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NCTS Summer School in Alg Geom

Hyperkähler mfds.

examples : • X - K3 $(4) \subset \mathbb{P}^3$, $(2,3) \subset \mathbb{P}^4$

$\omega^2 = f_6(x_0 : x_1 : x_2)$

• $S^2X = X \times X / S_2$ - sing along Δ

each surface slice is a A_1 - sing

$\mathbb{C}^2 / (x,y) \mapsto (-x,-y) \hookrightarrow \mathbb{C}^3 \quad w^2 = xy$

$\omega = dt_1 \wedge dt_2 - dw_1 \wedge dw_2$ (one part unchanged)

$\widetilde{(S^2X)}$ = blow up in Δ = hyperkähler

• $\widetilde{X \times X \cdots \times X / S_n}$ in a special way of blow-up

indeed, use Hilbert scheme of pts, on

$\dim X = 2$ it's a thm saying $\mathrm{Hilb}_n X$ is sm.

• $\mathbb{C}^2/\Lambda \quad x \mapsto -x$, $(\mathbb{C}^2/\Lambda)/\mathbb{Z}_2$ - Kummer surface

16 sing pts, order 2 in $\mathbb{C}^2/\Lambda$ $\quad = \widetilde{(\mathbb{C}^2/\Lambda)/\mathbb{Z}_2}$

Beauville's construction :

$S^n(A) \xrightarrow{\pi} A \qquad \widetilde{\pi^{-1}(+)}$ = hyperkähler mfd.

$(x_1, \cdots, x_n) \mapsto \Sigma x_i$

moduli of HK mfds :

Def'n : marked HK $\gamma_1, \cdots, \gamma_{b_2}$ basis in $H^2(N,\mathbb{Z})/\mathrm{tor}$

period map : $p : \{\text{marked HK}\} \to \mathbb{P}(H^2(N,\mathbb{Z}) \otimes \mathbb{C})$

$(N, \{\gamma_i\}) \mapsto (\cdots, \int_{\gamma_i} \omega_N, \cdots)$

equiv : γ_i^* dual basis then $\omega_N = \Sigma \left(\int_{\gamma_i} \omega_N\right) \gamma_i^*$.

Image of p : for K3, $\int_N \omega \wedge \omega = 0$, $\int_N \omega \wedge \bar{\omega} > 0$

p. 2 in terms of Poincaré duality,

get $\Leftrightarrow (c_1 \cdots c_{b_2}) (\langle \gamma_i, \gamma_j \rangle) \begin{pmatrix} c_1 \\ \vdots \\ c_b \end{pmatrix} = 0$ quadric in $\mathbb{P}^n$

and $\vec{c} (\langle \gamma_i, \gamma_j \rangle) \bar{\vec{c}}^t > 0$ (2 open sets)

For higher dim case, one way to do this is by picking a polarization (fixed) class $L \in H^2(N, \mathbb{Z})$ and $p : (N; L; (\gamma_1, \cdots, \gamma_{b_2})) \mapsto \mathbb{P}(H_0^2(N, \mathbb{Z}) \otimes \mathbb{C})^{\cap H^{11}(N)}$

primitive st $L^{2n-1} \alpha = 0$ (exclude L part)

$\langle \alpha, \beta \rangle = \int \alpha \wedge \beta \wedge L^{2n-2}$

and then $(\omega_N, \omega_N) = 0$, $\langle \omega_N, \bar{\omega}_N \rangle > 0$ easily.

the drawback is that we need to fix L

To put all HK together (not fixing L), we need to use Bogomolov-Beauville form on $H^2(N, \mathbb{Z})$.

$$B(\alpha) = \frac{n}{2} \int \alpha \wedge \alpha \wedge \omega_N^{n-1} \wedge \bar{\omega}_N^{n-1} - (1-n) \cdot \int \alpha \wedge \alpha \wedge \omega_N^{n} \wedge \bar{\omega}_N^{n-2} \cdot \int \alpha \wedge \alpha \wedge \bar{\omega}_N^{n} \wedge \omega_N^{n-2}.$$

Theorem: $\exists\, c$ st $cB(\alpha)$ is defined over $\mathbb{Z}$

B nondegenerate of signature $(3, b_2 - 3)$ $\#$

What we hope that is true:

moduli of marked HK is described by

$B(\omega, \omega) = 0$, $B(\omega, \bar{\omega}) > 0$ //

Some linear algebra:

assumption $\mathbb{R}^{3,p}$ or $\mathbb{R}^{2,p}$ (with quad form)

$\left\{ \begin{array}{c} \text{oriented 3-dim} \\ \text{positive space} \end{array} \right\} \cong SO_0(3,p) / SO(3) \times SO(p)$

similarly for $\mathbb{R}^{2,p}$. more is needed:

Thm $SO_0(2,p) / (SO(2) \times SO(p)) \cong \mathbb{R}^{1,p-1} + i V^+$

$V := \{ v \in \mathbb{R}^{1,p-1} \mid \langle v, v \rangle > 0 \} = V^+ \cup (-V^+)$.

Lemma : $SO_0(2,p)/SO(2)\times SO(p)$

$\hookrightarrow \mathbb{P}(\mathbb{R}^{2,p}\otimes\mathbb{C})$, as $\langle w,w\rangle = 0 = \langle w,\bar{w}\rangle$

(just repr a cpx line by 2 real vectors)

Lemma : $\mathbb{R}^{1,p-1} + iV^{+} \leftrightarrow \{ w \in \mathbb{P}(\mathbb{R}^{2,p}\otimes\mathbb{C})$

$\langle w,w\rangle = 0,\ \langle w,\bar{w}\rangle > 0$

$$\overset{\Psi}{v} \rightsquigarrow \left(v, -\frac{\langle v,v\rangle}{2}, 1\right) \in (\mathbb{R}^{1,p-1}\oplus\mathbb{R}^{1,1})\otimes\mathbb{C}.$$

This will be important later for Mirr. sym. (flat coor.

I. moduli space of $(N, L, \gamma_1, \cdots, \gamma_{b_2})$ marked polarized HK manifolds exists as a non-singular complex mfd $\mathcal{T}_{m,p}$, dim $= h^{1,1}-1$,

$\exists\ \mathcal{X} \to \mathcal{T}_{m,p}$ family. it has the following properties : $\mathfrak{X} \to U$ be a family then $\exists\ \varphi : U \to \mathcal{T}_{m,p}$ st $\mathfrak{X} \to U$ = pull back of $\mathcal{X}$, φ is defined up to automorphism $\Theta : N \to N$, $\Theta_* = id$ on $H^2(N,\mathbb{Z})$.

II. injectivity of period map after polarized :

$$p : \mathcal{T} \hookrightarrow SO_2(2, b_2-3)/SO(2)\times SO(b_2-3).$$

$H_0^2(N,\mathbb{R}) \ni \alpha \Rightarrow -\int \alpha\wedge\alpha\wedge L^{n-2} > 0$
since $*\alpha = -c\,\alpha\wedge L^{2n-2}$
for some constant c,

Take the Bogomolov-Beauville form
$q(\alpha) \equiv c\,B(\alpha)$.

$p((N;\gamma_1\cdots\gamma_{b_2})) \in \mathcal{P} = \left\{\begin{array}{l} B(u,u)=0 \\ B(u,\bar u)>0\end{array}\right. \cong \dfrac{SO_0(3,b_2-3)}{SO(2)\times SO_0(1,b_2-3)}$

however this gp to be quotient is not cpt; and $\mathcal{P}$ is only homogeneous

Will actually consider
$p(N;L;\gamma_1,\ldots,\gamma_{b_2}) \in \mathfrak{h}_{2,b_2-3} \in \mathbb{P}(H_0^2(X,\mathbb{Z})\otimes\mathbb{C})$.
Using Hilbert scheme, get

$\mathcal{X}_L' \to \mathcal{H}ilb(N,L)$ smooth
pass to univ. cover and quotient again
then get $\tilde{\mathcal{X}}_L \to \mathcal{T}_L(N) \leftarrow$ Teichmuller family.
Will show:

- $p : \mathcal{T}_L(N) \longrightarrow \mathfrak{h}_{2,b_2-3}$ embedding (Torelli)
- $\mathfrak{h}_{2,b_2-3} \setminus p(\mathcal{T}_H(N)) = \bigcup_{i=1}^{\infty} \mathcal{H}_i$ — for K3 these can be described explicitly.

Willmore: Isometric deformation.
Th (Yau). In $[L]$ can be realized as imaginary part of Ricci flat Kähler metric : i.e.

$$\partial\bar\partial \log\det(g_{i\bar j}) = 0 \Leftrightarrow \det(g_{i\bar j})|_U = |f|^2.$$

Bochner Principle : (N, g-K-Y metric)
φ holomorphic global tensor on $N \Rightarrow \nabla\varphi = 0$
$\nabla\omega_N = 0$, $\nabla\bar\omega_N = 0$, $\nabla\,\mathrm{Im}\,g = 0$.

p.5 $(\mathrm{Re}\,\omega_N, \mathrm{Im}\,\omega_N, \mathrm{Im}\,g) = (I, J, K)$
are of length 1. $I^2 = J^2 = K^2 = -id$, $IJ = K \ldots$
quaternion structure

$\sum (w_i)_{\alpha\beta}\, dx^\alpha \wedge dz^\beta \to \sum (w_i)^\mu_\alpha \frac{\partial}{\partial x^\mu} \otimes dx^\alpha$

$(w_i)^\mu_\alpha = \sum_\beta w_{\alpha\beta}\, g^{\beta\mu}$.

get a (twistor family): $\begin{matrix}\mathcal{X}\\ \downarrow\\ S^2\end{matrix}$

$E \subset H^2(N, \mathbb{R})$ span by $(\mathrm{Re}\,w, \mathrm{Im}\,w, L)$,

$E_2' \subset E$: E_2 two dim plane $e_1 \vee e_2$

$(e_1 + i e_2) = \omega^2_{N_{E_1}}$, $e_3 \perp E_2 \to I_{e_3}$ cpx str

(This is due to Calabi) since it is parallel

Notice that the family $\mathcal{X} \to S^2$ is C^∞-trivial.

Lemma: $\omega_{N_{E_1}}$ - holomorphic form wrt I_{e_3}

* $p : \begin{matrix}\mathcal{X}\\ \downarrow\\ \mathbb{CP}^1\end{matrix} \to SO_0(3, b_2 - 3)/SO(2) \times SO_0(1, b_2 - 3)$

notice $SO(3)/SO(2) = S^2$

and each embedding of $\mathfrak{h}_{2, b_2-3}$ to $\mathcal{P}$ corr to choice of the twistor family.

$p : \sigma_L(N) \to \mathcal{P}$ 1. this is étale

D^* 2. proper (via Grothendieck's valuative criterion)

3. and of degree 1.

2 is proved via isometric deformations.

Viehweg : $\mathcal{M}_L(N) \hookrightarrow \Gamma_L \backslash \mathfrak{h}_{2, b_2 - 3}$

Γ = mapping class gp

$= \mathrm{Diff}^+(N)/\mathrm{Diff}_0(N)$; $\Gamma_L = \{\varphi \in \Gamma \mid \varphi(L) = L\}$

eg. for torus, Γ is gen by Dehn twist and involution ι.

Baily-Borel compactification :

eg. for torus, $\mathbb{PSL}(2,\mathbb{Z}) \backslash \hat{\mathfrak{h}} = \mathbb{CP}^1$

$y^2 = x(x-1)(x-t)$, $\lambda = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ for $\bigcirc \to \bigcirc$

notice the rat'l pts (cusp) are precisely fixed pts under λ ~~and~~ by ~~its~~ conjugates.

now $\quad \Gamma_e \backslash \hat{\mathfrak{h}}_{2, b_2-3} \quad ; \quad H^2_0(N, \mathbb{Z}) \quad (2, b_2-3)$

$\qquad \gamma_0 \in H^2_0(N,\mathbb{Z})$ primitive, $B(\gamma_0, \gamma_0) = 0$

<u>no torsion</u>

Lemma: The monodromy transformation $\exists\, T \in \mathrm{Aut}(\Lambda_0 \otimes \mathbb{Q})$, $(T - id)^3 = 0$, $(T - id)^2 \neq 0$.

recall Grothendieck's thm : $(T^r - id)^{n+1} = 0$.

$$\begin{array}{ccccc} \mathcal{X} & \supset & \mathcal{X}_0 \text{ singular}, & & \mathcal{X}^\times \\ \downarrow & & \downarrow & & \downarrow \\ D & \ni & 0 & & D^\times \end{array}$$

$\pi_1(D^\times)$ acts on $H^*(\mathcal{X}_t, \mathbb{Q})$.

<u>B-B</u> comp : $(\Gamma \backslash \hat{\mathfrak{h}}_{2, b_2-3}) \setminus (\Gamma \backslash \mathfrak{h}_{2, b_2-3})$

$= \Gamma_i \backslash \mathfrak{h}_i \cup (* \cdots *)$

just 1-dim'l !!

cusps (= largest radius limit pt.)

B-B is always the "minimal model".

Thm : N : HK, $\mathrm{rk}\, H^2(N, \mathbb{Z}) \geq 5$, ~~then~~ $\Longleftrightarrow$ moduli space of H-K mfds admits large radius limit.

here means in wt 2, so $n = 3$ unipotency

because of Miles' result (19'th cent) qud. form with ≥ 5 variable has a $\mathbb{Q}$-pt.

Verbitsky :

$$S^*(H^2(N, \mathbb{Q})) \hookrightarrow H^{2k}(N, \mathbb{Q})$$

$k \leq n$. pf uses representation th.

2. Bogomolov have a much shorter pf.

3. Recently Voisin prove decomp of HK H^* into abelian v. Kuga Satake □

P.7 Geometric description of
γ_0-primitive $\beta(\gamma_0)=0$.

Th. (Clemens) in a semi-stable model $\begin{matrix} X \\ \downarrow \\ D \end{matrix}$
$\exists$ retraction $h_t : X \to X_0$

1) $X_t \setminus h_t^{-1}(\mathrm{Sing}\, X_0) \cong X_0 \setminus \mathrm{Sing}\, X_0$

2) $\exists\ (\mathrm{Sing} X_0)_1 \supset (\mathrm{Sing}\, X_0)_2 \supset \cdots\cdots$ st.

$x \in (\mathrm{Sing}\, X_0)_k \setminus (\mathrm{Sing}\, X_0)_{k+1}$, $h_t^{-1}(x) = (S^1)^k$!

let $X_0 = \bigcup C_i$, then

Thm.: $H_*(C_{i_0} \cap \cdots \cap C_{i_{k-1}}) \ni [\alpha]$

$$\downarrow$$

$H_{*-2}(C_{i_0} \cap \cdots \cap C_{i_k}) \ni [\alpha] \cap (C_{i_0} \cap \cdots \cap C_{i_k})$

# of Jordan blocks of dim $= k+1$

$= \dim H_{n-k}(C_{i_0} \cap \cdots \cap C_{i_k}) / \mathrm{Im}\, \gamma$ Gysin map

(Consequence of Clemens-Schmid exact sequence)

Thm: $N \to D$ with max unipotent monodromy in $H_2(N, \mathbb{Z})$, $(T^N - id)^2 \neq 0$,

$T^N \gamma_0 = \gamma_0 \ \exists\ \gamma_0 = h_t^{-1}(\cup P_{i_0} \cup P_{i_1} \cup \cdots)$.

Corollary: $P\gamma_0 = \delta_0$ poincaré dual

$\underbrace{\delta_0 \wedge \cdots \wedge \delta_0}_{k} \neq 0$, $k \leq n$, $\underbrace{\delta_0 \wedge \cdots \wedge \delta_0}_{n+\ell} = 0$, $\ell > 0$.

May count # of zero cusps
= # of orbits of all isotropic vectors γ_0
under the action Γ by conjugation.

Automorphic Theory in K3 case :

l polarization class , l - primitive vector in $\Lambda_{K3} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^3 \oplus (-E_8)^2$, $\langle l, l \rangle = 2n$

lemma : $l^\perp = \mathbb{Z} l^* + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 + (-E_8)^2 \cong \Lambda_{K3, l}$

$\langle l^*, l^* \rangle = -2n$. $\Gamma_{l} := \mathrm{Aut}^+(\Lambda_{K3}, l)$

Discriminant locus of K3 ,

$-\Delta_l = \{ \delta \in \Lambda_{K3, l} \mid \langle \delta, \delta \rangle = -2 \}$, -2 = min value can be achieved .