrow[d] \\ V_\alpha/G_\alpha \arrow[r] \arrow[d, "\varphi_\alpha"'] & V_\beta/G_\beta \arrow[d, "\varphi_\beta"] \\ U_\alpha \arrow[r, hook] & U_\beta \end{tikzcd}$$

commutes



example: $\mathbb{C}^2/\mu_2$ where μ_2 acts with weight $(1,1)$
As a variety $\mathbb{C}^2/\mu_2 = \operatorname{Spec} \mathbb{C}[x,y]^{\mu_2} = \operatorname{Spec} \mathbb{C}[x^2, xy, y^2]$
$= \operatorname{Spec} \mathbb{C}[a,b,c]/(b^2 - ac)$

$\mathbb{C}^2/\mu_2$ is singular as a variety, but smooth as an orbifold.

example: weighted proj space $\mathbb{P}(w_0, w_1, \dots, w_n)$

This is $(\mathbb{C}^{n+1} \setminus \{0\})/\sim$ where $\begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} \sim \begin{pmatrix} \lambda^{w_0} x_0 \\ \\ \lambda^{w_n} x_n \end{pmatrix}$
$\lambda \in \mathbb{C}^*$

for example $\mathbb{P}(1,2,3)$ have 3 charts

$U_x = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x \neq 0 \right\} / \sim \; = [1 ; y : z] \cong \mathbb{C}^2$

$U_y = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} . \; y \neq 0 \right\} / \sim \; = (x : 1 : z) \cong \mathbb{C}^2/\mu_2$

now have choices (stabilizer) — acting w weights $(1,1)$

$U_z = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : z \neq 0 \right\} / \sim \; = (x : y : 1) \simeq \mathbb{C}^2/\mu_3$ wt $(1,2)$

Need extra conditions on gluing maps:
want descent to work properly, i.e. I want to be able to define maps of orbifolds, or sheaves on orbifolds, or v.b. on orbifolds etc. by defining them locally (on U_α) and checking that they respect gluing maps. So need cocycle conditions on gluing maps to avoid pathologies ($\to$ Massey!)

2nd point of view: $\searrow$ harder to work more algebraic-geometric

Orbifolds as Deligne-Mumford stacks

Precisely, a category fibered in groupoids s.t. isomorphisms form a sheaf, descent data are effective, and with an étale representable surjection from a scheme

Ref: Stacks for everybody: Fantechi
Int. th on Alg. Stacks (appendix) Vistoli

example let X be a scheme

consider a cat $\mathcal{X}$ with obj are $S \to X$ (S scheme)

morphism in $\mathcal{X}$ = pull backs

("functor of points of X")

$\mathcal{X}$ maps to $\underline{Sch}$ $((S \to X) \mapsto S)$

For other $B \in \underline{Sch}$ is $\mathcal{X}(B) = \{\text{morphism } B \to X\}$
with identity morphism only

To specify a Stack we define $\mathcal{X}(B)$ for each scheme B, and think of $\mathcal{X}(B)$ as $\mathrm{Hom}(B, \mathcal{X})$

(isom form a sheaf
descent data are effective) — can define morphism locally and check they respect gluing maps

can define orbifold locally and check that they glue

example G = gp

$\mathcal{X} = BG$ stack of principal G-bundles

$\mathcal{X}(B)$ = principal G bundle on B
morphism are isomorphisms of G-bundles

$$\begin{array}{c} P \\ \downarrow \\ B \end{array} \quad B \in \underline{Sch}$$

This is a new example since
BG is NOT a scheme unless G = trivial gp

$pt \to BG$ = principal G bundle on pt
just one of "them" but $\mathrm{Aut} = G$

but if
$pt \to \mathcal{X}$ = point of X, with $\mathrm{Aut} = \{id\}$
"stack asso to scheme X"

example. fiber product, given $\mathcal{X} \xrightarrow{f} \mathcal{Z}$, $\mathcal{Y} \xrightarrow{g} \mathcal{Z}$

define stack $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ by

$$\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}(B) = \{(x, y, \alpha) : x \in \mathcal{X}(B),\ y \in \mathcal{Y}(B),\ \alpha \in \mathrm{Hom}_{\mathcal{Z}}(f(x), g(y))\}$$

Most important example.

X scheme

G = alg gp acting on X

The stack $\mathcal{X} = [X/G]$ has objects

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[loop above, phantom, "\circlearrowleft G"] \\ B & \end{tikzcd}$$

where $P \to B$ is a principal G bundle
$P \to X$ is a G-equiv map.

example: $[pt/G] = BG$

example: $X = \mathbb{C}^{n+1} \setminus \{0\}$
$G = \mathbb{C}^\times$ acting w wt $(1, \dots, 1)$

then $\mathcal{X} = [X/G]$ has $\mathcal{X}(B)$

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[loop above, phantom, "\circlearrowleft G"] \\ B & \end{tikzcd}$$

ie. $\mathcal{X}(B) = \{$ line bundles $L \to B$ together with "$\mathbb{C}$" base-point-free sections of $L^\vee \}$

$= \mathrm{Hom}_{\underline{Sch}}(B, \mathbb{P}^n)$ (Hartshorne II.7.1)

ie. $\mathcal{X}$ is $\mathbb{P}^n$.

By the way, changing wt gives moduli interpretation of weighted proj spaces!

Def^{n}. A morphism of stacks $f : \mathcal{X} \to \mathcal{Y}$ is representable iff for every morphism $S \to \mathcal{Y}$ from a scheme S, we have that

$$\begin{tikzcd} S \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ S \arrow[r] & \mathcal{Y} \end{tikzcd}$$

the fiber product $S \times_{\mathcal{Y}} \mathcal{X}$ is a scheme

Exercise. If f is representable then f is injective on automorphism gps

Def^{n}. A representable map $f : \mathcal{X} \to \mathcal{Y}$ is étale, proper, etc iff for every morphism $S \to \mathcal{Y}$ with S a scheme the induced map $S \times_{\mathcal{Y}} \mathcal{X} \to S$ of schemes is étale, proper, etc...

example $\mathcal{X} = [X/G]$

$$\begin{tikzcd} G \times X \arrow[r, "\text{action}"'] \arrow[d, "\text{proj}"'] & X \\ X & \end{tikzcd}$$

is an obj of $\mathcal{X}(X)$

i.e. is a morphism $X \to \mathcal{X}$

Now, given any scheme S, a map $S \to [X/G]$ is

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\circlearrowleft G"'] & X \\ S & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \boxed{\;} \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r] & {[X/G]} \end{tikzcd}$$

Exercise: the fiber product is P

so, $X \to [X/G]$ is representable

it is étale iff $P \to S$ is, i.e. $\Leftrightarrow$ G is finite

- Sheaves, vector bundles etc, on orbifolds

given an orbifold $\mathcal{X}$ and an open cover $\{S_i \to \mathcal{X}\}$ of $\mathcal{X}$ by schemes S_i, a vector bundle on $\mathcal{X}$ is a vector bundle E_i on each S_i, together with

$$\begin{tikzcd} S_i \times_{\mathcal{X}} S_j \arrow[r, "\varphi_i"] \arrow[d, "\varphi_j"] & S_i \leftarrow E_i \arrow[d] \\ E_j \to S_j \arrow[r] & \mathcal{X} \end{tikzcd}$$

isomorphism $\varphi_i^* E_i \xrightarrow{\sim} \varphi_j^* E_j$ satisfying cocycle conditions

example what are vector bundles on $[X/G]$?

now only one "open set", so to give a v.b on $[X/G]$

$$\begin{tikzcd} G \times X \arrow[r, "\text{action}"] \arrow[d, "\text{proj}"'] & X \arrow[d] & E \arrow[l] \\ X \arrow[r] & {[X/G]} & \end{tikzcd}$$

I need to give a v.b $E \to X$ and a G-action on E

while coh of orbifold is harder (next time)

but K-th is easy by above, it's just

$$K^\cdot([X/G]) \cong K^\cdot_G(X)$$

Third point of view

to be continued

$\to$ Exercise $\mathbb{P}(2,2)$ like $\mathbb{P}^1$, but every pt has isotropy gp μ_2

$\mathbb{P}^1 \times B\mu_2$ like $\mathbb{P}^1$, also every pt has isotropy gp μ_2, but

$\mathbb{P}(2,2) \not\cong \mathbb{P}^1 \times B\mu_2$

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Example: Does there exist a morphism $\mathbb{P}(2,2) \to \mathbb{P}^1$?

Yes, one way to const $[X/G] \to [Y/H]$ is to const $\rho : G \to H$ and a G-equivariant map $X \to Y$

In this case, $\mathbb{P}(2,2) = [\mathbb{C}^2 \setminus \{0\} / \mathbb{C}^\times]$ wt $(2,2)$

$\mathbb{P}^1 = [\mathbb{C}^2 \setminus \{0\} / \mathbb{C}^\times]$ wt $(1,1)$

so $\mathbb{C}^2 \setminus \{0\} \xrightarrow{id} \mathbb{C}^2 \setminus \{0\}$, $\mathbb{C}^\times \xrightarrow{\lambda \mapsto \lambda^2} \mathbb{C}^\times$ will do

Get $\mathbb{P}(2,2) \longrightarrow \mathbb{P}^1$ but not reverse!

This is an example of the coarse moduli space of an orbifold

Def'n: A scheme X is a coarse moduli space for an orbifold $\mathfrak{X}$ ~~iff~~ $\exists$ morphism $\mathfrak{X} \to X$ and X is universal among schemes with this property.

Coarse moduli space of orbifolds exists,
(Keel - Mori, Conrad's webpage for modern pf.)

example: The coarse moduli space of $[\mathbb{C}^2/\mu_2]$ is the variety $\mathbb{C}^2/\mu_2$.

A map $B \to [\mathbb{C}^2/\mu_2]$ is

$$\begin{array}{ccc} P & \xrightarrow[\circlearrowleft \mu_2]{} & \mathbb{C}^2 \\ \downarrow & & \\ B & & \end{array}$$

principal bundle

so we get $\mathcal{O}_{\mathbb{C}^2} \to \mathcal{O}_P$

hence $(\mathcal{O}_{\mathbb{C}^2})^{\mu_2} \to (\mathcal{O}_P)^{\mu_2}$

ie $\mathcal{O}_{\mathbb{C}^2/\mu_2} \to \mathcal{O}_B$, hence get $B \to \mathbb{C}^2/\mu_2$

Thus $[\mathbb{C}^2/\mu_2] \to \mathbb{C}^2/\mu_2$

There is a canonical isomorphism

$$H^\cdot(\mathfrak{X}; \mathbb{Q}) \xrightarrow{\sim} H^\cdot(X, \mathbb{Q})$$

[Abramovich - Graber - Vistoli]

just think of de Rham coh using diff forms of orbifold, everything goes through.

p. 2 Recap Orbifolds are locally $\mathbb{C}^n/G$, G finite

example : $\mathbb{P}(w_0, w_1, \cdots, w_n)$

BG

$[X/G]$

coarse moduli space, isotropy gp, how to give maps between orbifolds, and slightly subtlety $\mathbb{P}(2,2) \longrightarrow \mathbb{P}^1$ $\nexists$

But, WHY ORBIFOLDS ?

They arise in many places, esp in quotient construction

M sm space

$G \circlearrowright M$ $\quad M/G$ is often an orbifold if $\exists$ finite stabilizer

example $M_{g,n} = \{(\Sigma; x_1, \cdots, x_n)\}/\sim$

Birational geometry : Ruan pioneered the idea GW theory $\leftrightarrow$ birat'l geom [see eg. Lee-Lin-Wang]

GW theory : First, recall things about GW th of sm proj varieties.

X = variety

$X_{g,n,d}$ = moduli stack of stable maps to X from n-pointed genus g curve of deg $d \in H_2(X; \mathbb{Z})$

Have $ev_i : X_{g,n,d} \longrightarrow X \quad i = 1, 2, \cdots, n$

evaluation maps. Given $\alpha_1, \cdots, \alpha_n \in H^{\cdot}(X)$,

$$\langle \alpha_1, \cdots, \alpha_n \rangle^X_{g,n,d} = \int_{[X_{g,n,d}]^{vir}} \prod_{k=1}^{n} ev_k^* \alpha_k$$

"virtual count" of genus g deg d curve in X constrained to pass cycles PD ~~to~~ to $\alpha_1, \cdots, \alpha_n$.

$L_i \to X_{g,n,d}$ universal cotangent line bundle
$i = 1, \cdots, n$

$$L_i \big|_{[(\Sigma, x_1, \cdots, x_n, f : \Sigma \to X)]} = T^*_{x_i} \Sigma$$

$\psi_i := c_1(L_i) \in H^2(X_{g,n,d})$

$\langle \alpha_1 \psi_1^{i_1}, \dots, \alpha_n \psi_n^{i_n} \rangle^X_{g,n,d} = \int_{[X_{g,n,d}]^{vir}} \prod_{k=1}^{n} ev_k^*(\alpha_k).\psi_k^{i_k}$

gravitational descendents
(which also constraint the cx str of the source curve) via ψ_1

Correlators form a CohFT. (simply means there are MANY relations between all these inv.)

In particular,

$$\alpha_1 * \alpha_2 = \sum_{d \in H_2(X,\mathbb{Z})} q^d \langle \alpha_1, \alpha_2, \phi^\varepsilon \rangle^X_{0,3,d} \, \phi_\varepsilon$$

← summation convention on ε

in $\mathbb{C}[H_2(X,\mathbb{Z})]$

concretely: pick a basis $\beta_1, \dots, \beta_r$ for $H_2(X,\mathbb{Z})$, $d = d_1\beta_1 + \dots + d_r\beta_r$ for $d_i \in \mathbb{Z}$

$q^d = q_1^{d_1} \cdots q_r^{d_r}$ (indeed assume)

Here $\phi_1, \dots, \phi_N$ basis for $H^\cdot(X, \mathbb{C})$

dual basis $\phi^1, \dots, \phi^N$ so $\int_X \phi_i \cup \phi^j = \delta_i^j$.

over $\Lambda = \mathbb{C}[[q_1, \dots, q_r]]$.

The small quantum product on $H^\cdot(X;\Lambda)$ is graded commutative (i.e. $\mathbb{Z}_2$) and associative ← not so obvious WDVV eq'n

Big quantum product:

$t \in H^\cdot(X)$,
$$\alpha_1 *_t \alpha_2 = \sum_{d \in H_2(X,\mathbb{Z})} \sum_n \frac{q^d}{n!} \langle \alpha_1, \alpha_2, \underbrace{t, \dots, t}_{n}, \phi^\varepsilon \rangle^X_{0,n+3,d} \, \phi_\varepsilon$$

is associative

A less familiar perspective: associativity of the big QC etc (WDVV eq'ns) is equiv. to the solvability (consistency) of the system of PDE's

$$z \frac{\partial}{\partial t^\alpha} \blacksquare = (\phi_\alpha *_t) \blacksquare$$

$\forall z$

$\alpha = 1, 2, \dots N$
$t = t^1\phi_1 + \dots + t^N\phi_N$
$\blacksquare \in H^\cdot(X)$
z = parameter

In fact, can write down the solutions

p.4

$$z \frac{\partial}{\partial t^\alpha}\frac{\partial}{\partial t^\beta} J_X = (\phi_\alpha *_t)^\gamma_\beta \frac{\partial}{\partial t^\gamma} J_X$$

where $J_X = z + t + \sum \frac{q^d}{n!} \langle \underbrace{t,\dots,t}_{n}, \frac{\phi^\varepsilon}{z-\psi_{n+1}} \rangle^X_{0,n+1,d}\, \phi_\varepsilon$

$\in H(X)[z, z^{-1}]]$

$$= \sum_{k\geq 0} \phi^\varepsilon \frac{\psi_{n+1}^k}{z^{k+1}}$$

This is Givental's J function

Knowing $J_X \longleftrightarrow$ Knowing big QC

Can often compute J_X using mirror symmetry

pure in string: fermion 1-pt function

• tree level of Baker-Akhiezer function.

This is GW for sm manifolds.

GW of orbifolds:

Def^{n}: Let $\mathcal{X}$ be an orbifold, the "inertial stack" $I\mathcal{X}$ has

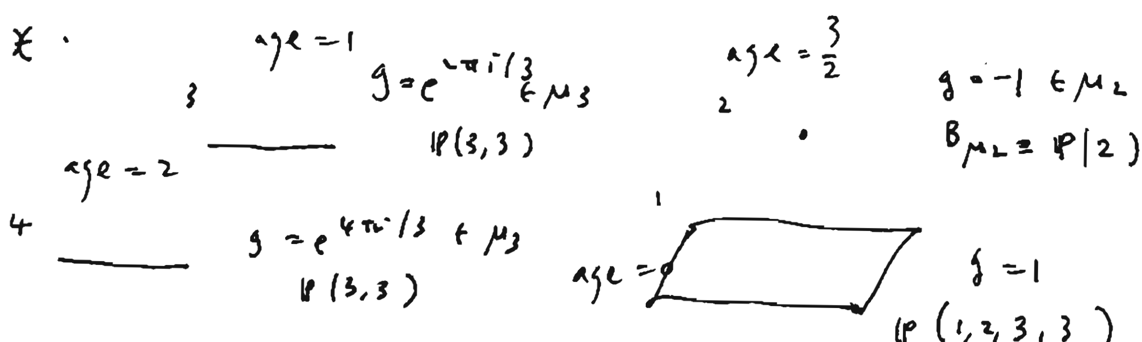
$I\mathcal{X}(B) = \{(x,g) \mid x\in \mathcal{X}(B),\ g\in \mathrm{Aut}_{\mathcal{X}}(x)\}$

with morphisms $(x,g)\to(x',g')$ given by

$x \xrightarrow{h} x'$ with $g' = hgh^{-1}$

Example $\mathcal{X} = \mathbb{P}(1,2,3,3)$

$I\mathcal{X}$:



Involution $I: I\mathcal{X}\to I\mathcal{X}$

sending (x,g) to (x,g^{-1}).

possible stabilizers are in μ_2 or μ_3

in above example it fixes 1, 2, switches 3, 4

The orbifold coh. or Chen-Ruan coh of $\mathcal{X}$ is $H^\cdot(I\mathcal{X}, \mathbb{Q})$ with new product (later) new grading

Given component C of $I\mathcal{X}$, let $\pi: C \to \mathcal{X}$ be the canonical map then C determines an automorphism g let $|g| = r$

Then $\pi^* TX = \bigoplus_{j=0}^{r-1} E_j$ ~ eigenspace for g with eigenvalue $\exp(2\pi i \cdot j/r)$

$$\mathrm{age}(C) := \sum_{j=0}^{r-1} \frac{j}{r} \dim E_j$$

shift the grading on $H^\cdot(C) \subset H^\cdot(I\mathcal{X})$ by age.

in the example, age $= 0 \quad \frac{3}{2} \quad 1 \quad 2$
for level $1 \quad 2 \quad 3 \quad 4$

under the shift (convolution) of $3 \leftrightarrow 4$ will related to <u>Strong Lefschetz property</u> (later)
$:\overset{\text{def}}{=}$ if the involution preserves age

why $H^\cdot_{CR}$?
It is the target for a Riemann-Roch (<u>chern character</u>) isomorphism

$$\mathbb{C} \otimes K^\cdot([X/G]) = K^\cdot_G(X) \otimes \mathbb{C}$$
$$= \bigoplus_{\substack{\text{conjugacy} \\ \text{class } [g]}} K^\cdot(X^g / Z_G(g))$$

by Atiyah-Segal ← ordinary one is too small

Exercise:
$$I[X/G] = \coprod_{\substack{\text{conj. class} \\ [g]}} [X^g / Z_G(g)]$$

More convincing: Semi-infinite S^1-equivariant coh of $LX \longleftrightarrow$ GW

$(LX)^{S_1} = X$ ok for sm.
↖ constant loops

but $(L\mathcal{X})^{S_1} = I\mathcal{X}$ (e.g. lovely work of Thaddeus "Floer coh". or CCLT)

6

The QC for orbifold goes the same way
just replace evaluation map to $I(\mathcal{X})$

$$ev_1 : \mathcal{X}_{g,n,d} \longrightarrow I(\mathcal{X})$$

stable map from orbifold curve to $\mathcal{X}$ under some condit.

$$\alpha *_t \beta = \sum_{d\in\Lambda} \sum_{n\geq 0} \frac{q^d}{n!} \langle \alpha, \beta, t, \dots, t, \phi^{\varepsilon} \rangle^{\mathcal{X}}_{0,n+3,d}\, \phi_{\varepsilon}$$

family of alg str on $H^{\cdot}_{CR}(X) \otimes \Lambda$

$t \in H^{\cdot}_{CR}(\mathcal{X})$

$\deg(q^d) = c_1(TX)\, d$ — the shifting of degree exactly match this

Chen-Ruan (symplectic)
Abramovich-Graber-Vistoli (algebraic)

Chen-Ruan product: Put $t=0=q$ in $*_t$
Get new product on $H^{\cdot}(I\mathcal{X};\mathbb{Q}) \cong H_{CR}(\mathcal{X})$
Different from cup product
this agree with CR product $\Leftrightarrow$ Hard Lefschetz property holds
(related to primitive coh cond. in Kähler)

Ruan's crepant resolution conj.
Vague principle from string th. String th on an orbifold is equiv to string th on a crepant resol.

$$\begin{tikzcd} Y \arrow[dr, "\pi"'] & & \mathcal{X} \arrow[dl] \\ & X & \end{tikzcd}$$

a crepant resol.

coarse moduli space of $\mathcal{X}$

How are $QC(Y)$ and $QC(\mathcal{X})$ related?

(Yasuda) $H^{\cdot}(Y) \cong H_{CR}(\mathcal{X})$ as graded v.s.
age shifted — in particular if Y exists then age $\in \mathbb{N}$

but Y has more quantum parameters:
$H^2(\mathcal{X}) \subset H^2_{CR}(\mathcal{X}) \cong H^2(Y)$

continue to lect 3

cont'd of CRC.

pick a basis $\underbrace{\beta_1, \dots, \beta_s}_{\text{basis for } H_2(X)}, \underbrace{\beta_{s+1}, \dots, \beta_r}_{\text{basis for } \ker \pi_*}$ for $H_2(Y)$

define quantum parameters $q_1, \dots, q_r$ on Y

Ruan: $QH^*(Y) \simeq QH^*_{CR}(\mathcal{X})$ after analytic continuation in $q_{s+1}, \dots, q_r$ and setting

$q_i = \omega_i \quad i = s+1, \dots, r, \quad \omega_i = \text{roots of unity}$

isom $\simeq$ as what ??

Ruan: isom as Frobenius algebras

Bryan-Graber: as Frobenius manifolds (and need Hard Lefschetz)

Next: A new version of CRC (Coates-Iritani-Tseng; Ruan)

almost $\Rightarrow$ Ruan (need semi-positivity, "quantum corrected" version of Ruan)

$\Rightarrow$ Bryan-Graber under HL.

explains: how Frobenius mfd str's on $QC(\mathcal{X})$ and $QC(Y)$ are related (NOT in general isom) actually after Birkhoff factorization (later)

Givental's symplectic formalism:

$\mathcal{H} := H_{CR}(\mathcal{X}) \otimes \mathbb{C}[z, z^{-1}]]$ coh.-valued Laurent series

$\Omega(f, g) = \operatorname{Res}_{z=0} (f(-z) \cdot g(z))\, dz$ sympl form

$\mathcal{H}_+ = H^{\cdot}_{CR}(\mathcal{X}) \otimes \mathbb{C}[z]$ isotropic for Ω, same as

$\mathcal{H}_- = z^{-1} H_{CR}(\mathcal{X}) \otimes \mathbb{C}[[z^{-1}]]$ max Lagrangian isotropic subspaces.

$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \simeq T^*\mathcal{H}_+$

Over basis $\phi_1, \dots, \phi_N, \phi^1, \dots, \phi^N$ for $H_{CR}(\mathcal{X})$

define Darboux coordinates $q_{i,\alpha}, p_{j,\beta}$

$$\mathcal{H} \ni \sum_{i,\alpha} q_{i,\alpha} \phi_\alpha z^i + \sum_{j,\beta} \frac{p_{j,\beta} \phi^\beta}{(-z)^{j+1}}$$

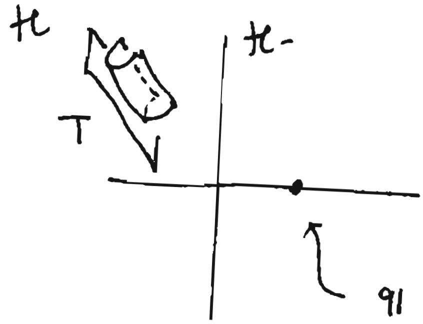
p.2 Put $q = q_0 + q_1 z + q_2 z^2 + \dots$ $\quad q_i \in H^{\cdot}_{CR}(\mathcal{X})$

generating functions:

$$\mathcal{F}^g_{\mathcal{X}}(t_0, t_1, t_2, \dots) \qquad t_i \in H^{\cdot}_{CR}(\mathcal{X})$$

$$= \sum_{n \geq 0} \sum_{d \in \Lambda} \sum_{i_1, \dots, i_n} \frac{q^d}{n!} \langle t_{i_1} \psi^{i_1}, \dots, t_{i_n} \psi^{i_n} \rangle^{\mathcal{X}}_{g,n,d}$$

knowing $\mathcal{F}^g_{\mathcal{X}}$ $\leftrightarrow$ knowing all genus g gravitational descendants of $\mathcal{X}$



$\mathcal{H}_+ = H^{\cdot}_{CR}(\mathcal{X}) \otimes \mathbb{C}[z]$

$q = q_0 + q_1 z + q_2 z^2 + \dots$

Set $q_i = \begin{cases} t_i & i \neq 1 \\ t_i - \mathbb{1} & i = 1 \end{cases}$, $\mathbb{1}$ = unit in $H^{\cdot}_{CR}(\mathcal{X})$

(Dilaton eq'n ; $\mathcal{F}^g_{\mathcal{X}}$ is homogeneous in q_i's)

Then $\mathcal{F}^g_{\mathcal{X}}$ becomes a function on $\mathcal{H}_+$

Set $\mathcal{L}$ = <u>graph of</u> $d\mathcal{F}^0_{\mathcal{X}} \subset T^*\mathcal{H}_+ \cong \mathcal{H}$

concretely $p_{i,\alpha} = \dfrac{\partial \mathcal{F}^0_{\mathcal{X}}}{\partial q_{i,\alpha}}$

Theorem (Coates-Givental, Tseng)

$\mathcal{L}$ is the germ of a Lagrangian cone with vertex at the origin st. each tangent space T to $\mathcal{L}_{\mathcal{X}}$ satisfying $\underline{T \cap \mathcal{L} = zT}$

This is geometric reformulation of

$\left\{ \begin{array}{l} \text{String eq'n} \\ \text{Dilaton eq'n} \\ \text{Topological recursion relation} \end{array} \right.$ i.e. axiomatic structure of $g=0$ GW theory (Givental)

The cone is very "fat" $T \cap \mathcal{L} = zT \Rightarrow T/zT \cong \mathcal{H}_+/z\mathcal{H}_+$

- Thm. $\exists$ $\mathbb{C}[z]$-linear symp transf $\mathbb{U}: \mathcal{H}_{\mathcal{X}} \to \mathcal{H}_{\mathcal{Y}}$ $\cong H^{\cdot}_{CR}(\mathcal{X})$ analytic continuation $\mathbb{U}(\mathcal{L}_{\mathcal{X}}) = \mathcal{L}_{\mathcal{Y}}$

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Aside · consider

$$\sum_{n\geq 0} \frac{\Gamma(a+2n)}{\Gamma(1+n)^2} x^n \qquad a \in \mathbb{C} \text{ parameter}$$

radius of conv. $= 1/2$

how can I analytically conti these to $x = \infty$?

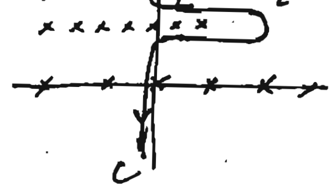
(Barnes 1908)

write as $= \sum_{n\geq 0} \mathrm{Res}_{s=n} \underbrace{\Gamma(s)\Gamma(1-s)}_{\frac{\pi}{\sin \pi s}} \frac{\Gamma(a+2s)}{\Gamma(1+s)^2} (-x)^s$

$\frac{\pi}{\sin \pi s}$ poles at $s \in \mathbb{Z}$ res $= \pm 1$

$$\star \quad = \int_C \underbrace{\Gamma(s)\Gamma(1-s) \frac{\Gamma(a+2s)}{\Gamma(1+s)^2}}_{\text{poles}} x^{-s}$$

poles $\to$ $a+2s = -k$, $k \geq 0$, $s = \frac{-k}{2} - \frac{a}{2}$



Barnes showed, exists and is analytic for all x

$|x| < \frac{1}{2}$, close contour to R get this

$|x| > \frac{1}{2}$, close contour to L,

$$\int_C \boxed{} \underset{|x|>\frac{1}{2}}{=} \cancel{\sum_{n\geq 0} \mathrm{Res}_{s=-1-n} \boxed{}} + \sum_{k\geq 0} \mathrm{Res}_{s=\frac{-k}{2}-\frac{a}{2}} \boxed{}$$

$$\parallel$$

$$\sum_{k\geq 0} \frac{\pi}{\sin \pi\left[\frac{k+a}{2}\right]} \frac{(-1)^k}{2\Gamma(1+k)} \frac{(-x)^{-a/2-\frac{k}{2}}}{\Gamma(1-\frac{a}{2}-\frac{k}{2})}$$

this is an explicit HG fun defined near $x = \infty$

- analytic conti of hypergeom fun is not so hard

Notice that $\star$ is globally defined analytic fun which specializes to a power series in $|x| < \frac{1}{2}$, $|x| > \frac{1}{2}$ respectively.

- $\mathbb{C} \subset \mathbb{P}(1,2) \supset \mathbb{C}/\mu_2$, "$x^{-1/2}$", the analytic conti know the orbifold str ! the $1/2$ will show up

where were we ? recall

$$\begin{array}{ccc} \tilde{X} & & Y \\ & \searrow \; X \; \swarrow^{\pi} & \end{array}$$

crepant resolution

$\Omega(f,g) = \mathrm{Res}_{z=0} (f(-z), g(z))\, dz$

in $\mathcal{H} \simeq T^* \mathcal{H}_+$

.2 convergence assumption :

$$J_Y^0 = \sum \frac{q^d}{n!} \langle t_{i_1}\varphi_1^{i_1}, \dots, t_{i_n}\varphi_n^{i_n} \rangle^Y_{0,n,d}$$

$q_1^{d_1}\cdots q_r^{d_r}$ where $d = d_1\beta_1 + \dots + d_r\beta_r$

$\in H_2(Y)$ where $\beta_1, \dots, \beta_r$ is basis for $H_2(Y)$

$\beta_{s+1}, \dots, \beta_r$ is a basis for $\ker \pi_*$

assume convergence in $q_{s+1}, \dots, q_r$

" radius of convergence in q_i is >1 for $s < i \le r$ "

recall $\beta_1, \dots, \beta_s$ gives a basis for $H_2(X)$ → but is the actual β+ only mod > 0.

⇝ need some coefficient ring

$\Lambda = \mathbb{C}[[q_1, \dots, q_s]]$ for $\mathcal{X}$ and Y.

for additional quantum parameter just set $q_i = 1$. $s+1 \le i \le r$

conjecture (Coates-Iritani-Tseng, Ruan)

$\exists$ $\mathbb{C}[z]$-linear, grading preserving, symplectic morphism $\mathbb{U} : \mathcal{H}_X \to \mathcal{H}_Y$ st, after analytic continuation,

$\mathbb{U}(\mathcal{L}_{\mathcal{X}}) = \mathcal{L}_Y$, Further

(a) $\mathbb{U}(1_{\mathcal{X}}) = 1_Y + O(z^{-1})$

(b) $\mathbb{U}(\rho^{\cup}_{CR}) = ((\pi^*\rho)\cup) \circ \mathbb{U}$ for all $\rho \in H^2(\mathcal{X}) \subset H^2_{CR}(\mathcal{X})$

(c) $\mathbb{U}(\mathcal{H}_{\mathcal{X}}^+) \oplus \mathcal{H}_Y^- = \mathcal{H}_Y$

(d) the matrix entries of $\mathbb{U}$ wrt our basis

$\phi_1, \dots, \phi_N$ for $H_{CR}(\mathcal{X})$

$\varphi_1, \dots, \varphi_N$ for $H^\bullet(Y)$

(which a priori are elements of $\Lambda[z, z^{-1}]$) are in fact elements of $\mathbb{C}[z, z^{-1}]$.

explain (a) fix scaling of cone

(b) is for monodromy compatibility

(c) makes the "same open set" possible as $\mathcal{X}$ and Y

(d) matrix of ~~numbers~~ (not Λ) since $\deg z = 2$ and its degree preserving !!

<u>special case</u>

$\mathbb{U}$ maps $\mathcal{H}_{\mathcal{X}}^{-}$ to $\mathcal{H}_{Y}^{-}$, then

$\mathbb{U} = u_0 + u_1 z^{-1} + \cdots + u_K z^{-K}$ where

$u_i : H_{CR}^{\cdot}(\mathcal{X}) \to H^{\cdot}(Y)$ is $\mathbb{C}$-linear, where

(i) u_0 is grading preserving

(ii) $u_0(\mathbb{1}_{\mathcal{X}}) = \mathbb{1}_Y$

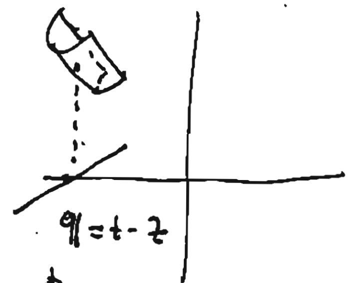
(iii) for $\rho \in H^4(\mathcal{X})$, $u_0 \cdot (\rho^{\cup}_{CR}) = (\pi^* \rho \cup) \cdot u_0$

(iv) u_0 intertwines the Poincaré pairing on $H_{CR}^{\cdot}(\mathcal{X})$ and $H^{\cdot}(Y)$.

(cf the old version of CRC)

CRC completely determines the relationship between $GW_0(\mathcal{X})$ and $GW_0(Y)$, so it implies a relationship between $QC(\mathcal{X})$ and $QC(Y)$. What is it?

- Turn off: $\underline{t_0 = t}$, $t_1 = t_2 = \cdots = 0$
  gravitational descendents, i.e. at $q = t - z$



unique point of the form

$-z + t + O(\frac{1}{z})$, this is

$-z + t + \sum \frac{q^d}{n!} \langle t, \overbrace{\cdots}^{n}, t, \frac{\phi^\epsilon}{-z - \psi_{n+1}} \rangle^{\mathcal{X}}_{0, n+1, d} \phi_\epsilon$

ie $= J_{\mathcal{X}}(t, -z)$

$z \frac{\partial}{\partial t^\alpha} \frac{\partial}{\partial t^\beta} J_{\mathcal{X}} = (\phi_\alpha *_t)^{\gamma}_{\beta} \frac{\partial}{\partial t^\gamma} J_{\mathcal{X}}$, $J_{\mathcal{X}} \rightsquigarrow QC(\mathcal{X})$

$\mathcal{L} \cap (-z + z\mathcal{H}_-) = \{ J_{\mathcal{X}}(t, -z) \cdot t \in H_{CR}^{\cdot}(\mathcal{X}) \}$

(another fancy way to write J)

CASE 1: $\mathbb{U}(\mathcal{H}_{\mathcal{X}}^{-}) = \mathcal{H}_{Y}^{-}$ $\leftarrow$ (always happens if $\mathcal{X}$ satisfies Hard Lefschetz: CIT Thm 5.14)

CASE 2: $\mathbb{U}(\mathcal{H}_{\mathcal{X}}^{-}) \neq \mathcal{H}_{Y}^{-}$

for case 1 $\mathbb{U}(\mathcal{L}_{\mathcal{X}} \cap (-z + z\mathcal{H}_-^{\mathcal{X}})) = \mathcal{L}_Y \cap (-z + z\mathcal{H}_-^{Y})$

$\Rightarrow$ $\mathbb{U} J_{\mathcal{X}}(\tau, -z) = J_Y(t(\tau), -z)$ for some $\tau \to t(\tau)$

p. 4 Also, $J_{\mathcal{X}} = -z + \tau + O(z^{-1})$

$J_{Y} = -z + t(\tau) + O(z^{-1})$

so $\mathbb{U} J_{\mathcal{X}} = -z + u_0(\tau) + c + O(z^{-1})$

where $\mathbb{U}(1_{\mathcal{X}}) = 1_Y - cz^{-1} + O(z^{-1})$

so $t(\tau) = u_0(\tau) + c$

So, $H^{\cdot}_{CR}(\mathcal{X}) \to H^{\cdot}(Y)$ intertwines $*^{\mathcal{X}}_{\tau} \leftrightarrow *^{Y}_{u_0(\tau)+c}$

if $c \neq 0$ then done.

in general $\deg c = 2$, Div eq'n : if $c \in H^2(Y)$ then

$*_{t+c} = *_t$ after $q_1 \mapsto e^{c_1} q_1, \cdots, q_r \mapsto e^{c_r} q_r$

where $c = c_1 \beta_1^{\vee} + \cdots + c_r \beta_r^{\vee}$

this gives an isomorphism after setting

$q_1 = e^{c_1} q_1, \cdots, q_s = e^{c_s} q_s$

$q_{s+1} = e^{c_{s+1}}, \cdots, q_r = e^{c_r}$.

if $c_i \in 2\pi i \mathbb{Q}$, these are roots of unity !

In general (case 2), it's not easy to say in this language

small QC algebras

If $\mathcal{X}$ is semi-positive, $\exists\, u_0(q) : H^{\cdot}_{CR}(\mathcal{X}) \to H^{\cdot}(Y)$

isomorphism of Frobenius alg's between small QC of $\mathcal{X}$, Y.

after setting $q_i = e^{c_i} q_i (1 + O(q))$ $i = 1 \ldots s$

$q_i = e^{c_i} (1 + O(q))$ $i = s+1, \ldots, r$

Frobenius mflds of $QC(\mathcal{X})$ and $QC(Y)$ are NOT isomorphic ! but we know the precise relation.

$$\underset{\substack{\uparrow \\ \text{positive } z}}{\mathbb{U}}\, S_{\mathcal{X}}\underset{\substack{\uparrow \\ \text{negative } z}}{(\tau, -z)}^{*} = \underbrace{S_Y(t(\tau), -z)^{*} R(\tau, z)}$$

Birkhoff factorization ! (∞-dim'l linear algebra)

determines $S_Y(t(\tau), -z)^*$.

$u_0(q) = R|_{\tau = z = 0}$.

How to prove CRC for toric examples?

Key ingredient: Mirror symmetry

$\mathfrak{X} \longleftrightarrow$ mirror variety $\check{\mathfrak{X}}$

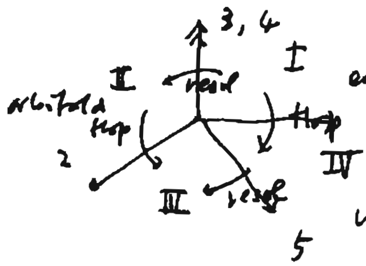
QC($\mathfrak{X}$) VHS($\check{\mathfrak{X}}$)

diff eq'n for QC($\mathfrak{X}$) Picard-Fuchs eq'n

A concrete example.

Toric variety: GIT quotient (or Cox ring constr.

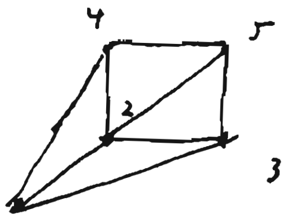
$\mathbb{C}^5/(\mathbb{C}^\times)^2$, $(\mathbb{C}^\times)^2 \curvearrowright \mathbb{C}^5$ $\Phi : \begin{pmatrix} 1 & 0 \\ -2 & -1 \\ 0 & 1 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}$



only chambers gives different varieties.

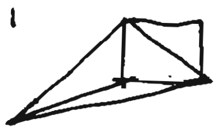
choose ξ in I

view at height 1 in $\mathbb{R}^3$:



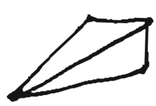
(2 corr to origin)
this is $K_{\mathbb{F}_1}$

ξ in IV



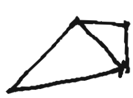
$\mathcal{O}(-1) \oplus \mathcal{O}(-3)$
$\downarrow$
$\mathbb{P}(2,2)$

ξ in II:



which is

ξ in III:



which is $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$
$\downarrow$
$\mathbb{P}(1,3)$

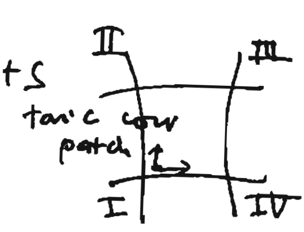
I $\to$ II satisfies HL, but IV $\to$ III does not!

So we learn something from this diag.

Want VHS on M_B: $\mathbb{C}$-str moduli space of mirror $\check{\mathfrak{X}}$

M_B = toric var corr to secondary fan

4 cones $\longleftrightarrow$ 4 toric fixed points



p. 6

$$I_{X_1} = \boxed{} \sum_{k,l \geq 0} \frac{q_1^k q_2^l}{\Gamma\left(1+\frac{p_1}{z}+k\right)\Gamma\left(1+\frac{-2p_1-2p_2}{z} - 2k - l\right)\Gamma\left(1+\frac{p_2}{z}+l\right)^2 \Gamma\left(\boxed{}\right)}$$

$1 + \frac{p_1 - p_2}{z} + k - l$

$\in H^*(X_1) \otimes \mathbb{C}[z, z^{-1}]]$

$= \mathcal{H}_{X_1}$

the pattern is completely det by the matrix

similarly, I get 4 HG functions in 4 charts:

MS ⇒ $I_{X_1} \in \mathcal{L}_{X_1}$

All satisfy the same differential eq'ns

$\tilde{I}_{X_1}$ a.c. of I_{X_1} compare to I_{X_4} → gives basis $\left(\vdots\right)$

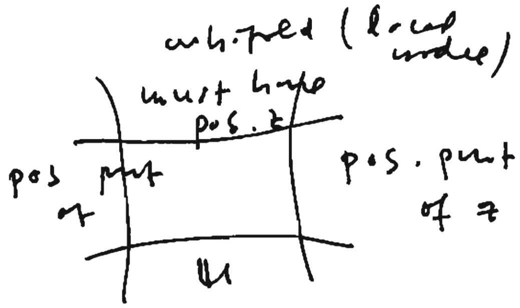
$\Rightarrow \tilde{I}_{X_1} = \left(\because\right)\left(\vdots\right)$

↑ linear z, but NO q

i.e. exactly the sympl transf $\mathbb{U}$

$\mathbb{U}$ for I → IV (flop case)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\pi^2}{3z^2} & \frac{i\pi}{z} & 0 & 1 \end{pmatrix}.$$



this explains the Ruan's paper discovery.

↑

End

has integral entries in terms of K-th basis of Iritani.

(integral str. of QC)