G. Schumacher 7/12

Analytic theory of Moduli spaces and Weil-Petersson metrics I : (All 6 lectures)

1. Introduction

1857 Riemann : "Theory of abelian functions"

alg. eq'n $f(z,w)=0$, classification alg'lly

parameter $3g-3$ "moduli" $\equiv$ mescesures ?

Geom: Teichmuller, Ahlfors, Bers, $g>1$

universal family $\mathcal{X}_g \longrightarrow \mathcal{T}_g \subset \mathbb{C}^{3g-3}$

Higher-dim: Solve local prob. deformation

A. Weil : $T_p \mathcal{T}_g = H^0(X, \Omega_X^{\otimes 2})^\vee = H^0(X, \mathcal{T}_X)$

using hyp. metric $ds^2 = g\, dV$

Harmonic theory : har. Beltrami differential

$$\mu \frac{\partial}{\partial z} d\bar{z} = \frac{\bar{\varphi}}{g} \frac{\partial}{\partial z} d\bar{z}$$

$\varphi\, dz^2$ hol.

$$(\mu_1, \mu_2) = \int \frac{\bar{\varphi}_1 \varphi_2}{g^2} g\, dV \qquad \text{(W-P metric)}$$

Weil's work never published, only appear in a report in his collected work.

Conjecture (Weil) : WP-metric is Kähler, Ric < 0.

solved by Ahlfors, Bers.

curvature tensor : Wolpert, Fock-Tromba

2. Existence of Moduli Spaces:

K class of compact cpx manifolds (with suitable structures)

$\mathcal{M} = \mathcal{M}_{\text{set-theory}} = \{ [X] \text{ isom. class}, X \text{ from } K \}$

holomorphic family of cpt cx mfd.

$\{\mathcal{X}_s\}_{s\in S}$: $f : \mathcal{X} \longrightarrow S$, $\mathcal{X}_s := f^{-1}(s)$.
proper smooth hol. map.

Get map
$$\begin{array}{cccc} \varphi_f : & S & \longrightarrow & \mathcal{M} \\ & \Psi & & \Psi \\ & s & \longmapsto & [\mathcal{X}_s] \end{array}$$

$\mathcal{M}$ topology: all such φ_f are continuous.

$\mathcal{M} = \bigcup_S S/\sim$, $r \sim s \Leftrightarrow \mathcal{X}_r \simeq \mathcal{Y}_s$.
quotient topology

Definition: $\mathcal{M}$ is a coarse Moduli space for K :$\Leftrightarrow$ for any hol. map $\mathcal{X} \xrightarrow{f} S$ with fibers from K, $\varphi_f : S \to \mathcal{M}$ is holomorphic.

Examples:

1). Families of Hopf surfaces:

$$\mathcal{X} = (\mathbb{C}^2 \setminus \{(0,0)\}) \times \mathbb{C} / \mathbb{Z} \longrightarrow \mathbb{C}$$

$$(z,w,s) \simeq \gamma(z,w,s), \quad \gamma = \begin{pmatrix} \alpha & s \\ 0 & \alpha \end{pmatrix}$$

for all $s \neq 0$, $\mathcal{X}_s \simeq \mathcal{X}_1$. ($S_3 \times S_1$) diffeom

$$\mathcal{X}_0 \not\simeq \mathcal{X}_1$$

This violate the requirement.

2) Hirzebruch surfaces:

$$\Sigma_m = (U_1 \times \mathbb{P}_1) \cup (U_2 \times \mathbb{P}_1), \quad U_j = \mathbb{C}$$

$$(z_1, \zeta_1) \sim \left(\frac{1}{z_1}, z_1^m \zeta_1\right) = (z_2, \zeta_2).$$

$\Sigma_0 = \mathbb{P}_1 \times \mathbb{P}_1$, $\Sigma_1 = \tilde{\mathbb{P}}_2$ (blow-up)

$$\mathcal{X} = (U_1 \times \mathbb{P}_1 \times \mathbb{C}) \cup (U_2 \times \mathbb{P}_1 \times \mathbb{C}) \longrightarrow \mathbb{C}$$

$z_2 = 1/z_1$, $\zeta_2 = z_1^2 z_2 + s z_1$ family

coor. transf. $\mathcal{X}_s = \Sigma_0$, $s \neq 0$, $\mathcal{X}_0 = \Sigma_2$ (jump).

Explanation:

1). Non-Kähler

2) Algebraic, but ruled

3) families of tori $\to$ need "polarization".

Notation:

$$(*) \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ R \arrow[r, "g"'] & S \end{tikzcd} \qquad X \to Y_R := Y \times_R S = \{(y, r) : f(y) = g(r)\}$$

comm. diagram

call $(*)$ Cartesian $:\Leftrightarrow$ $X \to Y_R$ isomorphism.

Example:

$$\xi : \begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ 0 \arrow[r] & S \end{tikzcd} \qquad \text{cartesian} \Leftrightarrow X \xrightarrow{\sim} \mathcal{X}_0$$

ξ called deformation of X over (S, s_0)

Base-change: $g : (R, r_0) \to (S, s_0)$ hol.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X}_R \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ 0 \arrow[r] & R \arrow[r, "g"] & S \end{tikzcd}$$

called $g^* \xi$. pull back.

η complete :$\Leftrightarrow$ any ξ is of the form $g^*\eta$.

η effective :$\Leftrightarrow$ $(Dg)_{r_0}$ unique if g exists.

η versal (= semi-universal)

:$\Leftrightarrow$ effective + complete

η universal :$\Leftrightarrow$ η complete and base change g unique.

Rmk: Since this is a local theory, always replace (S, s_0) by $(U(s_0), s_0)$ if necessary

$M := \bigcup S/\sim$

S base of versal deformations.

Douady Space (for imbeded deformations):

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[dr] & Z \times \mathcal{H} \arrow[d] \\ & \mathcal{H} \end{tikzcd}$$

Z complex space

$\mathcal{H} = \mathcal{D}$.

fibers are subspaces in Z.

Y_s run through all complex submanifolds

Ker $\to$ universal deformation of embedded sub-mfds.

X, Y complex mfds $\Rightarrow \exists\, I = \mathrm{Isom}(X, Y)$

$$I = \begin{cases} \emptyset \\ \simeq \mathrm{Aut}(X) \end{cases}$$ Lie group.

$$\begin{tikzcd} X \times I \arrow[rr, "\Phi"] \arrow[dr] & & Y \times I \arrow[dl] \\ & I & \end{tikzcd}$$

$I \subset$ Douady Space $(X \times Y)$

relative version:

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[dr] & Z_{\mathcal{H}} \arrow[r] \arrow[d] & Z \arrow[d] \\ & \mathcal{H} \arrow[r] & S \end{tikzcd}$$

classifies all subspaces over $R \to S$ in $Z_R = Z \times_S R$

i.e. subspaces in a "moving target".
this is what we need here.

Applications: given $\mathfrak{X}, \mathcal{Y} \to S$ two families

$\Rightarrow \exists$ space $\mathcal{J} = \mathrm{Isom}_S(\mathfrak{X}, \mathcal{Y}) \to S$

so that fiber $\mathcal{J}_s = \mathrm{Isom}(\mathfrak{X}_s, \mathcal{Y}_s) = \begin{cases} \emptyset \\ \mathrm{Aut}(\mathfrak{X}_s) \end{cases}$.

$$\begin{tikzcd} \mathfrak{X}_I \arrow[rr, "\Phi"] \arrow[dr] & & \mathcal{Y}_I \arrow[dl] \\ & I & \end{tikzcd}$$ universal isomorphism.

Theorem: K class of compact complex manifolds (with additional str: below)

(1). K closed under local deformations

(2). $\mathfrak{X}, \mathcal{Y} \to S$ families, then

$\mathrm{Isom}_S(\mathfrak{X}, \mathcal{Y})$ exists and is "proper".

$\Rightarrow \exists$ coarse moduli space for K.

Remark:

(2) $\Leftrightarrow$ $S_\nu \to S_0$, $\varphi_\nu : \mathfrak{X}_{s_\nu} \xrightarrow{\sim} \mathcal{Y}_{s_\nu}$ then $\exists$ subseq.

(P): $\nu(\mu)$ st. $\varphi_{\nu(\mu)} \xrightarrow[\mu \to \infty]{} \varphi_0 : \mathfrak{X}_{s_0} \xrightarrow{\sim} \mathcal{Y}_{s_0}$.

and (P) $\Rightarrow$ $\mathcal{M}$ is Hausdorff.

Proposition: Properness condition (P) $\Rightarrow$

$S/\!\sim$ has local model for $\mathcal{M}$ and

$S/\!\sim\; = S/G$, $G = \mathrm{Aut}(X)/\mathrm{Aut}^\varepsilon(X)$ $\leftarrow$ finite gp action on (S, s_0)

($\mathrm{Aut}^\varepsilon(X)$ = extendable automorphisms)

Short Argument:

$I = \mathrm{Isom}_{S\times S}(\mathfrak{X}\times S, S\times\mathfrak{X})_\sim \longrightarrow S\times S$

$\nu(I) = \Gamma = \{(r,s),\ r\sim s\}$

$\Gamma \to S \times S$ proper $\Rightarrow$ $\Gamma \subset S \times S$ analytic

$$\pi^{-1}(s_0) = \{ (r, s_0) : \mathfrak{X}_r \cong \mathfrak{X}_{s_0} \} \qquad \begin{tikzcd} \Gamma \arrow[dr, "\pi"'] \arrow[d, "pr_1"] \\ S & \end{tikzcd}$$

$= \{(s_0, s_0)\}$ some versal deformation

$\Rightarrow$ π finite map (shrink S if nec).

group action:

$\varphi \in \mathrm{Aut}(X)$ gives new deformation $\alpha^* \xi$

$$\begin{tikzcd}
X \arrow[drr] & X \arrow[l, "\sim"', "\varphi"] \arrow[r, "i"] \arrow[d] & \mathfrak{X} \arrow[r] \arrow[d, "s"] & \mathfrak{X} \arrow[d] \\
& 0 \arrow[r] & S \arrow[r, "\alpha"] & S
\end{tikzcd}$$

$\Leftrightarrow$

$$\begin{tikzcd}
& X \arrow[rr] \arrow[dd] & & \mathfrak{X} \arrow[dd] \\
X \arrow[ur, "\varphi", "\sim"'] \arrow[rr] \arrow[dr] & & \mathfrak{X} \arrow[ur] \arrow[dd] & \\
& 0 \arrow[rr, dashed, "f"] \arrow[dr] & & S \\
& & S \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

Problem: α need not be unique! otherwise done, just like action in Teichmuller space.

α gives a section of π, $s \mapsto (\alpha(s), s)$ by finitely many sections, because π finite map.

Let $\varphi, \psi \in \mathrm{Aut}(X)$ induces same α,

$\Rightarrow$ $\varphi \circ \psi^{-1}$ extends to all families

$\Rightarrow$ $\varphi \circ \psi^{-1} \in \mathrm{Aut}^{\varepsilon}(X)$, $\mathrm{Aut}(X)/\mathrm{Aut}^{\varepsilon}(X)$ finite set,

$\Rightarrow$ $\mathrm{Aut}^{\varepsilon}(X) \supset \mathrm{Aut}(X)$ $\Rightarrow$ $\dim \mathrm{Aut}(\mathfrak{X}_s) =$ constant

$\underset{\text{general theory}}{\Longrightarrow}$ any versal deformations is universal, in particular, φ determines α uniquely

$$\begin{tikzcd}
\mathrm{Aut}(X) \arrow[rr] \arrow[dr] & & \mathrm{Aut}(S, s_0) \\
& \mathrm{Aut}^{\varepsilon}(X) / \mathrm{Aut}^{\circ}(X) = G \arrow[ur] &
\end{tikzcd}$$

And $\Gamma = \bigcup_{\gamma \in G} \{ (s, \gamma s),\ s \in S \}$. #

When is (P) satisfied?

Def: (X, λ_X) polarized if $\lambda_X \in H^2(X;\mathbb{R})$ is a Kähler class. (e.g. $\mathrm{Im}(H^2(X;\mathbb{Z}))$ then is chern class of ample divisors, then in alg. geom.)

(P) is satisfied for non-uniruled polarized manifolds. (Fujiki).

Def: A cpx mfd is uniruled $:\Longleftrightarrow$
$\exists$ hol. v.b. $E \to Y$ s.t.

$$\begin{tikzcd} \mathbb{P}(E) \arrow[r, "\text{surj}"] \arrow[d] & X \\ Y \arrow[ur, dashed, "\nexists"'] & \end{tikzcd}$$

if X is Kähler, uniruled $\Rightarrow$

$$\begin{tikzcd} \mathbb{P}^1 \times Y \arrow[r, "\text{surj}"] \arrow[d] & X \\ Y \arrow[ur, "\nexists"'] & \end{tikzcd}$$

Kähler class: measure volumes of cpt subvarieties.

Bishop's thm $\Rightarrow$ compactness of Douady spaces.

$\exists\ \bar{I} \xrightarrow[\text{proper}]{} S$, $\bar{I}$ includes bimeromorphic maps.

By a classical thm of Mumford and Matsusaka then $\Rightarrow$ $\bar{I}$ is I itself. (details to be discussed later)

Rmk: (P) is clear for K·E manifolds.
in that case will see more closely.

lecture II 7/12 (conti.) II.

3. Families of Polarized Kähler manifolds.

Def: (1) X cpt cx mfd, $\lambda_X \in H^2(X;\mathbb{R})$ Kähler class. Call (X, λ_X) pol. K-mfd.

(2) $(\mathcal{X} \xrightarrow[f]{} S, \lambda_{\mathcal{X}/S})$ family of pol.-K. mfd

$\lambda_{\mathcal{X}/S} \in (R^1 f_* \Omega^1_{\mathcal{X}/S})(S)$ st.

$\lambda_{\mathcal{X}_s} = \lambda_{\mathcal{X}/S}|_{\mathcal{X}_s}$ is a Kähler class.

Prop: $\lambda_{\mathcal{X}/S}$ is equiv. to $\lambda \in (R^2 f_* \mathbb{R}_{\mathcal{X}})(S)$. st. $\lambda|_{\mathcal{X}_s} =: \lambda_{\mathcal{X}_s}$ is a Kähler class.

Rmk: If S "small" st. $\mathcal{X} \underset{\text{diffeo}}{\xrightarrow{\sim}} X \times S$, then $\lambda \in H^2(X;\mathbb{R})$ ...)

Idea: "real valued" holo section of locally free sheaf $\Rightarrow$ constant.

pf: 1) $\lambda \in (R^2 f_* \mathbb{R}_{\mathcal{X}})(S)$ given

$$0 \to f^{-1}\mathcal{O}_S \longrightarrow \mathcal{O}_{\mathcal{X}} \xrightarrow[d_{\mathcal{X}/S}]{} \Omega^1_{\mathcal{X}/S} \xrightarrow[d_{\mathcal{X}/S}]{} \Omega^2_{\mathcal{X}/S} \to$$

$\mathbb{R}_{\mathcal{X}} = f^{-1}\mathbb{R}_S \hookrightarrow f^{-1}\mathcal{O}_S$

break up into short exact seq.

$$0 \to f^{-1}\mathcal{O}_S \longrightarrow \mathcal{O}_{\mathcal{X}} \longrightarrow d_{\mathcal{X}/S}\mathcal{O}_{\mathcal{X}} \longrightarrow 0 \quad \text{etc.}$$

$$\Rightarrow \quad 0 \longrightarrow \underset{\mathcal{O}_S}{\underset{\|}{f_* f^{-1}\mathcal{O}_S}} \xrightarrow[\sim]{} \underset{\mathcal{O}_S}{\underset{\|}{f_*\mathcal{O}_{\mathcal{X}}}} \xrightarrow{0} f_* d_{\mathcal{X}/S}\mathcal{O}_{\mathcal{X}}$$

$$\longrightarrow \underset{\mathcal{O}_S^{h^{1,0}+h^{0,1}}}{\underset{\|}{R^1 f_*(f^{-1}\mathcal{O}_S)}} \xrightarrow{\text{surj}} \underset{\mathcal{O}_S^{h^{1,0}}}{\underset{\|}{R^1 f_*\mathcal{O}_{\mathcal{X}}}} \xrightarrow{0} R^1 f_* d_{\mathcal{X}/S}\mathcal{O}_{\mathcal{X}}$$

$$\xrightarrow{\text{inj}} R^2 f_* f^{-1}\mathcal{O}_S \longrightarrow R^2 f_*\mathcal{O}_{\mathcal{X}} \longrightarrow 0$$

$\alpha \longmapsto 0$ by type consideration!

$\lambda \longrightarrow \alpha \in (R^2 f_* f^{-1}\mathcal{O}_S)(S)$

$\Rightarrow \alpha \in (R^1 f_* d_{\mathcal{X}/S}\mathcal{O}_{\mathcal{X}})(S)$

so $\alpha \mapsto (R^1 f_* \Omega^1_{\mathcal{X}/S})(S)$

this may also been seen via spectral seq. arg.

2). $f_* \Omega^2_{\mathcal{X}/S} \subseteq R^1 f_* d_{\mathcal{X}/S}\mathcal{O}_{\mathcal{X}} \subseteq R^2 f_*(f^{-1}\mathcal{O}_S)$

$\quad \mathcal{O}_S^{h^{2,0}} \qquad \mathcal{O}_S^{h^{1,1}+h^{2,0}} \qquad$ etc.

Apply conjugates and take real part, get

$$0 \subseteq (R^1 f_* \Omega_{\mathcal{X}/S})_{\mathbb{R}} \subseteq \underbrace{R^2 f_* \mathbb{R}_{\mathcal{X}}}_{= f^{-1}\mathbb{R}_S}$$

gives $\lambda_{\mathcal{X}/S} \longmapsto \lambda$. $\square$.

Application: $\mathcal{X} \to S$ family, λ_X polarization for $X = \mathcal{X}_0 \Rightarrow \exists!$ maximum "subgerm" $R \subset S$ st. λ_X extends to polarization of $\mathcal{X}_R \to R$.

pf: $0 \to \mathbb{R} \xrightarrow{\sqrt{-1}} \mathcal{O}_{\mathcal{X}} \xrightarrow{Re} \mathcal{H}_{\mathcal{X}} \longrightarrow 0$

harmonic functions

assume $H^2(X;\mathbb{R}) = (R^2 f_* \mathbb{R})(S)$

$H^1(\mathcal{X}, \mathcal{H}_{\mathcal{X}}) \longrightarrow H^2(X;\mathbb{R}) \xrightarrow{a} (R^2 f_* \mathcal{O}_{\mathcal{X}})(S)$

$\qquad\qquad\qquad\qquad\qquad\qquad \mathcal{O}_S^{h^{0,2}}(S)$

$R = V(a(\lambda_X))$ zero set of $h^{0,2}$ hol. functions. Over R, λ_X comes from $(\rho_{ij}) \in H^1(\mathcal{X}_R, \mathcal{H})$, $\rho_{ij} = \rho_i - \rho_j$, $\rho_i : C^\infty$

$\omega_{\mathcal{X}} := \sqrt{-1}\,\partial\bar\partial \rho_i$ real C^∞, $(1,1)$ form with $\partial\bar\partial$ potential, $\omega_{\mathcal{X}} \rightsquigarrow [\lambda_{\mathcal{X}_R/R}]$.

claim: it is a polarization.

$\omega_{\mathcal{X}}|_{\mathcal{X}_{s_0}} \sim \omega_{\mathcal{X}}$ Kähler form

$\qquad \mathcal{X}_{s_0} = X$

$\omega_{\mathcal{X}}|_X + \sqrt{-1}\,\partial\bar\partial\varphi = \omega_X$

next : extend φ to $\mathfrak{X}_R$ in C^∞ way

$\omega_{\mathfrak{X}} + \sqrt{-1}\,\partial\bar\partial\,\Phi|_{\mathfrak{X}_{s_0}}$ Kähler

so also for $\mathfrak{X}_s$; s near s_0. $\square$

Applications : Every polarized Kähler mfd possess a versal deformation (in this class)

Pf : Take versal deformation S, and then restrict to RCS as above. $\square$

Theorem : $\exists$ moduli space of non-uniruled polarized Kähler manifolds.

Rmk : Uniruledness and Non-uniruledness are all closed conditions. Good! And they are expected to be distinguished by numerical invariants.

4. Polarizations and Distinguished Metrics.

Recall convergence property (P) :

$$\varphi_\nu : \mathfrak{X}_{s_\nu} \longrightarrow \mathcal{Y}_{s_\nu} \quad , s_\nu \longrightarrow 0$$

Q : $\exists$ subsequence st. $\varphi_{\nu(\mu)} \longrightarrow \varphi_0 : \mathfrak{X}_{s_0} \to \mathcal{Y}_{s_0}$

Notation :

$$\omega_X = \sqrt{-1} \sum g_{\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^\beta ,$$

$$\omega_X^n := \frac{1}{n!}\,\omega_X \wedge \cdots \wedge \omega_X = \det(g_{\alpha\bar\beta})\, dV = g\, dV .$$

$$\mathrm{Ric}(\omega_X) := \mathrm{Ric}(\omega_X^n) = -\sqrt{-1}\,\partial\bar\partial \log g$$

case I : $\omega_X = -\mathrm{Ric}(\omega_X)$

case II : $\omega_X \in [\lambda_X]$, $\mathrm{Ric}(\omega_X)$.

for Calabi-Yau metrics.

I). $\mathcal{X} \to S$ family of can. polarized mfds.

$\lambda_{\mathcal{X}_s} = -2\pi c_1(\mathcal{X}_s)$: "canonical" because any bi-holomorphic map respects can. polarization.

$\omega_{\mathcal{X}_s}$ Kähler-Einstein form in $\mathcal{X}_s$ — give rise to $\omega_{\mathcal{X}/S}$ rel. form

$\omega_{\mathcal{X}} = -\mathrm{Ric}(\omega^n_{\mathcal{X}/S})$

$= 2\pi c_1(K_{\mathcal{X}/S}, g^{-1})$, $g\,dV$ is rel. K-E vol. form.

by definition, "$\omega_{\mathcal{X}}$ in $\mathcal{X}$ $\partial\bar\partial$ local potential"

$$\omega_{\mathcal{X}}|_{\mathcal{X}_s} = \omega_{\mathcal{X}_s} \text{ is Kähler-Einstein form}$$

and has "nothing" in the S direction! But in Applications, it is enough.

II). $(\mathcal{X} \to S, \lambda_{\mathcal{X}/S})$ polarized family with $c_1(\mathcal{X}_s) = 0$,

$\omega_{\mathcal{X}_s} \in \lambda_{\mathcal{X}_s}$ the Ricci flat metric $\Rightarrow \exists\, \omega_{\mathcal{X}}$ with local $\partial\bar\partial$ potential and $\omega_{\mathcal{X}}|_{\mathcal{X}_s} = \omega_{\mathcal{X}_s}$.

Aim: $T_p M$, locally $M = S/G$,
sufficient to consider $T_p S$, since G acts on S and "everything" compatible with this G action.

(Kodaira-Spencer theory).

Back to existence of M, and (P):
solutions of Calabi Problem is unique,

I). $c_1(X) = c_1(Y) = 0$. $\varphi : (X, \lambda_X) \to (Y, \lambda_Y)$
biholomorphic map of polarized mfds i.e.
$\varphi^* \lambda_Y = \lambda_X$. but $\omega_X \in \lambda_X$, $\omega_Y \in \lambda_Y$.
Ricci flat $\Rightarrow [\varphi^* \omega_Y] = \varphi^* [\omega_Y] = \varphi^* \lambda_Y$
so $\varphi^* \omega_Y$ Ricci flat (K-E) $\Rightarrow \varphi^* \omega_Y = \omega_X$.
$(\varphi^* \mathrm{Ric}(\omega_Y) = \mathrm{Ric}(\varphi^* \omega_Y))$

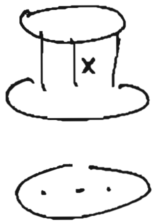
II). Also by uniqueness $\varphi^*\omega_Y = \omega_X$.

Any biholomorphic map is an isometry. with respect to distance function

classical thm: convergence to isometry wrt distance function $\Rightarrow$ wrt diff. metric $\Rightarrow$ convergence to isomorphism.

This $\Rightarrow$ (P).

Problem: Intrinsic hermitian metric on S.



$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{A} & \mathcal{X} \\ \downarrow & & \downarrow \\ S & \xrightarrow{\alpha} & S \end{array} \qquad \begin{array}{l} A|_X = \varphi \\ A|_{\mathcal{X}_s} : \mathcal{X}_s \to \mathcal{X}_{\alpha(s)} \end{array}$$

$$\mathrm{Aut}(X) \to \mathrm{Aut}(S, s_0)$$
$$\varphi \longmapsto \alpha$$

So, Any intrinsic hermitian metric on S will be G-inv. As such a metric, it will descend to $M \to S/G$.

This means M has a structure of V-space.

$T_{s_0}S$ characterized by (S may be singul

Kodaira-Spencer map: infinitesimal deform

$D = (0, \mathcal{O}_D)$; $\mathcal{O}_D = \mathbb{C}\langle z\rangle/(z^2)$ the famous space "double points"

then $T_{s_0}S = \mathrm{Hol}(D, S)$

$v \rightsquigarrow \rho(v) : D \to S$; ξ deformations over

$\Rightarrow \rho(v)^*(\xi)$ deformation over D. this gives a finite dim'l v.s.

$$= \begin{cases} H^1(X, T_X) \\ H^1_{\lambda_X}(X, T_X) \subset H^1(X, T_X) \text{ for polarization.} \end{cases}$$

where $H^1_{\lambda_X}(X, T_X) = \ker\left(H^1(X, T_X) \xrightarrow{\cup \lambda_X} H^2(X, \mathcal{O}_X)\right)$.

Geometric Version of K-S map: ρ

(usually one may take "any coh. theory" and do the K-S theory in that version.)

Here we want metrics:

$$\mathcal{X} \to S \text{ hol. family } \quad (T_X \equiv \Theta_X)$$

$$(*) \quad 0 \to T_{\mathcal{X}/S} \to T_{\mathcal{X}} \to f^{-1} T_S \to 0$$

edge homomorphism: $T_{s_0} S \to H^1(X, T_X)$ equals ρ.

compute: take $\left.\frac{\partial}{\partial s}\right|_{s_0}$ any tangent vector. take C^∞ lift to $\mathcal{X}$:

$$\frac{\partial}{\partial s} + b^\alpha(z) \frac{\partial}{\partial z^\alpha} \quad \text{wrt. local coor. } (z, s).$$

then $\bar\partial\left(\frac{\partial}{\partial s} + b^\alpha \frac{\partial}{\partial z^\alpha}\right)\Big|_X = \frac{\partial b^\alpha}{\partial \bar z^\beta} d\bar z^\beta \cdot \frac{\partial}{\partial z^\alpha}$

is a global T_X-valued $(0,1)$ form on X and is $\bar\partial$ closed.

Here: take horizontal lift wrt. $\omega_{\mathcal{X}}$ (it is suffi: that $\omega_{\mathcal{X}}$ is positive definite in vertical direction)

5. Weil - Petersson Metrics:

Approach: $T_s S \xrightarrow{\sim} H^1(X, T_X)$

and L^2-inner product for $A^{p,q}$ forms with value in T_X (or $\Lambda^r T_X$).

the minimum is achieved for har. representatives.

$$\left\|\frac{\partial}{\partial s}\right\|^2_{WP} := \int \left\|A\left(\tfrac{\partial}{\partial s}\right)\right\|^2 g\, dV.$$

where $A\left(\frac{\partial}{\partial s}\right)(z)$ har. repr of $\rho\left(\frac{\partial}{\partial z}\right)$.

classical case: of curves.

$\mu \frac{\partial}{\partial z} d\bar{z}$, $g|dz|^2 = ds^2_{\text{hyp}}$. metric

Christelffel symbol: only one $\Gamma^z_{zz} = \frac{1}{g} \cdot \frac{\partial g}{\partial z}$

μ harmonic $\Leftrightarrow$ $\bar{\mu} g = \varphi$ is holomorphic:

since $\frac{\partial \varphi}{\partial \bar{z}} = \overline{\frac{\partial}{\partial z}(\mu \cdot g)} = \overline{g\left(\frac{\partial \mu}{\partial z} + \mu \frac{1}{g} \frac{\partial g}{\partial z}\right)}$

$= \overline{g \mu_{;z}}$ cov. derivative

$= 0$ iff μ harmonic.

$\mu = \frac{\bar{\varphi}}{g}$ $\varphi\, dz^2$ hol. quad. differential.

$\sim T_S^\vee = H^0(X, \Omega_X^{\otimes 2})$,

WP: $(\varphi, \psi) = \int \frac{\varphi \cdot \bar{\psi}}{g^2} \cdot g\, dV$

compare to infinitesimal Teichmüller metri

$$\|\varphi\|^2_{\text{Teich}} = \int |\varphi|\, dV.$$

#

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§5. Weil-Petersson metrics:

$$\begin{array}{ccc} f^{-1}(s) = \mathcal{X}_s & \hookrightarrow & \mathcal{X} \\ \downarrow & & \downarrow f \text{ proper smooth} \\ \{s\} & \hookrightarrow & S \end{array}$$

$(\mathcal{X}_s, \omega_{\mathcal{X}_s})$ distinguished metric

Kodaira-Spencer map: $\rho : T_s S \longrightarrow H^1(\mathcal{X}_s, T_{\mathcal{X}_s})$

L^2 inner product on $A^{(0,1)}(\mathcal{X}_s, T_{\mathcal{X}_s})$:

$$\left\|\frac{\partial}{\partial s}\right\|^2_{WP} := \int_{\mathcal{X}_s} \|A(z)\|^2 g\, dV$$

$$= \int A^{\alpha}_{s\bar\beta} A^{\bar\delta}_{\bar s\gamma}\, g^{\bar\beta\gamma} g_{\alpha\bar\delta}\, g\, dV$$

A_s har. repr of $\rho\left(\frac{\partial}{\partial s}\Big|_s\right)$.

cases: I: $\omega_{\mathcal{X}_s} \in \lambda_{\mathcal{X}_s}$ (polarized), $Ric(\omega_{\mathcal{X}_s}) = 0$

II$^{\mp}$: $Ric(\omega_{\mathcal{X}_s}) = \mp\omega_{\mathcal{X}_s}$.

$\omega^n_{\mathcal{X}_s} = \det(g_{\alpha\bar\beta})\, dV =: g\, dV$

$\omega_{\mathcal{X}_s} = \sqrt{-1}\, g_{\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^\beta$, $\omega_{\mathcal{X}}$ (1,1) form on $\mathcal{X}$ st. $\omega_{\mathcal{X}}|_{\mathcal{X}_s} = \omega_{\mathcal{X}_s}$.

computes

$g_{\alpha\bar\beta}$, $g_{s\bar\beta}$, $g_{\alpha\bar s}$, $g_{s\bar s}$ or

$g_{\alpha\bar\beta}$, $g_{i\bar\beta}$, $g_{\alpha\bar j}$, $g_{i\bar j}$ wrt. $dz^\alpha, d\bar z^\beta$ fiber, $ds^i, d\bar s^j$ base

$$\begin{array}{c} (z,s) \in \mathcal{X} \\ \downarrow f \\ s \in S \end{array}$$

$z = (z_1, \cdots, z_n)$
$s = (s_1, \cdots, s_m)$

let $V = \frac{\partial}{\partial s} + a^\alpha_s \frac{\partial}{\partial z^\alpha}$ horizontal lift of $\frac{\partial}{\partial s}$ wrt. $\omega_{\mathcal{X}}$.

ie. $0 = \left(V, \frac{\partial}{\partial \bar z^\beta}\right) = g_{s\bar\beta} + a^\alpha_s g_{\alpha\bar\beta}$. $V \perp \mathcal{X}_s$

$V = \frac{\partial}{\partial s} - g^{\bar\beta\alpha} g_{s\bar\beta}$.

$\bar\partial V|_{\mathcal{X}_s} \in \mathcal{A}^{(0,1)}(\mathcal{X}_s, T\mathcal{X}_s)$ repr $\rho(\frac{\partial}{\partial s}|_s)$.

$\bar\partial V|\mathcal{X}_s = a^\alpha_{s;\bar\beta} =: A^\alpha_{s\bar\beta}$

(in here ";" covariant derivative on fiber)

Prop : (i) $\bar\partial(A^\alpha_{s\bar\beta} \frac{\partial}{\partial z^\alpha} dz^{\bar\beta}) = 0$

(ii) $A_{s\bar\beta\bar\delta} = A_{s\bar\delta\bar\beta}$

(iii) $\bar\partial^*(A^\alpha_{s\bar\beta} \frac{\partial}{\partial z^\alpha} d\bar z^\beta) = 0$

(harmonicity).

Pf of (iii) :

$-\bar\partial^* A_s = g^{\bar\beta\gamma} A^\alpha_{s\bar\beta;\gamma}\, \partial_\alpha$

$(*) \quad = -g^{\bar\beta\gamma} g^{\bar\delta\alpha} g_{s\bar\delta;\bar\beta\gamma} = -g^{\bar\beta\gamma} g^{\bar\delta\alpha}(g_{s\bar\delta;\gamma\bar\beta} - g_{s\bar\tau} R^{\bar\tau}_{\bar\delta\bar\beta\gamma}$

$\underset{(\mathrm{II}^-)}{=} -g^{\bar\beta\alpha}(g^{\bar\delta\gamma} \frac{\partial}{\partial s} g_{\bar\delta\gamma})_{\bar\beta} + g_{s\bar\delta}\, g^{\bar\delta\alpha}$

$- g^{\bar\beta\alpha}(\frac{\partial}{\partial s} \log g)_{\bar\beta} + g_{s\bar\delta}\, g^{\bar\delta\alpha} = 0$

$g_{s\bar\beta}$ by def. of $\omega_{\mathcal{X}} = \sqrt{-1}\,\partial\bar\partial \log g$.

(I) : Add assumption $H^1(\mathcal{X}_s, \mathcal{O}_{\mathcal{X}_s}) = 0$

($\mathrm{Aut}^{\lambda\mathcal{X}}(x)$ compact if moduli space exists this further assumption make Aut finite'

on $\mathcal{X}$, $\eta = 2\pi c(\mathcal{X}/s, g)$, g her. metric : $-K_{\mathcal{X}/}$

$\eta_{\alpha\bar\beta} = R_{\alpha\bar\beta} = 0$ (with $*$)

$g^{\bar\delta\gamma} A^\alpha_{s\bar\beta;\gamma} = -g^{\bar\beta\alpha} \eta_{s,\bar\beta}$

$\eta_{s\bar\beta} dz^{\bar\beta}$: $\bar\partial$ closed (0,1) form.

$$\bar\partial^* \eta_{s\bar\beta}\, d\bar z^\beta = -\eta_{s\bar\beta;\alpha}\, g^{\bar\beta\alpha} = -\frac{\partial}{\partial s}\left(\eta_{\alpha\bar\beta}\right) g^{\bar\beta\alpha}$$

$\equiv 0$ (Ricci flat)

$\eta_{s\bar\beta}\, d\bar z^\beta$ harmonic $\Rightarrow\ = 0$.

Fiber Integrals:

- Fact: $\mathcal{X} \to S$ smooth proper (S not nec. smooth)

ω_j (1,1) form on $\mathcal{X}$ with local $\partial\bar\partial$ potential, C^∞

$n = \dim_{\mathbb{C}} \mathcal{X}_s$.

$$\int_{\mathcal{X}/S} \omega_1 \wedge \cdots \wedge \omega_{n+1} = \omega_S : (1,1) \text{ form on } S$$

$= \sqrt{-1}\,\partial\bar\partial\varphi$ locally φ C^∞ fcn.

$X \times S$, α a $(2n+2k)$-form:

$\int_{X\times S/S} \alpha$ is a $2k$-form, $\omega^{n+1} = \frac{1}{(n+1)!}\underbrace{\omega\wedge\cdots\wedge\omega}_{n+1}$

Theorem: $\mathrm{Ric}(\omega_{\mathcal{X}_s}) = -\omega_{\mathcal{X}_s}$. then

$$\omega_{WP} = \int_{\mathcal{X}/S} \omega_{\mathcal{X}}^{n+1}.$$

Application: ω_{WP} is Kähler (also for S singular)
construction compatible with Base change
without lose of generality ass. $\dim S = 1$.

$n = 1$: (Wolpert)

pf: $\varphi = \| V(z) \|^2 = \| \frac{\partial}{\partial s} + a^\alpha_s \frac{\partial}{\partial z^\alpha} \|^2 (z)$

$$= g_{s\bar s} - g_{s\bar\beta}\, g_{\alpha\bar s}\, g^{\bar\beta\alpha}$$

$\omega_{\mathcal{X}}^{n+1} = \varphi(z)\, g\, dV \cdot \sqrt{-1}\, ds\wedge d\bar s$

because $\det\begin{pmatrix} g_{s\bar s} & g_{s\bar\beta} \\ g_{\alpha\bar s} & g_{\alpha\bar\beta}\end{pmatrix} = \varphi(z)\, g(z)$.

Lemma: $(\Box + id)(\varphi) = \| A^{\alpha}_{s\bar\beta}(z) \|^2$.

Application: $\| \frac{\partial}{\partial s} \|^2_{WP} = \int \| A_s \|^2 g\, dV$

$= \int (\Box \varphi + \varphi) g\, dV = \int \varphi g\, dV \Rightarrow$ Thm.

pf of lemma:

$$g_{s\bar s} = \frac{\partial^2}{\partial s \partial \bar s} \log g = \frac{\partial}{\partial s}\left(g^{\bar\tau\sigma} \frac{\partial}{\partial \bar s} g_{\sigma\bar\tau} \right)$$

$$= - g^{\bar\tau K} g^{\bar\lambda\sigma} g_{s\bar\lambda ; K}\, g_{\sigma\bar s, \bar\tau} + g^{\bar\tau\sigma} g_{s\bar s; \bar\tau\sigma}$$

on the other hand,

$$\Box\varphi = - g^{\bar\tau\sigma} g_{s\bar s, \bar\tau\sigma} + g^{\bar\beta\alpha} g^{\bar\tau\sigma} (g_{s\bar\beta}\, g_{\alpha\bar s})_{;\bar\tau\sigma}$$

$$= (\text{——} \cdots \text{——}) + g^{\bar\beta\alpha} g^{\bar\tau\sigma} (g_{s\bar\beta, \bar\tau\sigma}\, g_{\alpha\bar s}$$

$$+ g_{\alpha\bar\beta, \bar\tau}\, g_{\alpha\bar s, \sigma} + g_{s\bar\beta, \sigma}\, g_{\alpha\bar s, \bar\tau} + g_{s\bar\beta}\, g_{\alpha\bar s, \bar\tau\alpha}$$

$$= (\text{——} \cdots \text{——}) + \underset{\text{A har.}}{\underset{\uparrow}{0}} + A_s \cdot A_{\bar s} + (\text{——} \cdots \text{——})$$

$$+ \underset{\text{K-E condition}}{0} + g_{s\bar\beta}\, g_{\alpha\bar s}\, g^{\bar\beta\alpha}$$

$$= -\varphi + \| A \|^2 . \quad ☆$$

the K-E condi just eliminates all bad term

Ricci-flat case:

$\omega_{\mathcal{X}}$ as above, $\eta_{\mathcal{X}} = 2\pi c_1(\mathcal{X}/S, g)$, $\omega_{\mathcal{X}}|_{X_s}$

Theorem:
$$\omega_{WP} = \int_{\mathcal{X}/S} \eta_{\mathcal{X}} \wedge \omega_{\mathcal{X}}^n .$$

(with no extra assumptions).

pf: Assuming $H^1(\mathfrak{X}_s, \omega_{\mathfrak{X}_s}) = 0$.

(*) $\eta_{\alpha\bar\beta} = R_{\alpha\bar\beta} = 0$

$\eta_{s\bar\beta}\, dz^{\bar\beta} = 0$ (see above), $(\eta_{\alpha\bar s} = 0)$.

lemma: $\Box\varphi = \eta_{s\bar s} + \|A_s\|^2(z)$.

pf similar with $\varphi = g_{s\bar s} - g_{s\bar\beta}\, g_{\alpha\bar s}\, g^{\bar\beta\alpha}$.

pf thm: $\displaystyle\int \|A_s\|^2 g\, dV \underset{\text{lemma}}{=} -\int \eta_{s\bar s}\, g\, dV$

$\displaystyle\underset{(*)}{=} \pm\int \eta\wedge\omega_{\mathfrak{X}}^{n}$.

General case needs a little more work. ※

6. Curvature of WP metric:

Restrict to case $k = \mp 1$: $Ric(\omega_{\mathfrak{X}_s}) = k\,\omega_{\mathfrak{X}_s}$

$\omega_{WP} = \sqrt{-1}\, G^{WP}_{i\bar j}(s)\, ds^i \wedge d\bar s^j$

$$G^{WP}_{i\bar j} = \int_{\mathfrak{X}_s} A^{\alpha}_{i\bar\beta} A^{\bar\delta}_{\bar j\gamma}\, g^{\bar\beta\gamma} g_{\bar\delta\alpha}\, g\, dV$$

$$\underset{\substack{\text{symmetry}\\ A_{\bar\beta\bar\delta} = A_{\bar\delta\bar\beta}}}{=} \int_{\mathfrak{X}_s} A^{\alpha}_{i\bar\beta} A^{\bar\beta}_{\bar j\alpha}\, g\, dV$$

Need to compute: $\dfrac{\partial G_{i\bar j}}{\partial s^k}$, $\dfrac{\partial^2 G_{i\bar j}}{\partial s^{\bar l}\partial s^k}$.

The idea is "Direct".

$$\begin{tikzcd} \mathfrak{X} \arrow[d] & X\times S \arrow[l, "\Phi"', "\sim C^\infty"] \arrow[dl] \\ S & \end{tikzcd}$$

"$\displaystyle\frac{\partial}{\partial s}\int_{\mathfrak{X}_s}\Omega = \frac{\partial}{\partial s}\int_X \Phi^*\Omega = \int_X \frac{\partial}{\partial s}\Phi^*\Omega$

$\displaystyle= \int_X \Phi^*(L_{\tilde v}\Omega) = \int_{\mathfrak{X}_s} L_{\tilde v}\Omega$"

$\tilde v = \frac{\partial}{\partial s}$ on X, $L_{\tilde v}$ Lie derivatives.

Basic: B tensor $\Rightarrow L_v B$ tensor

(0) $L_v(B) = [v, B]$

(1) $L_v h = v(h)$

(2) BC contraction: $L_v(BC) = L_v(B)C + B L_v(C)$

$$v = \frac{\partial}{\partial s} + a_s^\alpha \frac{\partial}{\partial z^\alpha}$$

$v_i = \partial_i + a^\alpha_i \partial_\alpha$, $i = 1 \dots m$, $\alpha = 1 \dots n$

Formulas:

(1) $L_{v_i}(g_{\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^\beta) = A_{i\bar\beta\bar\delta}\, d\bar z^\beta \wedge d\bar z^\gamma \;(=0)$

$L_{v_i}(g_{\alpha\bar\beta}) = A_{i\bar\beta\bar\delta}$

i.e. L_v not type preserving.

(just for curiosity: let's compute

$$\begin{aligned} L_{v_i}(g_{\alpha\bar\beta})_{\alpha\bar\beta} &= [\partial_i + a^\gamma_i \partial_\gamma, g_{\alpha\bar\beta}]_{\alpha\bar\beta} \\ &= \partial_i g_{\alpha\bar\beta} + a_i^\gamma g_{\alpha\bar\beta,\gamma} + a^\gamma_{i,\alpha} g_{\gamma\bar\beta} \\ &= g_{i\bar\beta,\alpha} - g_{i\bar\beta,\alpha} = 0 . \end{aligned}$$)

in particular, $L_v(g\, dV) = 0$.

(2) $L_{v_k}(A^\alpha_{j\bar\beta}\, \partial_\alpha\, d\bar z^\beta) = (v_k, v_{\bar j})^{j\bar\beta}{}_\alpha\, \partial_{\bar\beta}\, dz^\alpha$

$$- (A^\gamma_{k\bar\beta} A^{\bar\beta}_{\bar j\alpha})\, \partial_\gamma\, dz^\alpha + (A^{\bar\beta}_{\bar j\alpha} A^\alpha_{i\bar\delta})\, \partial_{\bar\delta}\, d\bar z^\beta$$

From (2) + $\bar\partial^* A = 0$.

$$\partial_k G^{WP}_{i\bar j} = \int L_{v_k}(A_i) \cdot A_{\bar j}\, g\, dV = 0$$

(4) $L_{v_k}(A_i) = L_{v_i}(A_k)$ ($\Rightarrow$ Kähler property)

This is the only method for proving Kähler property without using fiber integral

(Koiso, etc ... diff language for this)

$$\partial_{\bar\ell}\partial_k G^{WP}_{i\bar j} = \int \left(L_{V_{\bar\ell}} L_{V_k} A\right)\cdot A_{\bar j}\, g\, dV$$

$$+ \; L_{V_k}(A_i)\cdot L_{V_{\bar\ell}}(A_{\bar j})\, g\, dV$$

Need

(5). $[V_{\bar j}, V_k] = -(V_k, V_{\bar j})^{;\alpha}\partial_\alpha + (V_k, V_{\bar j})^{;\bar\beta}\partial_{\bar\beta}$

for any function h,

$[h^{;\alpha}\partial_\alpha, A^{\alpha}_{i\bar\beta}] = h^{;\gamma} A^{\alpha}_{i\bar\beta,\delta} - h^{;\alpha}_{\gamma} A^{\gamma}_{j\bar\beta}$

Put together, (5) $\Rightarrow$

(6) $\displaystyle\int \left(L_{[V_{\bar\ell}, V_k]} A_i\right)\cdot A_{\bar j}\, g\, dV$

$$= -\int (V_k, V_{\bar\ell})\cdot \Box(A_i - A_{\bar j})\, g\cdot dV$$

$\Rightarrow \partial_{\bar\ell}\partial_k G^{WP}_{i\bar j} = \frac{\partial}{\partial s_k}\int L_{V_{\bar\ell}} A_i \cdot \overset{=0}{A_{\bar j}} - \int L_{V_{\bar\ell}} A_i \cdot L_{V_k} A_{\bar j}$

$$-\int (V_k, V_{\bar\ell})\,\Box(A_i, A_{\bar j}) + \int (L_{V_k} A_i)(L_{V_{\bar\ell}} A_{\bar j}).$$

Use $(\Box - k)(V_k, V_{\bar\ell}) = A_k\cdot A_{\bar\ell}$

Try to invert this. Let

$Q := \Box(\Box - k)$.

$Qf = f_{,\alpha\gamma\bar\beta\bar\delta}\, g^{\bar\beta\alpha} g^{\bar\delta\gamma}$

Kernel $\equiv 0$ (as $\mathrm{Aut}^{\gamma}(X)$ is compact)

$\Rightarrow$ no eigen fcn in $(0, k]$ for $\Box$.

$$\Rightarrow \partial_{\bar\ell}\partial_k G^{WP}_{i\bar j} = -k\int (\Box - k)^{-1}(A_i A_{\bar j})\,(A_k\cdot A_{\bar\ell})$$

$$-k\int (\Box - k)^{-1}(A_i\cdot A_{\bar\ell})(A_k\cdot A_{\bar j})$$

$$+\int L_{V_k}(A_i)\, L_{V_{\bar\ell}}(A_{\bar j}) \;\underline{+}\;\text{''}\int (A_i A_{\bar j})(A_k\cdot A_{\bar\ell})$$

$$+\int A_i A_{\bar j} A_k A_{\bar\ell} - \int (A_i A_{\bar\ell})(A_k\cdot A_{\bar j}) + \int A_i A_{\bar\ell}\cdot A_k A_{\bar j}\text{''}.$$

"—" is in fact $\int (A_i \wedge A_k)(A_{\bar j}\wedge A_{\bar\ell})$.

$\exists$ more direct computation?

$$
\begin{aligned}
R_{i\bar{j}k\bar{\ell}} = \; & k \int (\Box - k)^{-1} (A_i, A_{\bar{j}})(A_k, A_{\bar{\ell}}) \\
& + k \int (\Box - k)^{-1} (A_i, A_{\bar{\ell}})(A_k, A_{\bar{j}}) \\
& + k \int (\Box + k)^{-1} \underbrace{(A_i \wedge A_k)}_{A^{(0,2)}(\Lambda^2 T_X)} \cdot (A_{\bar{j}} \wedge A_{\bar{\ell}}) \, .
\end{aligned}
$$

contri
comes
dim X =

☺ : Will be no more computations.

(to be continued)

G. Schumacher 7/26

Analytic Theory of Moduli Spaces and Weil-Petersson Geometry IV.

§7. Moduli of Canonically polarized varieties and the W-P form:

Riemann-Roch thm:

- $\dim X = 1$, D divisor,

$h^0(X, \mathcal{O}_X(D)) - h^1(X, \mathcal{O}_X(D)) = 1 - g + \deg D$

$\iff \chi(X, \mathcal{O}_X(D)) = c_1(\mathcal{O}_X(D)) + \frac{1}{2} c_1(X)$.

- $\dim X = 2$: Noether's formula:

$\chi(X, \mathcal{L}) = \frac{1}{2}\left(c_1^2(\mathcal{L}) + c_1(\mathcal{L}).c_1(X)\right) + \frac{1}{12}\left(c_1^2(X) + c_2(X)\right)$

(or evaluated at $[X]$).

- General formula: (Hirzebruch)

$\chi(X, \mathcal{L}) = \left(td(X).ch(\mathcal{L})\right)[X]$.

td: polynomial in Chern classes.

$T_0 = 1$, $T_1 = \frac{1}{2} c_1$, $T_2 = \frac{1}{12}(c_1^2 + c_2)$, $T_3 = c_2.c_1$ etc ...

When $\mathcal{L}$. $rk = 1$, $ch(\mathcal{L}) = \exp(c_1(\mathcal{L}))$

get prev. formula easily.

Chern Character: $ch(\mathcal{F} \oplus \mathcal{G}) = ch(\mathcal{F}) + ch(\mathcal{G})$

$ch(\mathcal{F})_0 = rk\, \mathcal{F}$

$ch(\mathcal{F} \otimes \mathcal{G}) = ch(\mathcal{F}).ch(\mathcal{G})$

More generally for $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$

$ch(\mathcal{F}) = ch(\mathcal{F}') + ch(\mathcal{F}'')$. (as ch is a top'l notion?)

For $\mathcal{F} \in Coh(X)$, finite ~~tor dim~~ resolution

$$0 \to P_k \to P_{k-1} \to \cdots \to P_0 \to \mathcal{F} \to 0$$

P_j locally free,

$ch(\mathcal{F}) := \sum_{j=0}^{\infty} (-1)^j \, ch(P_j)$, is consistent.

Next step: bounded complex of coh. sheaves $\mathcal{F}^\cdot$

$ch(\mathcal{F}^\cdot)$ well-defined.

Grothendieck HRR Thm:

$f: X \to Y$ morphism

"$f_! : K(X) \to K(Y)$"

$\mathcal{F} \mapsto Rf_* \mathcal{F}$

represented in Y by a bounded complex $\mathcal{H}^\cdot$ of locally free sheaves. In particular,

$$\mathcal{H}^q(\mathcal{H}^\cdot) = R^q f_* \mathcal{F}$$

$$ch(f_! \mathcal{F}) = ch(\mathcal{H}^\cdot)$$

if $R^q f_* \mathcal{F}$ also locally free, then

$ch(f_! \mathcal{F}) = \sum_{q=0}^{\infty} (-1)^q \, ch(R^q f_* \mathcal{F})$.

In this case,

$$R^q f_* \mathcal{F} \otimes_{\mathcal{O}_Y} \mathbb{C}(y) = H^q(X_y, \mathcal{F}_y)$$

Theorem (G.-H.-R.-R.):

$$ch(f_! \mathcal{F}) = f_*(ch(\mathcal{F}) . td(X/Y))$$

(In general, $ch(f_! \mathcal{F}) . td(Y) = f_*(ch(\mathcal{F}) . td(X)$

where: $f_* : H^*(X, \mathbb{Q}) \longrightarrow H^*(Y, \mathbb{Q})$.

a) In degree 0 :

LHS . $ch(f_! \mathcal{F}) = rk(f_! \mathcal{F})$

$= \sum_{q=0}^{\infty} (-1)^q \, rk(R^q f_* \mathcal{F})$

If $X \to Y = (0)$ is a constant map,

f_* is evaluation over $X = \chi(X, \mathcal{F})$.

RHS . is $(ch(\mathcal{F}) . td(X))[X]$.

b) degree 2 : (which we are interested in),

"$ch(f_! \mathcal{F})$" $= c_1(\det(f_! \mathcal{F}))$

$= \otimes (\Lambda^{max} R^q f_* \mathcal{F})^{(-1)^q}$

In general, define on Y

$(\det \mathcal{F})_y = \otimes (\underbrace{\Lambda^{max} H^q(X_y, \mathcal{F}_y)}_{\text{simply } \mathbb{C}})^{(-1)^q}$

on Y

$\lambda(F) := (\det \mathcal{F})^{-1}$

Analytically, $f_*(ch(\mathcal{F}) . td(X/Y))$

$= \int_{X/Y} ch(\mathcal{F}) . td(X/Y)$

// Rmk : (later) (fiber integration)

Bismut - Gilet - Soulé - GHRR Thm : Kähler case

$(\mathcal{F}, h_F)$ hermitian vector bundle. //

Extend GHRR to formal differences :

$ch(E_1 - E_2) = ch(E_1) - ch(E_2)$, virtual v.b.

<u>But</u> : $\det(E_1 - E_2)$ ($\det(f_!(E_1 - E_2)))$

always a holo. line bundle

$= \det E_1 \otimes (\det E_2)^{-1}$. #

RHS . is $c_1(\det \mathcal{F})$,

i. Kähler case with metric

Assume $\exists$ Kähler g, $\mathcal{F}$ hermitian $h_{\mathcal{F}}$

$H^q(X_y, \mathcal{F}_y)$ carries a metric : L^2 inner product

k, on $(\lambda(F, k))$.

ii. $\left(\int_{\mathcal{X}/\mathcal{Y}} ch(\mathcal{F}, h)\, td(\mathcal{X}/\mathcal{Y}, g)\right)_{(2j)}$ is a (j,j) form on Y.

Unfortunately, these 2 do not agree!

Reason: Short exactness of hol. str. is not exactness of hermitian structure

the "Error term" is represented by analytic torsion

k^Q : Quillen metric on $\lambda(\mathcal{F})$, $k^Q = k\, e^{-\tau/2}$.

Analytic Torsion :

$\tau = \sum (-1)^q \tau_q$, $\tau_q = \zeta_q'(0)$,

$H^q(X_y, \mathcal{F}_y)$ with Laplace $\Box_y = \bar{\partial}^* \bar{\partial} + \bar{\partial}\bar{\partial}^*$

Zeta function : $\zeta(s) = \sum_{\lambda > 0} \frac{1}{\lambda^s}$. $\operatorname{Re} s \gg 0$

this extends to whole $\mathbb{C}$.

Application :

$\mathcal{X} \to S$ family of canonically polarized var.

$K_{\mathcal{X}/S}$, virtual bundle $\qquad n = \dim \mathcal{X}_s$

$$E = \underbrace{\left(K_{\mathcal{X}/S} - K_{\mathcal{X}/S}^{-1} \right)}_{\text{"}rk = 0\text{"}}{}^{\otimes n+1}$$

$$\begin{aligned} ch\, E &= ch\left(K_{\mathcal{X}/S} - K_{\mathcal{X}/S}^{-1}\right)^{n+1} \\ &= \left(\left(1 + c_1 + \frac{c_1^2}{2} + \cdots\right) - \left(1 - c_1 + \frac{c_1^2}{2} - \cdots\right)\right)^{n+1} \\ &= 2^{n+1} c_1 (K_{\mathcal{X}/S})^{n+1} + \cdots \text{ (higher order)} \end{aligned}$$

$$c\left(\det E, h^Q\right) = -\Big(\int_{\mathcal{X}/S} \qquad \Big)_{(2)}$$

$\exists$ inside term is only need $(n+1,n+1)$ part.

$$2^{n+1}\left(c(K_{\mathcal{X}/S})^{n+1} + \cdots\right)\left(1+\tfrac{1}{2}c(\mathcal{X}/S, g) + \cdots\right)$$

$$= -2^{n+1}\int_{\mathcal{X}/S} c\left(K_{\mathcal{X}/S}, g\right)^{n+1},$$ we have seen this,

$c(K_{\mathcal{X}/S}, g) = \omega_{\mathcal{X}}$, $\omega_{\mathcal{X}}|_{\mathcal{X}_s} = \omega_{\mathcal{X}_s}$ K-E form.

So $= -2^{n+1}\int_{\mathcal{X}/S}\omega_{\mathcal{X}}^{n+1} = \omega_{WP}$ on S.

up to a factorial constant

// On the other hand, In Mumford's calculation in order to get ample line bundle on Moduli space he uses only $K_{\mathcal{X}/S}^m$ via high power.

$\lambda_m := \det(f_* \omega_{\mathcal{X}/S}^m)$ and use GRR. eg λ_{12} in dim 1 case. //

$$G = (\mathcal{L} - \mathcal{L}^{-1})(K_{\mathcal{X}/S} - K_{\mathcal{X}/S}^{-1})^n$$

$(\mathcal{X} \to S, \mathcal{L})$ polarized family of Ricci flat mfds,

$\mathrm{ch}(G) = 2^{n+1} c(\mathcal{L}).c(K_{\mathcal{X}/S})^n \;\exists$

same calculation as before $\quad c(\det G, h^Q) = 2^{n+1}\int_{\mathcal{X}/S} c(\mathcal{L})\, \underbrace{c(K_{\mathcal{X}/S})}_{\omega_{\mathcal{X}}}{}^n$

$$= \omega_{WP}.$$

§8. Moduli of Embeded Manifolds (Douady Space). (= Hilbert scheme in alg case.)

Z : cpx mfd (space) with Kähler form ω_Z

$X \subset Z$ submfd, Douady space parametrizes embeded submfds.

Theorem (Douady): Z, $\exists H$ complex space and $\mathcal{X} \hookrightarrow Z \times H$ st. $pr_2|_{\mathcal{X}}$ is proper & smooth

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook] \arrow[dr] & Z \times H \arrow[d, "pr_2"] \\ & H \end{tikzcd}$$

universal property: $\mathcal{Y} \hookrightarrow Z \times S$ then $\exists!\ S \to H$ st. $\mathcal{Y} = \mathcal{X} \times_H S$.

$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook] \arrow[dr] & Z \times S \arrow[d] \\ & S \end{tikzcd}$$

. Weil-Petersson form ω_{WP} on H:

$T_{s_0} H = H^0(X, N_{X/Z}) = \mathrm{Hom}(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_X)$

$Z \supset X \longleftrightarrow s_0$, or $X = V(\mathcal{J})$, $\mathcal{J} \subset \mathcal{O}_Z$, $\mathcal{J}/\mathcal{J}^2$ as an $\mathcal{O}_X$ module.
ideal

Analytically, $\frac{\partial}{\partial s}$ tangent vector

$\mathcal{X} \to S$, lift to $N_{X|\mathcal{X}}$, $\mathcal{X} \subset Z \times S$

project section to Z get section of $N_{X/Z}$.

ω_Z pulled back to $\mathcal{Z} = Z \times_H S$ still positive definite in fibers of $\mathcal{X}$, take C^∞ lift of $\frac{\partial}{\partial s}$ to perpendicular to X wrt. $\omega_{Z \times S}$.

Now, $\omega_{\mathcal{X}} = \omega_{Z \times S}|_{\mathcal{X}}$

in local coordinates: $\omega_{\mathcal{X}} = \frac{\sqrt{-1}}{2} \cdot ($

$$g_{\alpha\bar\beta}\, dz^\alpha\, d\bar z^\beta + g_{s\bar\beta}\, ds\, d\bar z^\beta + g_{\alpha\bar s}\, dz^\alpha\, d\bar s + g_{s\bar s}\, ds\, d\bar s$$

$|v(z)|^2 = g_{s\bar s} - g^{\bar\beta\alpha} g_{s\bar\beta} g_{\alpha\bar s}$

$\|v\|^2_{WP} = \int (\text{------})\, g\, dV$

$\omega_{WP} = \int_{\mathcal{X}/H} (\omega_{Z\times H}|_{\mathcal{X}})^{n+1}$

Algebraic Case :

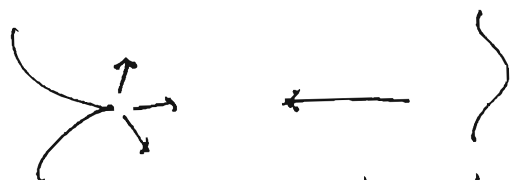
$$\omega_Z = c_1(\mathcal{L}, h)$$

$$\omega_{WP} = -c_1(\lambda, h^Q)$$

$$\lambda = -\det\left((\mathcal{L} - \mathcal{L}^{-1})^{n+1}\right).$$

Notice that : the Kähler form constructed in the Douady space does not comes from a line bundle in general !

Also, the fiber integral in the singular case need to be taken care :

eg. 

singular space has bigger tangent space if do resolution, will "lost important structure", should really see K-E structure in the infinitesimal sense, or in non-integrable sense.

G. Schumacher 8/2 '2000.

Lecture VI : (last one)

Situation: X cpt $\supset D$ smooth hyp. surf.

(N) $K_X + D > 0$, $X' = X - D$

$\omega_{X'}$ complete, $\mathrm{Ric}(\omega_{X'}) = -\omega_{X'}$

Let $\dim_{\mathbb{C}} X = 2$ (which is not really nec.)

$0 < \alpha < 1$. $D = V(\sigma)$

$$g_{\sigma\bar\sigma} = \frac{2}{|\sigma|^2 \log^2\left(\frac{1}{\|\sigma\|^2}\right)}\left(1 + \frac{\tilde g_{\sigma\tilde\sigma}}{\log^\alpha\left(\frac{1}{\|\sigma\|^2}\right)}\right) \quad \in C^{k,\lambda}(X')$$

$$g_{\sigma\bar w}, g_{w\bar\sigma} = \frac{\tilde g_{\sigma\bar w}}{\sigma \log^{1+\alpha}\left(\frac{1}{\|\sigma\|^2}\right)}$$

$$g_{w\bar w} = g^0_{w\bar w}\left(1 + \frac{\tilde g_{w\bar w}}{\log^\alpha\left(\frac{1}{\|\sigma\|^2}\right)}\right), \quad g^0_{w\bar w} \ C^\infty \text{ on } \bar X$$

and on $\underline{C}$
KE tensor.

Rmk: $g^{\sigma\bar\sigma} \sim |\sigma|^2 \log^2\left(\frac{1}{|\sigma|^2}\right)$

$$g^{\sigma\bar w}, g^{w\bar\sigma} = O\left(|\sigma| \log^{1-\alpha}\left(\frac{1}{\|\sigma\|^2}\right)\right)$$

$$g^{w\bar w} = O(1).$$

Surprising Thing: since the assymptotic is strong, Ric in this case gives rise to estimate of curvature tensor:

$$R_{\sigma\bar\sigma\sigma\bar\sigma} = g^2_{\sigma\bar\sigma}\left(1 + O\left(\frac{1}{\log^\alpha\left(\frac{1}{\|\sigma\|^2}\right)}\right)\right)$$

$$R_{w\bar w w\bar w} = g^2_{w\bar w}\left(1 + O(\text{ .... same})\right)$$

$$R_{\_\_} = O\left(\frac{1}{|\sigma|^j \log^{j+\alpha}\left(\frac{1}{\|\sigma\|^2}\right)}\right)$$

$j = \#\ \sigma$'s in $R_{\_}$.

Theorem: $\exists$ nbd of C st. on $U - C$ has hol. sect. curv. bounded from above by $-c < 0$.

Kobayashi Conj: $C \subset \mathbb{P}^2$, $\deg C \geq 5 \Rightarrow \mathbb{P}_2 \setminus C$ hyperbolic for generic C.

or: $f : \mathbb{C} \to \mathbb{P}^2 - C$ holo. $\deg C \geq 4$
$\Rightarrow f$ algebraically deg.

$\mathbb{P}_2$, $\deg C \geq 4$, $K_{\mathbb{P}^2} + C > 0$, $u(C) - C$ hyperbolic

i.e. Partial evidence of the conj.

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§ 10. Moduli of framed Manifolds:

Def: X cpt $\supset C$ smooth hyp. (X, C) is called a framed mfd.

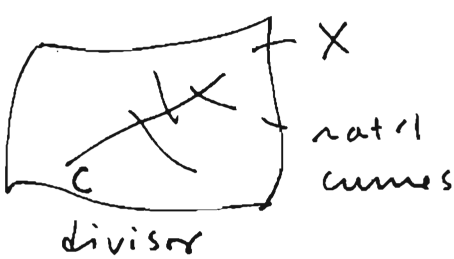
X uniruled over $C \iff$

$$\begin{tikzcd} \mathbb{P}^1 \times Y \arrow[r, dashed, "\text{mero. onto}"] \arrow[d, "pr_Y"'] & X \\ Y & H \arrow[l] \arrow[u, phantom] \end{tikzcd}$$

$H \longrightarrow C$ proper transform

no factorization via Y

st. $pr_Y|_H : H \to Y$ is a modification



generically $\#(C \cap \text{rat curve}) = 1$

Def: (X, C) given, un-uniruled $:=$
$X - C$ non-uniruled, C non-uniruled
(X, C, λ_X) polarized if λ_X pol. for X.

Thm: $\exists\ M_{f,r}$ moduli space of non-uniruled pol. framed mfds.

pf: Statement of Matsusaka - Mumford thm:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\sim"', "\psi"] \arrow[d] & \mathcal{D} \arrow[d] \\ \mathcal{X} \arrow[r, "\varphi"] \arrow[dr, "f"'] & \mathcal{Y} \arrow[d, "g"] \\ & S \end{tikzcd}$$

φ : bimeromorphic map

$(X, C) = f^{-1}(0)$

$(Y, D) = g^{-1}(0)$

" $\varphi|_X : X \to Y$ " meaningless.

$S = \Delta$ $\quad\varphi$ isom over $\Delta^* = \Delta - \{0\}$ by assumption.

$\Gamma = \Gamma_\varphi \subset \mathcal{X} \times_S \mathcal{Y}$

$\Gamma \cap (X \times Y) = \Gamma_0 \cup \Gamma_R$

if (X,C), (Y,D) non-uniruled

$pr_X(\Gamma_0) = X$, $pr_X(\Gamma_R) \subsetneq X$

$pr_Y(\Gamma_0) = Y$, $pr_Y(\Gamma_R) \subsetneq Y$

(pf: Hironaka resol. of singularities)

with polarizations $\Rightarrow$ φ isom.

$M_{fr} \supset \tilde{M}_{fr}$ (where X is not uniruled + $\lambda_X = [C]$)

$$* \to \downarrow \quad (X,C) \mapsto M(X,[C]) \quad \text{surj. map.}$$

Given M, may replace λ_X by a v.a. divisor (Big Matsusaka thm)

Fujita thm: $K_X + (n+2)C$ is ample (>0):

replace C by $(n+2)\cdot C$ and repr by smooth C.

assume $K_X + C > 0$

Def: (X,C) canonically polarized $\Leftrightarrow K+[C]>0$

$M_{fr} \xrightarrow[\text{surj}]{} M$
component

$\{(X,C);\ X$ fixed, $C\subset X$ smooth div.
$[C]=\lambda_X\}$

conversely, $K_X+C>0 \Rightarrow (X,C)$ not uniruled.

pf: $K_C>0 \Rightarrow C$ non-uniruled.

Assume $X\setminus C$ uniruled, $\mathbb{P}_1\times Y \xrightarrow{\varphi} X$ surj. mero.

pick $U\subset Y$ st. $\mathbb{P}_1\times U$ does not intersect indeterminacy set of φ and $(\mathbb{P}_1\times U)\cap H$ does not intersect the exc. set of $H\to Y$.

$$\mathbb{P}_1\times\{y\} \xrightarrow{\sim} \text{image in } X$$

$$(K_X+C)|_{\mathbb{P}_1\times\{y\}}$$

$$\deg(K_{\mathbb{P}_1}+\{y\}) = -2+1=-1 \quad \not\leftarrow$$

* in last page $\hat{M}_{fr}$ would have fiber = moduli space of hyp. surf.
$\downarrow$
M

Next Step: Deformation Theory. (rel.)

$\varphi: X\to Y$ holo. $M\in \mathrm{Coh}(X)$

Want "tangent cohomology" — coherent sheaves

$$T^0(X/Y, M) = \mathrm{Der}_{\varphi^{-1}(\mathcal{O}_Y)}(\mathcal{O}_X, M)$$

$$T^1(X/Y, M) = \mathrm{Ext}^1_{\varphi^{-1}(\mathcal{O}_Y)}(\mathcal{O}_X, M) \quad (\neq \mathrm{Ext}^1$$

which parametrizes

$$0 \to M \to \mathcal{A} \to \mathcal{O}_X \to 0$$

$$\begin{tikzcd} & \mathcal{A} & \mathcal{O}_X \\ & & \varphi^{-1}\mathcal{O}_Y \arrow[ul] \arrow[u] \end{tikzcd}$$

$\mathcal{A}$ $\mathbb{C}$-alg on X.

$T^1(X/Y, \mathcal{O}_X)$ infinitesimal def of X over Y.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\ Y \arrow[r] \arrow[d] & Y \times D \arrow[d] \\ 0 \arrow[r] & D \end{tikzcd}$$

$T^q(\varphi)$ here X and Y are deformed.

$\varphi := i : C \hookrightarrow X$

$\mathcal{T}^0(C/X, \mathcal{O}_C) = 0$

$\mathcal{T}^1(C/X, \mathcal{O}_C) = N_{C/X}$

$\mathcal{T}^q(C/X, \mathcal{O}_C) = 0 \quad \forall q > 1$.

$$0 \to T^0(i) \to T^0(X) \to H^0(C, N_{C/X})$$

$T^0(X) = H^0(X, T_X)$ $\qquad T^1(C/X) = H^0(C, N_{C/X})$.

$$\to T^1(i) \to T^1(X) \to H^1(C, N_{C/X})$$

$\Omega_X^1(\log C)$ generated by $\frac{dz^1}{z_1}, dz^2, \dots, dz^n$, $C = V(z_1)$

$\Omega_X^1(\log C)^\vee$ gen. by $z_1 \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z^2}, \dots, \frac{\partial}{\partial z^n}$

$T_X(-C)$:

$$0 \to T_X(-C) \to \Omega_X^1(\log C)^\vee \to T_C \to 0$$

$$0 \to \Omega_X^1(\log C)^\vee \to T_X \to (N_{C/X})^* \to 0$$

leads into long exact seq of tangent coh.

$$T^q(i) = H^q(X, \Omega_X^1(\log C)^\vee)$$

$K_X + C > 0$ + Kodaira-Nakano vanishing thm

$\Rightarrow T^0(i) = 0$.

$$0 \to H^0(X, T_X) \to H^0(C, N_{C/X}) \to T^1(i) \to T^1(X)$$

$T^1(i) = H^1(X, \Omega_X^1(\log C)^\vee)$

Hilbert space technique : $X' = X \setminus C$.

$\omega_{X'} \sim \sqrt{-1}\left(\dfrac{dz_1 \wedge d\bar{z}_1}{|z|^2 \log^2(|z'|^2)} + dz_2 \wedge d\bar{z}_2 + \cdots \right)$

$L^{0,q}_{(2)}(X', \mathcal{F}) = \{ (0,q)\text{-form } \alpha,\ \alpha, \bar{\partial}\alpha \text{ square int'l}$

$\rightsquigarrow$ sheaves $\mathcal{L}^q$ on X

Andreotti - Vesentini, Zucker :

$$0 \to \mathcal{F} \to \mathcal{L}^0 \to \mathcal{L}^1 \to \cdots \mathcal{L}^n \to 0$$

Cor : $H^0_{(2)}(X', \mathcal{T}_{X'}) = H^0(X, \Omega_X^1(\log C)^\vee)$

$H^q_{(2)}(X', \mathcal{T}_{X'}) = H^q(\text{——})$.

Another important part of A-V :

the formal adjoint $\bar{\partial}^*$ is the adjoint.

(No prob. about ∞ - boundary, Stokes's thm)

(Based on Hörmander's L^2 - thm)

Pointwise Estimate :

Banach Space Technique.

$\omega_{X'}$ Kähler - Einstein, also in family

$C^{k,\lambda}(X')$ - spaces (with parameter), solution of

$C^{k,\lambda}(X')$ - tensor,

$w, z^2, \cdots, z^n$ quasi - coord. $\dfrac{d\sigma}{\sigma \log \sigma} = \dfrac{2}{1-w^2}\, dw$, $|w| \le R <$

- Use implicit function thm with parameter to get variation of KE metric tensor wrt. parameter

$\rightsquigarrow C^{k,\lambda}(X')$ - tensors.

Kodaira - Spencer forms :

$V = \dfrac{\partial}{\partial s} + a^\alpha_j \dfrac{\partial}{\partial z^\alpha}$ $\omega_{X'}$ - horizontal lift

$\bar\partial v|_X = A^\alpha_{s\bar\beta}\, \frac{\partial}{\partial z^\alpha}\, d\bar z^\beta$, $C^{k,\lambda}(X')$ - tensor

Fact: A bounded $C^{k,\lambda}(X')$ tensor $\Rightarrow A \in L^2$.

Furthermore, $\bar\partial^* A = 0$, so

$A^\alpha_{s\bar\beta}\, \partial\alpha\, d\bar z^\beta$ harmonic K-S form.

$\Rightarrow$ M_{fr} carries generalized W-P metric,

$$\int_{\mathfrak{X}/S} \omega_{\mathfrak{X}}^{n+1} = \text{const. } \omega^{WP}$$

Also curvature form formula:

$$R^{WP}_{i\bar j k \bar l} = -\int(\Box+1)^{-1}(A_i\cdot A_{\bar j})(A_k\cdot A_{\bar l})\, g\, dV$$

$$-\int(\Box+1)^{-1}(A_i\cdot A_{\bar l})(A_k\cdot A_{\bar j})\, g\, dV$$

$$+\int(\Box-1)^{-1}(A_i\wedge A_k)\cdot(A_{\bar j}\wedge A_{\bar l})\, g\, dV$$

the contribution of positivity can only comes from the harmonic part.

THEOREM: $X = X_0$ fixed, moduli space (X_0, C) with $K_{X_0} + C > 0$ is hyperbolic.

Pf: hol. sect. curv. of ω_{WP} is $\leq \dfrac{-2}{\mathrm{Vol}(X_0)}$:

$$0 \to H^0(X, T_X) \to H^1(C, N_{C/X}) \xrightarrow{c} T^1(X) \to T^1(X)$$

needs to show $c(A)\wedge c(A) = 0$ in $H^{0,q}_{(2)}(X', \Lambda^2 T_{X'})$ (bec. it is harmonic) and is true.

Now $\int(\Box+1)^{-1}(A\cdot\bar A)(A\cdot\bar A)\, g\, dV$

$$= \sum \frac{1}{1+\lambda_\nu}\,\psi_\nu\bar\psi_\nu \geq \int \psi_0\bar\psi_0\, g\, dV.$$

$(A\cdot\bar A) = \sum\psi_\nu$, $\Box(A\cdot\bar A) = \sum\lambda_\nu\psi_\nu$

$$\int_X (A\cdot\bar A)\, g\, dV = \lambda_0\cdot\mathrm{vol}(X) = G^{WP}_{s\bar s}$$

$$\int \psi_0 \bar{\psi}_0 = \lambda_0^2 \operatorname{vol}(X) = \frac{(G_{ss}^{wp})^2}{\operatorname{vol}(X)}$$

So the main result is:

$$\begin{tikzcd} \mathcal{M}_{fr}, \omega_{wp} \arrow[d] \\ \mathcal{M} \end{tikzcd}$$

fiber →

(understandable) $\approx \int \omega_{wp}^{N+1}$ only along fibers.

This may eventually help to understand $\mathcal{M}$!?

<u>End</u>.