Logarithmic GW th

Steffen Marcus lect 1 6/17 TIMS

§0 Motivation

X sm proj var.

$GW(X)$: (virtual) curve counting th.

↳ moduli space of stable maps

$\bar{M}_\Gamma(X)$ proper DM-stack, Discrete inv $\Gamma = g, n, \beta$

↳ evaluation

$ev_i : \bar{M}_\Gamma(X) \to E(X)$ where $E(X)$ is our evaluation space (usual X, but for orbifold or log th get "$E(X)$")

↳ $[\bar{M}_\Gamma(X)]^{vir}$ virtual class

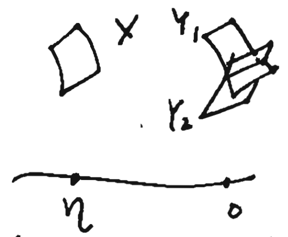
Get: $\gamma_i \in H^*(E(X))$

$$\langle \gamma_i \rangle_\Gamma = \int_{[M_\Gamma(X)]^{vir}} \prod_i ev_i^* \gamma_i$$

counts (virtually) curves in X w/ incidence conditions with $[Z_i] = \gamma_i^\vee$

Techniques: localizations
degenerations.

Degeneration $W \downarrow B$



$X_0 = Y_1 \cup_D Y_2$

D sm. Y_i sm.

Would like. $GW(X_\eta) \overset{?}{=} GW(X_0)$?

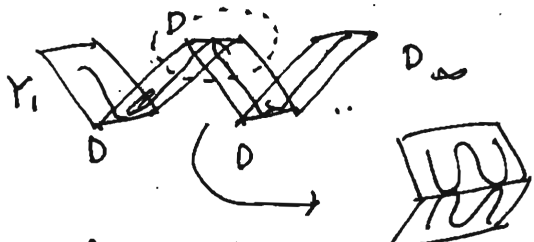
Rel $GW(Y_i/D)$ → count (virtually) curves in Y_i (relative invariants) w/ tangency conditions along D.

Degeneration formula:

"$GW(X) = \text{Rel } GW(Y_1/D) * \text{Rel } GW(Y_2/D)$"

some complicated convolution depending on "gluing" along D

p. 2 Widely used approach:
"Expanded degenerations"
A.-M. Li - Ruan (2001)
Ionel - Parker (2001/2002) } symplectic
Jun Li (2001/2002) Algebraic
Idea deform target X and curve together:
replace X_0, Y_i's by "expansions"



D_∞ tangency condition along D

pre-deformability condition

The theory is great but there are problems with this th:

- Families of expansions are non-trivial (deformation th is hard)
- Pre-deformability is locally closed

→. D sm But we want to generalize

Siebert (2001 lecture): Big idea
Use log geometry !!!

Advertisement
Main Existence theorem
(Gross - Siebert 2011, Abramovich - Chen 2011)
2 gps get exactly the same moduli space

$X = (\underline{X}, M_X)$ log variety
log sm $\underline{X}$ projective.

$M^{\log}_\Gamma(X) = (\underline{M}^{\log}_\Gamma(X), M_{\overline{M}_\Gamma(X)})$ stack of log stable map

algebraic, $\underline{M}^{\log}_\Gamma(X)$ is a proper DM stack,

has $[M^{\log}_\Gamma(X)]^{vir}$

$\exists$ Evaluation map (Abramovich - Chen - Gillam - M)

$ev_i \quad M_{\Gamma}^{log}(X) \longrightarrow \Lambda_{\mathbb{N}}(X)$

comparison thm (Abramovich, - M, - Wise)

$\underline{X}$ sm proj, D sm. then

Rel. GW(X/D) = Log GW($\underline{X}$, M_X).

Outline.

§1 Log geometry

§2 Log Smoothness

§3 Log curves

§4. Log stable maps

§5 Some statements above

In progress: gluing formula, degeneration, localization

§1 Log Geometry

Monoid: Commutative semi-gp w/unit

$(+, 0)$ or $(\cdot, 1)$ (depending on preference)

morphism of monoids $M \to N$ st $0 \mapsto 0$

will define X log scheme $(\underline{X}, M_X)$

Let $\underline{X}$ be a scheme. underlying scheme

Pre-log str Sheaf of monoids M_X log structure

M_X with a morphism "exponential"

$$\underset{(+,0)}{M_X} \xrightarrow{\alpha} \underset{(\cdot,1)}{\mathcal{O}_{\underline{X}}}$$

Morphisms:
$$\begin{tikzcd} M_X \arrow[rr] \arrow[dr, "\alpha"'] & & N_X \arrow[dl, "\beta"] \\ & \mathcal{O}_X & \end{tikzcd}$$

Call it a log structure if $\alpha^{-1}(\mathcal{O}_{\underline{X}}^*) \underset{\alpha}{\xrightarrow{\sim}} \mathcal{O}_{\underline{X}}^*$.

so $\mathcal{O}_X^* \hookrightarrow M_X$

$X = (\underline{X}, M_X)$ is a <u>log scheme</u>.

1.4 Characteristic monoid.

$\overline{M}_X := M_X / \mathcal{O}_X^*$ (i.e. $1 \to \mathcal{O}_X^* \to M_X \to \overline{M}_X \to 0$)

why ?? eg.

$$\begin{array}{lll} \underline{X} & \text{sm irred} & \supset D \quad \text{reduced SNC} \\ \downarrow & \text{dominant map} & \downarrow \qquad D = Y_1 \cup \cdots \cup Y_m \\ \underline{S} & \text{sm curve} & \ni s \end{array}$$

Then $\Omega_{\underline{X}/\underline{S}} := \Omega_X / f^*\Omega_S$ not locally free where f is singular

But, $\Omega_{\underline{X}}(\log D)$ diff forms w/ at worst log poles along Y_i's

$\quad \bigcup \leftarrow$ then get inclusion

$f^*\Omega_S(\log(s))$ and quotient is locally free!

In terms of "log D", f is "as good as" smooth

If $\underset{M_X}{\overset{m}{\top}} \overset{\alpha}{\longmapsto} x \neq 0$ in $\mathcal{O}_X$, think m gives a branch of $\log x$, will allow "$\frac{dx}{x}$"

Associated log structure

given $\alpha : M_X \to \mathcal{O}_{\underline{X}}$ a pre-log str

define $(M_X \xrightarrow{\alpha} \mathcal{O}_{\underline{X}})^a$ the associated as a push out

$$\begin{array}{ccccc} \alpha^{-1}\mathcal{O}_{\underline{X}}^* & \hookrightarrow & M_X & \xrightarrow{M_X^a} & \mathcal{O}_X \\ \downarrow & & \downarrow & & (m,u) \mapsto \alpha(m)\cdot u \\ \mathcal{O}_X^* & \longrightarrow & M_X^a & = & M_X \oplus \mathcal{O}_X^* \Big/ \{(\alpha^{-1}(p), p) \mid p \in \alpha^{-1}(\mathcal{O}_X^*)\} \end{array}$$

glue the things that need to be glued to get a log-str

Some Fundamental examples:

- $(\underline{X}, \mathcal{O}_{\underline{X}}^*)$ trivial log str
- $(\operatorname{spec} k, k^* \oplus \mathbb{N})$ standard log-point

$$\alpha : k^* \oplus \mathbb{N} \longrightarrow k$$
$$(u,n) \longmapsto u \cdot 0^n = \begin{cases} u & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Seems silly but is actually very important

Can similarly make Q log pt

w/ any "sharp" monoid Q ($Q^* = \{0\}$).

here 6/18 lect 2 continued)

More meaningful example

$\underline{X}/k$ scheme $D \in X$ Divisorial log structure

$M_{(X,D)}(U) = \{ g \in \mathcal{O}_{\underline{X}}(U) \mid g \text{ is invertible on } U \setminus D \}$

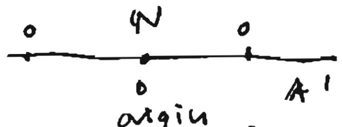
$\subseteq \mathcal{O}_{\underline{X}}(U)$

$D = V(x) \overset{i}{\hookrightarrow} \mathbb{A}^1 = \mathrm{Spec}(k[x])$

$M_{(\mathbb{A}^1, D)}$ generated by $\mathcal{O}^*_{\mathbb{A}^1}$ and $\{x\}$

$M_{(\mathbb{A}^1, D)} \longrightarrow \overline{M}_{(\mathbb{A}^1, D)} = i_* \mathbb{N}_\Delta$

$f \longmapsto$ order of vanishing at $x = 0$.



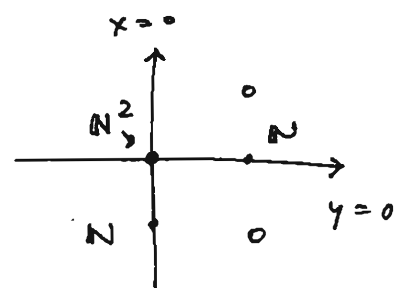
Stalks at $\overline{M}_{(X,D)} = \begin{cases} \mathbb{N} & \text{at } 0 \\ 0 & \text{otherwise} \end{cases}$

$D = V(x,y) \overset{(i)}{\subset} \mathbb{A}^2 = \mathrm{Spec}\, k[x,y]$ (i) inclusion of 2 lines resp

$M_{(\mathbb{A}^2, D)}$ gen by $\mathcal{O}^*_{\mathbb{A}^2}$ and $\{x, y\}$

$M_{(\mathbb{A}^2, D)} \longrightarrow \overline{M}(\mathbb{A}^2, D) = i_* \mathbb{N} \oplus j_* \mathbb{N}$

$f \mapsto (\mathrm{ord}_{V(x)} f, \mathrm{ord}_{V(y)} f)$



Key example:

- P a monoid, $\underline{X}$ scheme

  $P \to \Gamma(\underline{X}, \mathcal{O}_{\underline{X}})$ morphism of monoids, "P_X denote the const sheaf on X"

  get $P_X \xrightarrow{\alpha} \mathcal{O}_{\underline{X}}$ a pre-log str. get

  $\alpha^a : P^a \to \mathcal{O}_X$ associated log str

- $\hookrightarrow$ R a ring, $R[P]$ monoid algebra

  write $X = \mathrm{Spec}(P \to R[P])$

For the underlying scheme, $\underline{X} = \mathrm{Spec}(R[P])$

w. asso log $(P \to R[P])^a$

exercise: describe $\mathrm{Spec}(\mathbb{N}^n \to k[\mathbb{N}^n])$.

To describe the log category, need:

- inverse image log structure given $\underline{f}: \underline{X} \to \underline{Y}$, M_Y on $\underline{Y}$.

  define $f^* M_Y$ on X as $(f^{-1}M_Y \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_{\underline{X}})^a$.

Now, morphisms of log schemes

a map $f: X \to Y$ is $\underline{f}: \underline{X} \to \underline{Y}$ and

$f^b \quad f^* M_Y \to M_X$ over $\mathcal{O}_{\underline{X}}$

Get LogSch, a category!

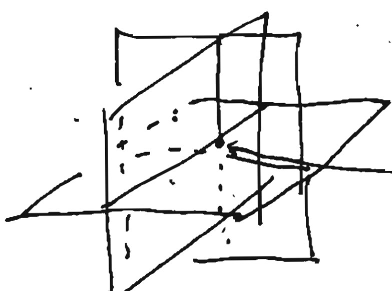
eg 1 $\quad D = V(x_1, \dots, x_r) \subset \mathbb{A}^n = \underline{X}$, M_X Divisorial log str

$i: \underline{\mathrm{Spec}\, k} \xrightarrow[\text{origin}]{} \underline{X}$

$i^{-1}M_X = M_X|_0 = \{$ germs of $u\, x_1^{n_1} \cdots x_r^{n_r} \mid n_i \geq 0$, u invertible

$\alpha: i^{-1}M_X \to i^{-1}\mathcal{O}_{\underline{X}} \to \mathcal{O}_{\underline{\mathrm{Spec}\, k}} = k$

$$u\, x_1^{n_1} \cdots x_r^{n_r} \longmapsto \begin{cases} u(0) & n = 0 \\ 0 & n > 0 \end{cases} \qquad n = \vec{n} \text{ (multi-index)}$$



assoc log str is $\mathbb{N}^r$ log pt

at origin $k^* \oplus \mathbb{N}^r$

Exercise: $\mathrm{Hom}_{\mathrm{LogSch}}((X, M_X), \mathrm{Spec}(P \to \mathbb{Z}[P]))$

$\simeq \mathrm{Hom}_{\mathrm{Monoid}}(P, \Gamma(X, M_X))$

cont'd to last 2 page.

charts a chart for P on X is :

P : monoid
$$P_X \xrightarrow{\alpha} M_X$$
P_X : constant sheaf

st. $(P_X \xrightarrow{\alpha} M_X \to \mathcal{O}_X)^a$ gives $P^a \simeq M_X$

looks alot like "$M_X \to \mathcal{O}_X$" ie

a chart for P on X

$$P_X \to M_X$$
$$\Updownarrow$$
$f : X \to \mathrm{Spec}(P \to \mathbb{Z}[P])$ where $f^b : f^* P^a \xrightarrow{\sim} M_X$

This is very suitable to think about toric varieties

A chart for $Q \to P$ on $X \to Y$ is a comm. diagram

map of monoid

$$\begin{tikzcd} Q_X \arrow[r, "\text{chart}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & f^* M_Y \arrow[d] \\ P_X \arrow[r, "\text{chart}"'] & M_X \end{tikzcd} \quad \text{on } X$$

• Caution Here "charts" may not exist at all. It only capture the part of log str which looks like const.

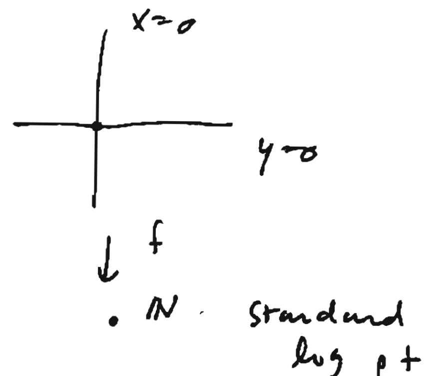
eg

node in $\mathbb{A}^2$, $X = V(xy)$, M_X is pull back of

$M(\mathbb{A}^2, V(xy))$

$$\begin{tikzcd} (n,m) \arrow[r, mapsto] & \{x^n y^m\} \\ (n,n) \in \mathbb{N}^2 \arrow[r] & k[xy]/(xy) \\ \mathbb{N} \arrow[u, "\Delta"] \arrow[r] & k \arrow[u] \\ n \arrow[u] \arrow[r, mapsto] & 0^n \end{tikzcd}$$

gives a chart on f



Exercise ; understand this example.

monoids $P \rightsquigarrow P^{gp} := \{(a,b) \mid (a,b) \sim (c,d)$ if $\exists s \in P$ st. $s+a+d = s+b+c\}$

ie. Grothendieck gp of P

integral if $P \hookrightarrow P^{gp}$

saturated : integral and $\forall p \in P^{gp}$, $np \in P \Rightarrow p \in P$

fine integral & f g

toric if fine, saturated, sharp, no torsion

free if $P \simeq \mathbb{N}^r$

A log scheme X is "$*$" if étale locally $\exists$ a chart $P \to M_X$ with P being with property "$*$"

In the following, assume all monoid are fine, saturated

Goal: log smoothness

Need: $\mathcal{D}_X^{log}$ derivations

Follow Grothendieck's formalism.

strict: $f: X \to Y$ is strict if $f^\flat: f^* M_Y \xrightarrow{\sim} M_X$

eg $\underline{X} \to \mathbb{A}^1$, $M_{(X,D)} \simeq f^* M_{(\mathbb{A}^1, 0)}$

$D = f^{-1}(0) \mapsto 0$, so f is strict.

§2. log Smoothness

Let $f: X \to Y$ be in Log Sch

must consider some lifting problem

EGA Grothendieck / (eg Hartshorne)

$$\begin{tikzcd} T_0 \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ T_1 \arrow[r] \arrow[ur, dashed, shift left, "g_1"] \arrow[ur, dashed, shift right, "g_2"'] & Y \end{tikzcd}$$

strict closed immersion

i.e. J st $J^2 = 0$ (dual number)

$g_1 - g_2$ a derivation

the problem is "in which category"?

Grothendieck's def^n of formal smoothness in EGA

(he proved formal sm $\Rightarrow$ sm)

Say $T_0 = pt$

$T_1 = T_0[\varepsilon]$

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \cdot \arrow[r] \arrow[ur, dashed, "?"] & Y \end{tikzcd}$$

lifting

lifting problem on level of $f^\# : f^* \mathcal{O}_Y \to \mathcal{O}_X$

$g_1^\# - g_2^\#$ is a scheme derivation

Now, at level of $f^\flat$, $g_1^\flat - g_2^\flat$ instead of $\#$.

conti on log-smoothness

$f: X \to Y$

log derivation : let I be an $\mathcal{O}_X$-mod

A log-derivation of X/Y to I is $(\partial, D\log)$

- $\partial : \mathcal{O}_X \to I \in Der_Y(X, I)$ ordinary derivation
- $D\log : M_X \to I$ additive map st.

  $D\log(a+b) = D\log(a) + D\log(b) \quad \forall a, b \in M_X$

  $\hookrightarrow \alpha(a) D\log(a) = \partial(\alpha(a)) \quad \forall a \in M_X$

  $\hookrightarrow D\log(a) = 0 \quad \forall a \in f^{-1}(M_Y)$

  ie $f\, d\log f = df$

log differentials $\exists$ an $\mathcal{O}_X$-module $\Omega^1_{X/Y}$ with a universal log-derivation $(d, d\log) \in Der_Y(X, \Omega^1_{X/Y})$

st $\forall I$, $\mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/Y}, I) \xrightarrow{\sim} Der_Y(X, I)$

$u \mapsto (u \circ d, u \circ d\log)$

Explicit construction

$$\Omega^1_{X/Y} = [\Omega^1_{\underline{X}/\underline{Y}} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} M_X^{gp})]/K$$

where K is generated by local sections

- $(d\alpha(a), 0) - (0, \alpha(a) \otimes a)$ $\quad a \in M_Y$
- $(0, 1 \otimes a)$ $\quad a \in Im(f^{-1}M_Y \to M_X)$

This is a big thm, but will not be proved here.

Rmk the 1st one is just $d e^{\log f} = e^{\log f} d\log f$

and just $D\log(a) = 0$ for a from the base.

Have $\Omega^1_{X/Y}$, $T_{X/Y} = \mathcal{H}(\Omega^1_{X/Y}, \mathcal{O}_X)$

log differential $\quad$ log tangent

Eg P monoid

- $X = Spec(P \to k[P])$

then $\Omega^1_{X/\mathrm{Spec}\,k} = \Omega^1_{\underline{X}/k} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} P^{gp})/K$

Exercise. $= \mathcal{O}_X \otimes P^{gp}$.

p.2 for $P = \mathbb{N}^r$, get $\Omega^1_{\mathbb{A}^r/\mathrm{Spec}\, k} = \mathcal{O}_{\mathbb{A}^r} \otimes \mathbb{Z}^r$

i.e. $d x_i = x_i (1 \otimes e_i)$

this is the fundamental example

In general, $h : P \to Q \rightsquigarrow h^{gp} : P^{gp} \to Q^{gp}$

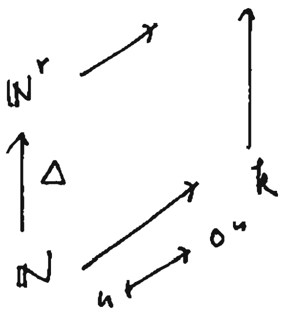
$X = \mathrm{Spec}(P \to \mathbb{Z}[P])$ get $f : X \to Y$

$Y = \mathrm{Spec}(Q \to \mathbb{Z}[Q])$

Then $\Omega^1_{X/Y} = \mathcal{O}_X \otimes \mathrm{cok}(h^{gp})$

another example

$k[x_1, \dots, x_n] / (x_1, \dots, x_r)$



Standard log pt

then $\Omega^1_{X/Y}$: generated by

$\frac{dx_1}{x_1}, \frac{dx_2}{x_2}, \dots \frac{dx_r}{x_r}, dx_{r+1}, \dots, dx_n$

with relation $\sum_{i=1}^{r} \frac{dx_i}{x_i} = 0$

Exercise: what if 0

log-smooth. $f : X \to Y$ log smooth (log étale)

if f is loc of finite presentation, and

$\exists$ a sol (nec. unique) to the lifting prob in Log Sch.

$$\begin{tikzcd} T_0 \arrow[r] \arrow[d, "J^2=0"'] & X \arrow[d, "f"] \\ T_1 \arrow[r] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

Facts: log étale $\Leftrightarrow$ log sm + $\Omega^1_{X/Y} = 0$

log smooth $\Rightarrow$ $\Omega^1_{X/Y}$ locally free

(the $\Leftarrow$ side is easy to be true under more conditions

not true even in Scheme ??)

check

There are 2 Kato's involved in the log th

Kazuya Kato's <u>structure theorem</u>

f log sm $\iff$ étale locally on X and Y, f fits into

$$\begin{tikzcd} X \arrow[r, "\text{chart}"] \arrow[d, "f"'] & \operatorname{Spec}(\mathbb{Z}[P]) \arrow[d] \\ Y \arrow[r, "\text{chart}"] & \operatorname{Spec}(\mathbb{Z}[Q]) \end{tikzcd}$$

with

1) Induced $\underline{X} \to \underline{Y} \times_{\operatorname{Spec}\mathbb{Z}[Q]} \operatorname{Spec}\mathbb{Z}[P]$ is étale

2) kernel and torsion part of cok of $Q^{gp} \to P^{gp}$ are finite gps of <u>order</u> invertible on $\underline{X}$

Log sm "means" étale locally toric (toroidal)

eg $\underline{X} = \operatorname{Spec}(P \to k[P])$

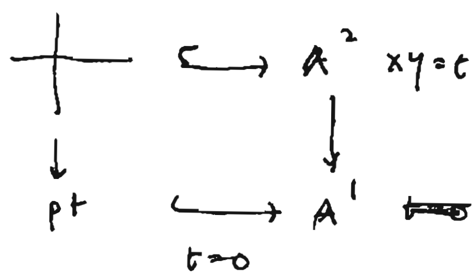
$Y = \operatorname{Spec} k$ trivial log structure

$\underline{X} \simeq \underline{Y} \times_{\operatorname{Spec}\mathbb{Z}[Q]} \operatorname{Spec}\mathbb{Z}[P]$ and X is log sm over Y even if $\underline{X}$ is not sm

eg node in $\mathbb{A}^2$, $\underline{X} = V(xy)$ chart version

$$\begin{tikzcd} \mathbb{N}^2 \arrow[r] & k[x,y]/(xy) \\ \mathbb{N} \arrow[r] \arrow[u] & k \arrow[u] \end{tikzcd}$$

$\underline{\text{Standard}}$ log pt $\rightsquigarrow$



ie. the log str. remembers its deformations (this is the content of the str. thm)

deform of nodes

Exercise: not log sm over a trivial log pt

extend thm: $\mathbb{N} \to P$, $1 \to \rho$

get $\operatorname{Spec}(P \to k[P]) \xrightarrow{f} \operatorname{Spec}(\mathbb{N} \to k[\mathbb{N}])$

log sm $\quad = X \qquad\qquad = Y$

Base change

$$\begin{tikzcd} \underline{X_0} \arrow[r, hook, "i"] \arrow[d] & \underline{X} \arrow[d, "f"] \\ \operatorname{Spec} k \arrow[r, hook] & \underline{Y} \end{tikzcd}$$

0: origin

gives $\underline{X_0}$ log str $i^* M_X$

$\operatorname{Spec} k \cdot 0^* M_Y$

get log sm $X_0 \to \operatorname{Spec} k$

• weight be reducible and non-reduced!

Warning: $\underline{f}$ sm in scheme $\Rightarrow$ $\underline{f}$ flat

but f log sm in log Sch $\not\Rightarrow$ $\underline{f}$ flat

(examples: $\mathbb{N}^2 \to \mathbb{N}^2$ $\rightsquigarrow$ $\operatorname{Spec} \mathbb{Z}[\mathbb{N}^2] \to \operatorname{Spec} \mathbb{Z}[\mathbb{N}^2]$)

maybe a wrong ex. check

$(a,b) \mapsto (a+b, b)$ log sm

not flat: of some patch of a blow up

Integral: $h: Q \to P$ integral if $\mathbb{Z}[Q] \to \mathbb{Z}[P]$ is flat, fine saturated.

$f: X \to Y$ integral if $\forall x \in X$, $h: (f^{-1}M_Y)_x \to \overline{M}_{X,x}$ is integral

Prop: f log sm and integral $\Rightarrow$ $\underline{f}$ flat.

NB Many semi-stable obj are log sm

philosophy: MODULI SPACES OF LOG SM OBJECTS (+INTEGRAL) ARE ALREADY COMPACT !!

(really log sm to mpl lg pt)

§3. Log sm curves

log scheme

Def$^{\text{n}}$: A log curve is a log sm integral $f: C \to W$ st geometric fibers are reduced, 1-dim, connected.

Q: what if we don't require fine / satu?

Ex $P = \mathbb{N} \setminus \{1\} = \langle 2, 3 \rangle$. Then $\underline{X} \cong \operatorname{Spec} k[x,y]/(y^2 - x^3)$

$X = \operatorname{Spec}(P \to k[P])$.

Fumihara Kato's Structure Theorem (the 2nd Kato)

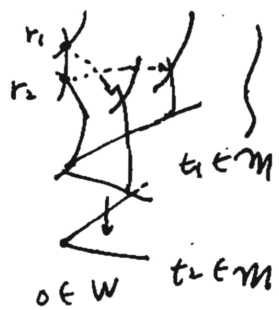
If you study a log curve $f: C \to W$, get

- singular fiber at worst nodal;
- natural divisorial log str on C (images of sections)
- natural choices $\rho \in M_W$ "smoothing" nodes of C_0 singular

Idea: local description is:

p. 2

$$\begin{array}{c} C \\ \downarrow f \\ W \end{array}$$



partial smoothings

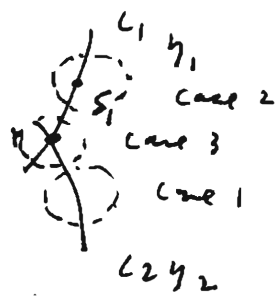
$\underline{W} = \operatorname{Spec} A$

(A, m) complete local ring

This will give natural log str but the base can have any log str, this cause a problem!

Let $Q = \overline{M}_{W,0}$ $\quad \sigma : Q \to A$ a chart on W

Then étale locally at $x \in C_0$, C is $\simeq$ to one of $V \to W$.



- $x = \eta_i$ 1). $V = \operatorname{Spec} A[z]$ W log str

gen pt $\overline{M}_{C,\eta_i} = Q \longrightarrow Q_V$

$$q \longmapsto \sigma(q)$$

- $x = s_j$ 2). $V = \operatorname{Spec} A[z]$

(section) $\overline{M}_{C,s_j} = Q \oplus \mathbb{N} \longrightarrow Q_V$

$$(q, a) \mapsto \sigma(q)\, z^a$$

- $x = r$, 3). $V = \operatorname{Spec} A[z,w]/(zw - t)$ $t \in m$

$$\overline{M}_{C,r} = Q \oplus_{\mathbb{N}} \mathbb{N}^2 \longrightarrow Q_V \ ; \ (q, (a,b)) \mapsto \sigma(q) z^a w^b$$

log str on base

$$\begin{array}{ccc} \mathbb{N}^2 \hookrightarrow Q \oplus_{\mathbb{N}} \mathbb{N}^2 & \longrightarrow & A[z,w]/zw-t \\ \uparrow \Delta & & \uparrow \\ \mathbb{N} \hookrightarrow Q & \xrightarrow{\sigma} & A \\ n \mapsto n\rho_r & \longrightarrow & t^n \end{array}$$

$(1,1) = \rho_r$

$zw = t$

are eq'ns at $Q \oplus_{\mathbb{N}} \mathbb{N}^2$

- get $\rho_r \neq 0$ for each node.
- get $\mathbb{N}$ for section on C_0.

These $\underline{3 \text{ charts}}$ are essential to do log stable curves.

Notice get "generalization" morphisms

$$r \rightsquigarrow \eta_1 \qquad \chi_{\eta_i} : \overline{M}_{C,r} \longrightarrow \overline{M}_{C,\eta_i}$$
$$r \rightsquigarrow \eta_2 \qquad Q \oplus_{\mathbb{N}} \mathbb{N}^2 \longrightarrow Q$$

$\eta_1 \quad (q,(a,b)) \mapsto q + b\rho_r$

$\eta_2 \quad (q,(a,b)) \mapsto q + a\rho_r$

Exercise: $\iota : Q \oplus_{\mathbb{N}} \mathbb{N}^2 \xrightarrow[\chi_{\eta_1} \times \chi_{\eta_2}]{} Q \times Q$ is injective.

(good exercise in monoid)

Get a moduli space:

$M^{\log}_{g,n} = \{ \text{log sm curve } f : C \to W \}$

stable: $H^0(T_{C/\operatorname{spec} k}) = 0$ on geom pts

Problem CFG / Log Sch So not yet a stack

Want: $\overline{M}^{\log}_{g,n} = \left(\underline{\overline{M}^{\log}_{g,n}}, M_{\overline{M}^{\log}_{g,n}} \right)$ a log alg stack

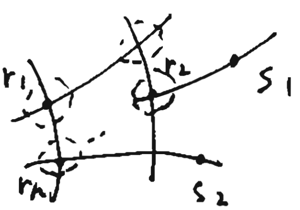
$\rightarrow \overline{M}^{\log}_{g,n}$ DM stack / Sch

$M_{\overline{M}^{\log}_{g,n}}$ natural / universal

CFG = category fibered in groupoids

For Kato's solution log curves are nodal, pointed, looks a lot like Deligne-Mumford stable.

Build a ~~canonical log str~~ on DM stable curves $\underline{C}$.



$\downarrow$

pt

- Nodes need charts

$$\begin{array}{ccc} \mathbb{N}^2 & \longrightarrow & k[z,u,t]/zu - t \\ \Delta \uparrow & & \uparrow \\ \mathbb{N} & \longrightarrow & k[t] \end{array}$$

- div. log str on S_i's

$Q = \mathbb{N}^{\#\text{nodes}(n)}$ on base

$\uparrow$ the smallest number can be chosen

p. 4 2 punchlines :

#1 $f : X \to S$ a DM stable curve with can log str $f' : X' \to S'$ a log curve (stable)

Assume

$$\begin{tikzcd} \underline{X}' \arrow[r, "\underline{a}"] \arrow[d, "\underline{f}'"'] \arrow[dr, phantom, "\square"] & \underline{X} \arrow[d, "\underline{f}"] \\ \underline{S}' \arrow[r, "\underline{b}"'] & \underline{S} \end{tikzcd}$$

then $\underline{a}$ and $\underline{b}$ extend uniquely to a diagram in Log Sch

$$\begin{tikzcd} X' \arrow[r, "a"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ S' \arrow[r, "b"'] & S \end{tikzcd}$$

This is a universal property of "basic" objects in an CFG / Log Sch

call f satisfying this "basic". Result is

$$\overline{M}^{log}_{g,n} = \left(\overline{M}^{basic}_{g,n} , M_{\overline{M}^{basic}_{g,n}} \right)$$

and $\overline{M}^{basic}_{g,n} \simeq \overline{M}^{DM}_{g,n}$ also . Divisorial !!!

So log geom get DM directly without knowing DM's work

solves a categorical problem

CFG / Log Sch   Log Alg Stacks

$\{ (S, M_S) \to (\mathcal{X}, M_{\mathcal{X}}) \} \longleftarrow (\mathcal{X}, M_{\mathcal{X}})$

Conversely, given $\mathcal{Y}^{log}$, $\longmapsto$?

Answer : study minimality / Basicness

- F Kato
- Kim (1st attempt to study log curve)
- Gillam

§4. Stable Log Maps :

$X = (\underline{X}, M_X)$ "nice" : log sm
$/pt$

(M_X SNC Divisorial)

enough for most applications

moduli problem

$$M_\Gamma^{\log}(X) = \left\{ \begin{array}{l} \text{log curve} \\ \text{(pre-stable)} \end{array} \quad \begin{array}{ccc} C & \xrightarrow{f} & X \\ \downarrow{\scriptstyle \pi} & & \\ W & & \end{array} \;\middle|\; \begin{array}{l} \text{underlying maps} \\ \text{of schemes} \\ \text{is stable} \end{array} \right\}$$

would like to understand geom points.

Our setting:

k alg closed, character $= 0$

$\underline{W} = \operatorname{Spec} k$

M_W - Q log pt, Q toric, sharp

$k^\times \oplus Q \longrightarrow k \qquad (u,q) \mapsto \begin{cases} u & q \neq 0 \\ 0 & q = 0 \end{cases}$

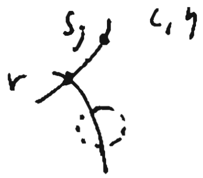
need to understand

$$\overline{\pi^* M_W} \xrightarrow{\psi} \overline{M}_C \xleftarrow{\varphi} \overline{M}_X$$

stalks at Q $\qquad\qquad$ stalks are P_x

Let $x \in C$ be a closed pt

3 possibilities étale locally



x a sm pt $p \in \eta_1, \eta_2$ [1]

x a marked pt [2]

x is a node [3]

[1] $x = \eta_i$ Then ψ_{η_i} is identity and

$\varphi_{\eta_i} : P_{\eta_i} \to Q$ (map for data on X for divisor D_i in Q)

[2] $x = s_j$ Then $\overline{M}_{C,s_j} = Q \oplus \mathbb{N}$, ψ_{s_j} inclusion of Q,

$$\begin{tikzcd} & & Q \\ P_{s_j} \arrow[r] \arrow[urr, bend left, "\varphi_{s_j}"] \arrow[drr, bend right, "pr_2"'] & Q \oplus \mathbb{N} \arrow[ur, "pr_1"'] \arrow[dr] & \\ & & \mathbb{N} \end{tikzcd}$$

Call this u_{s_j}, then will record contact order data
(tangency condi)

(see eg $X = \mathbb{A}^1$ with 0 the standard log str, divisorial) p 6

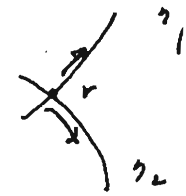
[3] $x = r$, Then $\bar{M}_{C,r} = Q \oplus_{\mathbb{N}} \mathbb{N}^2$ $\qquad$ $\mathbb{N} \xrightarrow{\Delta} \mathbb{N}^2$

$\mathbb{N} \longrightarrow Q$

$1 \longmapsto \rho_r$

get a diagram

(**)
$$\begin{tikzcd}
 & P_{\eta_1} \arrow[r, "\varphi_{\eta_1}"] & Q \\
P_r \arrow[ur, "x_1"] \arrow[r] \arrow[dr, "x_2"'] & Q \oplus_{\mathbb{N}} \mathbb{N}^2 \arrow[ur] \arrow[r, "\iota"] \arrow[dr] & Q \times Q \arrow[u, "pr_1"'] \arrow[d, "pr_2"] \\
 & P_{\eta_2} \arrow[r, "\varphi_{\eta_2}"] & Q
\end{tikzcd}$$



2 generalization maps

Exercise : ι is injective

$\iota : Q \oplus_{\mathbb{N}} \mathbb{N}^2 \xrightarrow{x_1 \times x_2} Q \times Q$

$(m, (a,b)) \longmapsto (m + a\rho_r,\ m + b\rho_r)$

if $\iota(m,(a,b)) = 0$, then $m + a\rho_r = m + b\rho_r$ with $\rho_r \neq 0$

$\Rightarrow a = b$

$(m,(a,b)) = (m,(a,a)) \cdot (m \oplus a(1,1))$

$= (m + a\rho_r, (0,0)) = 0$

so get sub monoid

Thus, $Q \oplus_{\mathbb{N}} \mathbb{N}^2 \hookrightarrow Q \times Q$

$\cong \{ (m_1, m_2) \in Q \times Q \mid m_1 - m_2 \in \mathbb{Z}\rho_r \text{ in } Q^{gp} \}$

in fact, if $m_1 - m_2 = (a-b)\rho_r$

$(m_1, m_2) = \begin{cases} \iota(m_2, (a-b, 0)) & a-b \geq 0 \\ \iota(m_1, (0, b-a)) & a-b \leq 0 \end{cases}$

i.e. a, b are relative "speed" of smoothing and wrt each comp

get data $u_r : P_r \to \mathbb{Z}$ defined by eq'n

$$\varphi_{\eta_2}(x_2(m)) - \varphi_{\eta_1}(x_1(m)) = u_r(m)\rho_r \in \mathbb{Z}\rho_r :$$

determined by the commu diag (**)

S. Marcus lect 5 (final)

6/21 at TIMS. 2013

Recall:

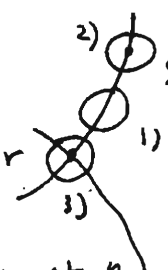
$(\underline{X}, M_X) = X$ log Sm M_X-divisorial SNC

$$M^{\log}_{\beta}(X) = \left\{ \begin{array}{c} C \xrightarrow{f} X \\ \text{pre-stable log curves} \downarrow \pi \\ W \end{array} \;\middle|\; \begin{array}{c} \text{Kontsevich stable} \\ \text{of } \underline{f}, \underline{\pi} \end{array} \right\}$$

Study geometric pt (over (pt, Q))

consider — toric sharp monoid on the base

$x \in C$



$$\begin{array}{ccccc} \pi^*\overline{M}_{W,x} & \xrightarrow{\psi_x} & \overline{M}_{C,x} & \xleftarrow{\varphi_x} & f^*\overline{M}_{X,x} \\ \| & & \| & & \| \\ Q & & ? & & P_x \end{array}$$

1) x a Sm pt η_i, Then $\overline{M}_{C,\eta_i} = Q$, $\psi_{\eta_i} = id$, $\varphi_{\eta_i}: P_{\eta_i} \to Q$

2) x a marked pt s_j, $\overline{M}_{C,s_j} = Q \oplus \mathbb{N}$, ψ_{s_j} inclusion of Q

$$\begin{tikzcd} & & Q \\ P_{s_j} \arrow[urr, bend left] \arrow[r, "\varphi_{s_j}"] \arrow[drr, bend right, "u_{s_j}"'] & Q \oplus \mathbb{N} \arrow[ur] \arrow[dr] & \\ & & \mathbb{N} \end{tikzcd}$$

contact order data

3) x a node r

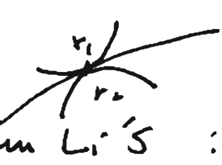
$\overline{M}_{C,r} = Q \oplus_{\mathbb{N}} \mathbb{N}^2$ $\quad 1 \mapsto (1,1)$, $1 \mapsto \rho_r$

$$\begin{tikzcd} & P_{\eta_1} \arrow[rr] & & Q \\ P_r \arrow[ur, "\chi_{\eta_1,r}"] \arrow[r, "\varphi_r"] \arrow[dr, "\chi_{\eta_2,r}"'] & Q \oplus_{\mathbb{N}} \mathbb{N}^2 \arrow[urr] \arrow[rr, hook, "\iota"] \arrow[drr] & & Q \times Q \arrow[u] \arrow[d] \\ & P_{\eta_2} \arrow[rr] & & Q \end{tikzcd}$$

image of ι is $\{(m_1, m_2) \in Q \times Q \mid m_1 - m_2 \in \mathbb{Z}\rho_r$ in $Q^{gp}\}$ get $P_r \to \mathbb{Z}$ from

$$\varphi_{\eta_2}(\chi_2(m)) - \varphi_{\eta_1}(\chi_1(m)) = \underset{\uparrow \mathbb{Z}}{u_r(m)}\, \rho_r$$

ie contact order data of node relative to that component.



Notice This is the big difference from that of Jun Li's: his pre-deformability require $r_1 = r_2$, but now could be different

Def^{n}: the type of a stable log map $f: C \to X$ is

$$u(f: C \to X) = \begin{cases} \cdot\ u_{s_j} \in \mathrm{Hom}(P_{s_j}, \mathbb{N}) & \forall j = 1, \dots, n \\ \cdot\ u_{r_\ell} \in \mathrm{Hom}(P_{r_\ell}, \mathbb{Z}) & \forall \text{ nodes } r_1, \dots, r_k \\ \cdot\ \Gamma_C & \text{dual graph of } C \end{cases}$$

This identify all discrete data

Next Q $\overline{M}^{\log}_{\Gamma}(X) = (\overline{M}^{bas}_{\Gamma}(X), \mathcal{M}^{bas}_{\Gamma}(X))$

what is "basic" log str on base ?

Have $X_{\eta_i, r}$ $P_r \to P_{\eta_i}$. For $m \in P_r$,

$$a_r(m) := \left((0 \cdots 0, \underline{X_{\eta_1, r}(m)}, 0 \cdots 0, \underline{-X_{\eta_2, r}(m)}), (0 \cdots 0, u_r(m), 0 \cdots 0)\right)$$

$$\in \Big(\prod_{\eta : \text{cmp of } C} P_\eta \times \prod_r \mathbb{N}\Big)^{gp}$$

data from target X — power of nodes. data on base

this gives the relation needed to take care of the Data of smoothing a node.

Let $R \subseteq (\prod_\eta P_\eta \times \prod_r \mathbb{N})^{gp}$ saturated subgp generated by

$\langle a_r(m) \mid r: \text{a node},\ m \in P_r \rangle$.

Def^{n}: Basic monoid for a stable log map is :

$$\Big(\prod_\eta P_\eta \times \prod_r \mathbb{N}\Big) \overset{i}{\hookrightarrow} \Big(\prod_\eta P_\eta \times \prod_r \mathbb{N}\Big)^{gp} \xrightarrow{\ q\ } \Big(\prod P_\eta \times \prod_r \mathbb{N}\Big)^{gp} / R$$

$$Q^{bas} = \Big[\, q \circ i \Big(\prod_\eta P_\eta \times \prod_r \mathbb{N}\Big) \Big]^{sat}$$

why ??

$$\begin{array}{ccc} & P_{\eta_1} \longrightarrow Q \\ P_r \nearrow & \\ \quad\searrow & P_{\eta_2} \longrightarrow Q \end{array}$$

construct Q using one eq^n

$$X_{\eta_2, r}(m) - X_{\eta_1, r}(m) = u_r(m) \rho_r$$

- Before we have Q, there is NO stack can exists ! This is the most difficult relation in Gross-Siebert program to construct the "basic" Q !!

Have canonical morphisms :

$$\varphi_y : P_y \xrightarrow[\text{of factors}]{\text{morphism}} \prod_y P_y \times \prod_r \mathbb{N} \xrightarrow{\text{"quotient"}} Q^{bas}$$

$$\mathbb{N} \longrightarrow \prod_y P_y \times \prod_r \mathbb{N} \longrightarrow Q^{bas}$$

Theorem : This choice satisfies our universal pull back property for "basicness"

Def^{n}. A basic S.log Map $\begin{array}{ccc} C & \xrightarrow{f} & X \\ {\scriptstyle \pi}\downarrow & & \\ W & & \end{array}$ is one where on geom pts of W,

The chart on W is Q^{bas}, and φ, f are of this type.

(This is Gross-Siebert's version, Qile Chen also has a independent construction with similar result)

Thm : Basicness is open condition on W

Fact : Basic log str on stable log maps do not introduce new automorphisms : Log Aut $\subset$ Aut

We have : $\overline{M}^{log}_{\Gamma}(X) = (\overline{M}^{bas}_{\Gamma}(X), \overline{M}_{M^{bas}_{\Gamma}(X)})$.

Need to prove : • $M^{bas}_{\Gamma}(X)$ alg DM stack.

• Boundedness : $\left\{ \begin{array}{l} \text{Discrete data are} \\ \beta_i \in H_2(X,\mathbb{Z}),\ g,\ n \\ \{ u_{s_j} : D_{s_j} \to \mathbb{N} \} \end{array} \right.$

fix this, show # of "types" are finite (This is DIFFICULT, in J. Li's models, this is achieved by pre-deformability, here we use "easier" data and get same bdd result)

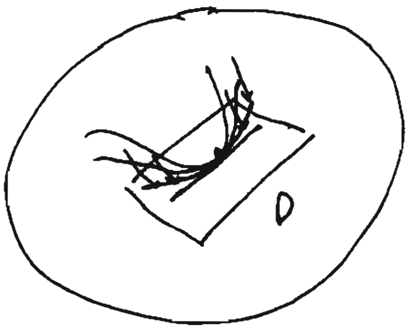
• Properness : Stable reduction (use properness of $\overline{M}_{g,n}(X,\beta)$)

properness + locally constant $u_{s_j} : P_{s_j} \to \mathbb{N}$

$\Rightarrow$ tangency data remains recorded at limits

• virtual fund class exists. (same theory holds using Behrend Fantechi's construction)

P.4 Picture



a curve falls into the div D

$u_{s_j} : P_{s_j} \longrightarrow Q \oplus \mathbb{N} \longrightarrow \mathbb{N}$

locally constant in families

in this case, just

$$\begin{array}{ccc} \mathbb{N} & \longrightarrow & \mathbb{N} \\ 1 & \longmapsto & 2 \end{array}$$

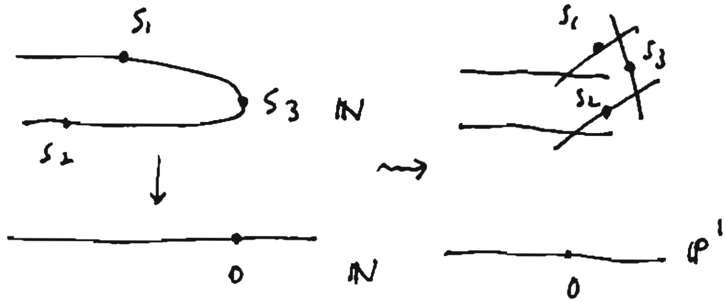
so even when the curves falls into D, the image is still 2

eg. $X = \mathbb{P}^1 \supset 0$

$$\begin{array}{c} C \\ \downarrow \\ \mathbb{P}^1 \end{array}$$

$s_3 : \mathbb{N} \longrightarrow \mathbb{N} : 1 \longmapsto 2$

$s_1, s_2 : 0 \to \mathbb{N} : 0 \longmapsto 0$



$s_3 : 1 \longmapsto 2$

J. Li's solution: use expanded degeneration

New solution: fix by consistency

§5 Other properties

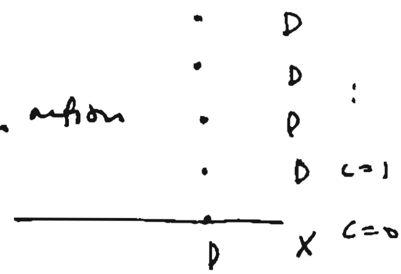
Evaluation spaces: $\Lambda_{\mathbb{N}}(X)$

$ev_i : M^{\log}_{\Gamma}(X) \longrightarrow \Lambda_{\mathbb{N}}(X)$

$\Lambda_{\mathbb{N}}(X, M_X) = \bigsqcup_{c \in \mathbb{N}} \Lambda^c_{\mathbb{N}}(X)$

$\setminus \operatorname{sing}(X, D)$ in classical notation

$\mathbb{G}_m$ action



level 0: $\Lambda^0_{\mathbb{N}}(X) = X \times B\mathbb{G}_m$

$[\mathcal{O}(D)|_D / \mathbb{G}_m] \overset{?}{\longleftarrow} \Lambda^c_{\mathbb{N}}(X) =$ "D" twisted by contact order c.

using same (normal) line bundle

$B\mathbb{G}_m = [pt/\mathbb{G}_m] = \left\{ \begin{array}{c} T \xrightarrow{f} pt \\ \downarrow \\ U \end{array} \;\middle|\; \begin{array}{l} \mathbb{G}_m \text{ torsor} \\ f\ \mathbb{G}_m \text{ equiv} \end{array} \right\}$

$\mathbb{G}_m$ torsor over a base

$= \left\{ \begin{array}{c} L \\ \downarrow \\ U \end{array} \;\middle|\; L \text{ a line bundle} \right\}$

Exercise:

$\mathcal{A} = [\mathbb{A}^1/\mathbb{G}_m] = \left\{ \begin{array}{c} (L, s) \\ \downarrow \\ U \end{array} \;\middle|\; L \text{ line bundle, } s \in \Gamma(U, L^\vee) \right\}$

$B\mathbb{G}_m \subset \mathcal{A}$ as a divisor at origin

$(\mathcal{A}, \mathcal{B}, \mathbb{G}_m)$ – universal smooth pair //

Write Relative stable maps

$\overline{Li}(X,D)$ rel maps with / expansions

- expansions
- predeformability
- difficult obs th

] 3 hard properties

but there are 4 other theories:

- $\overline{Kim}(X,D)$ – log maps with expansions
- → $\overline{ACGS}(X,D)$ – log maps ← $\boxed{\text{This is what we talk about}}$
- $\overline{AF}(X,D)$ – orbifold maps with expansions
- $\overline{Cad}(X,D)$ – orbifold maps for tangency

All 4 are trying to remove the 3 hard properties

$\overline{ACGS}$ is the 1st to be completely general, namely to allow D SNC. All others need D smooth

Maps: let (X,D) sm pair:

$$\begin{tikzcd}
& \overline{AF}(X,D) \arrow[dl, "\Phi"'] \arrow[dr, "\Psi"] & & \overline{Kim}(Y,D) \arrow[dl, "\theta"'] \arrow[dr, "\Upsilon"] & \\
\overline{Cad}(X,D) & & \overline{Li}(X,D) & & \overline{ACGS}(X,D)
\end{tikzcd}$$

Thm (Abramovich–Cadman–Wise 2008~)

- $\Phi_*[AF]^{vir} = [Cad]^{vir}$ $g=0$
- equivalent descendent rel GW(X) in $g=0$.

Thm (Abramovich–, Wise)

- $\Psi_*[AF]^{vir} = [Li]^{vir}$ all genus
- $\theta_*[Kim]^{vir} = [Li]^{vir}$
- $\Upsilon_*[Kim]^{vir} = [ACGS]^{vir}$
- All primary / descendant GW inv's coincide

Diff moduli, but all the same invariants

1.6. for M, M' any 2 of these theories, Costello's theorem (very new)

$$\begin{tikzcd} M(X,D) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & M'(X,D) \arrow[d] \\ M(A, B\mathbb{G}_m) \arrow[r, "\text{equiv}"'] & M'(A, B\mathbb{G}_m) \end{tikzcd}$$ is Cartesian.

gives the comparison of obstruction theory