Qile Chen (Columbia)

2013 TIMS Mini workshop in GW

lect 1 log geom. 7/8

△ monoids

Def. A monoid P is a semi-gp comm

$\exists!$ $0 \in P$ i.e. $0 + a = a$

Ex 1. $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{N}^n$

. $P = \sigma^\vee$ σ rat'l convex cone in $\mathbb{Z}^n$ $\to$ toric

if σ is strongly convex $\Rightarrow$ $\sigma^\vee = P \supset P^* = \{0\}$

call such P sharp

All these are good monoid.

Def'n: P is integral if $P \mapsto P^{gp}$ is injection

($\Leftrightarrow$ cancelation law)

Ex 2: Bad monoids

$P = \langle a, b, c \mid a+b = a+c \rangle$ not int

Def'n: P is saturated if

($\forall a \in P^{gp}$, $na \in P$ for some n $\Rightarrow$ $a \in P$)

Ex 3 $\{0, 2, 3, 4, \cdots\} \subset \mathbb{N}$ not saturated

Def'n: P is fine if P is f.g and int.

P is f.s. if fine & saturated.

Ex 4 P is f.s $\Leftrightarrow$ $P \hookrightarrow P^{gp}$ ~~and~~ $P = (P^\vee)^\vee$ $\Leftrightarrow$ P is toric

a finite rk lattice

△ log str / $\mathbb{C}$.

Def'n M_X is log str / $\underline{X}_{et}$ if

. M_X is a sheaf of monoids

. $\alpha : M_X \to \mathcal{O}_{\underline{X}}$ monoid under mult.

st $\alpha^{-1}\mathcal{O}_X^* \xrightarrow{\sim} \mathcal{O}_X^*$

Ex. $D \hookrightarrow X$
closed subsch

$M_X = \{f \in \mathcal{O}_X \mid f|_{X \setminus D} \in \mathcal{O}^*\} \xrightarrow{\alpha} \mathcal{O}_X$

Rmk : M_X is bad, unless (X, D) is locally toroidal.

So far people only can handle loc. toroidal, otherwise even cannot define smoothness

Def^{n}: $\beta : P \to M_X$ is a chart if "formally" it consists of all non-invertible sections.

ie $\dfrac{P \oplus \mathcal{O}_X^*}{\beta^{-1}\mathcal{O}_X^*} \simeq M_X$

- A log str is coherent if locally charts exist.

Ex $\underline{X} = \operatorname{Spec} k[P]$, P polytope, D = toric boundary

$\rightsquigarrow M_X \rightsquigarrow X = (\underline{X}, M_X)$ called a log scheme denoted by $\operatorname{Spec}(P \to k[P]) = A_P$

charts

$$\begin{tikzcd} P \arrow[rr] \arrow[dr, "\beta"'] & \circlearrowleft & k[P] \\ & M_X \arrow[ur] & \end{tikzcd}$$

To do GW, need the notion of smoothness

$X = (\underline{X}, M_X)$ $\quad P \xrightarrow{\beta} M_X \xrightarrow{\alpha} \mathcal{O}_X$

$\Rightarrow k[P] \longrightarrow \mathcal{O}_X \Rightarrow \underline{A}_P \xleftarrow{\varphi} \underline{X}$

in fact, $\varphi^* M_{A_P} = M_X$.

Def^{n}: X is log sm. if φ is smooth (locally) in the usual sense.

Smoothness of a morphism

$f : X \to Y$ $\quad$ locally take charts

$\underline{f} : \underline{X} \to \underline{Y}$

$f^b : f^* M_Y \longrightarrow M_X$

$$\begin{tikzcd} f^* M_Y \arrow[rr, "f^b"] \arrow[dr, "f^*\alpha_Y"'] & \circlearrowright & M_X \arrow[dl, "\alpha_X"] \\ & \mathcal{O}_X & \end{tikzcd}$$

$$\begin{tikzcd} P \arrow[r] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowright"] & M_Y \arrow[d, "f^b"] \\ Q \arrow[r] & M_X \end{tikzcd} \rightsquigarrow \begin{tikzcd} A_P & Y \arrow[l] \\ A_Q \arrow[u] \arrow[ur, phantom, "\circlearrowright"] & X \arrow[l] \arrow[u] \end{tikzcd}$$

get
$$\begin{tikzcd} A_P & Y \arrow[l] \\ A_Q \arrow[u] & Y \times_{A_P} A_Q \arrow[l] \arrow[u] & \\ & & X \arrow[ul, "\Psi"] \arrow[uul, bend right] \arrow[ull, bend left] \end{tikzcd}$$

Def: f is log sm (ét) if

① ker and torsion of coker of h^{gp} is finite ~~(ét)~~ (ét ⟺ ker, coker of h^{gp} is finite.)

② Ψ is smooth (ét) in the usual sense.

To define differential, need characteristic of M_X.

$\overline{M}_X := M_X/\mathcal{O}_X^*$ ~ combinatorics of M_X

Lemma: ét locally near $x \in X$

$\exists$ a chart $\beta: P := \overline{M}_X \to M_X$ st: $P \xrightarrow{\beta} M_{X,x} \to \overline{M}_{X,x} = P$ is id

(Under the assumption that x is good.)

Rmk: This gives a canonical choice of P.

$$\begin{matrix} G_P \circlearrowright A_P \xrightarrow{\Psi} [A_P/G_P] = \mathcal{A}_P \\ \| \\ \mathrm{Spec}(k[P^{gp}]) \end{matrix}$$

a toric stack with log str

· M_{A_P} descends to $M_{\mathcal{A}_P}$

all boundary are stable under the G_P action

· Ψ is strict, $\Psi^* M_{\mathcal{A}_P} = M_{A_P}$

· Ψ is sm in the usual sense

It's good to use this Artin stack

A chart $P \to M_X \iff X \to A_P$

(*) $\exists$ A global map $P \to \overline{M}_X$ locally lifts to a chart $\iff$
$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & A_P \arrow[d, "\Psi"] \\ X \arrow[r] & \mathcal{A}_P \end{tikzcd}$$

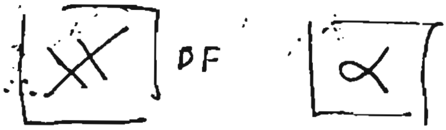
"canonical locally"

Def M_X is called Deligne-Faltings (DF) if satisfies (*)

p.4

Ex If M_X is coming from SNC divisor then M_X is PF

Rmk: Not true for NC



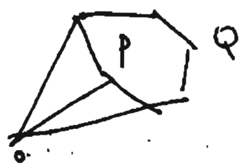
• X proj toric

$T \hookrightarrow X \rightsquigarrow \underline{X}\setminus T =: \partial X$

$(\underline{X}, \partial X) \rightsquigarrow X = (\underline{X}, M_X)$

claim: M_X is PF

Pf: $\underline{X} \rightsquigarrow Q$ $\quad \underline{X} = \mathrm{Proj}(k[P])$



$$\begin{tikzcd}
(U, M_U) \arrow[r, "\text{str}"] \arrow[d, "\text{complement of inv sat}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{A}_P \arrow[d] \\
X \arrow[r, "\text{strict}"] & \mathcal{A}_P
\end{tikzcd}$$

Observation. $x \in X$; $x \in U$.

$P = \overline{M}_{X,x} \longrightarrow M_X$ - descend $P \to \overline{M}_X$ locally lifts to chart over $V \ni x \rightsquigarrow V \to \mathcal{A}_P$ "canonical" at x

1st Def of differential $\quad X$ log scheme

$\Omega_X :=$ sheafification of $\Omega_{U/\mathcal{A}_P}$

X is log sm $\Leftrightarrow U \to \mathcal{A}_P$ sm $\Rightarrow \Omega_X$ is locally free

• Olsson's stack $/\mathbb{C}$: very big stack, highly non-separated but OK

All we need is the étale cover

$$\begin{array}{l} \mathcal{L}og(\underline{S}) = \{\text{f.s log str over } \underline{S}\}/\sim \\ \quad \big\downarrow \text{ fiber cat.} \\ Sch_{\mathbb{C}} \end{array}$$

Thm (Olsson): $\mathcal{L}og$ is an alg stack

• ~~étale~~ smooth covering $\bigcup_{P \text{ toric}} \mathcal{A}_P \xrightarrow{\text{ét}} \mathcal{L}og$

$$\begin{array}{ccc} x \in V & \longrightarrow & \mathcal{A}_p \\ & \searrow \quad \circlearrowleft \quad \swarrow^{pt} & \\ & \mathcal{L}og & \end{array}$$

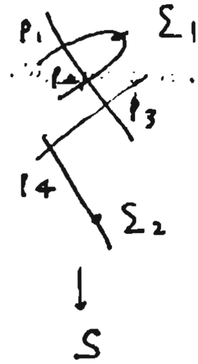
$\Omega_{V/\mathcal{A}_p} = \Omega_{V/\mathcal{L}og}$

2nd def'n $\Omega_X := \Omega_{X/\mathcal{L}og}$.

Δ log curve.

Ex Smoothing of a node:

$$\begin{array}{ccccc} \mathcal{A}_{\mathbb{N}^2}\langle x,y\rangle & x\cdot y & \log x + \log y & \mathbb{N}^2 & C \\ \downarrow & \uparrow & \uparrow & \downarrow & \downarrow \\ \mathcal{A}_{\mathbb{N}}\langle t\rangle & t & \log t & \mathbb{N} & pt \end{array}$$



$\mathbb{N}^{\#\text{ nodes}} = \mathbb{N}^4 \langle t_1, t_2, t_3, t_4 \rangle$
$\rightsquigarrow$ sm parameters.

$\overline{M}_C = \bigoplus_{i=1}^{4} \overline{M}_{p_i} \oplus \bigoplus_j M_{\Sigma_j}$ canonical log str

Def'n Such a log str is the canonical log str on curves

Properties
1) uniquely determined by underlying str.
2) forms an open family

Underlying stack $\underline{C}_{g,n} \supset \overbrace{\pi^{-1}\Delta \sqcup \bigcup_{i=1}^{n} \Sigma_i}^{N.C.} \rightsquigarrow \mathcal{C}_{g,n} = (\underline{C}_{g,n}, M_{\underline{C}_{g,n}})$

$\pi \downarrow$

$\underline{M}_{g,n} \ni \Delta$ = (locus of sing fiber)
NC-divisor

$\rightsquigarrow \mathcal{M}_{g,n} = (\underline{M}_{g,n}, M_{\underline{M}_{g,n}})$

gives the canonical log str of the universal family

Def $C \to S$ is a log curve if

$$\begin{array}{ccc} C & \longrightarrow & C_{can} \\ \downarrow & \lrcorner & \downarrow \\ S & \longrightarrow & S_{can} \end{array}$$ with can log str

$$\begin{array}{l} \exists\ \mathcal{M}_{g,n}(S) = \{ (g,n) \text{ log curves} \ldots \text{ over } S \} \\ \qquad \Big\downarrow \ \text{fiber cat} \\ \qquad \mathrm{Log\,Sch} \end{array}$$

- Summary: log curve
  → canonical log str

good properties
1) universality (uniquely det. by underlying)
2) openness

o Log map X : an f.s log. sch

Def A (g,n)-log map is

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\pi"'] & X \\ S & \end{tikzcd}$$

st $C \xrightarrow{\pi} S$ is a (g,n) log-curve

: It is stable if $\underline{f}$ is stable in the usual sense.

Arrows

$$\begin{tikzcd} C_2 \arrow[r] \arrow[d] \arrow[rr, bend left, "f_2"] \arrow[dr, phantom, "\circlearrowleft"] & C_1 \arrow[r, "f_1"'] \arrow[d] & X \\ S_2 \arrow[r] & S_1 & \end{tikzcd}$$

$M_{g,n}(X,\beta)(S) := \{ (g,n,\beta) - \text{stable log map} / S \} / \sim$

$\downarrow$ - f.b. cat when S is an arbitrary f.s log sch

Log Sch

Theorem (Abramovich, —, Gross, Seibert)

(1) $M_{g,n}(X,\beta)$ is rep by a DM stack with a nat log str

(2) If $\underline{X}$ is proj. then $M_{g,n}(X,\beta)$ is proper

(3) There is an open and closed decomp

$$M_{g,n}(X,\beta) = \coprod_{\Gamma} M_\Gamma(X)$$

$\Gamma = \{ g, n, \beta, \{c_i\}_{i=1}^n \}$

for log str from div, there are tangency conds along each div

Idea of the proof

0. Minimality What does the underlying space parametrize?

For each

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\pi"'] & X \\ S & \end{tikzcd}$$

get (1) construct

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C_m \arrow[r, "f_m"] \arrow[d] \arrow[dr, phantom, "*"] & X \\ S \arrow[r, "g"] & S_m & \end{tikzcd}$$

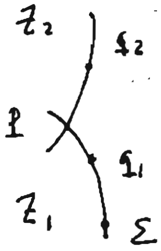
minimal model

(2) $((\nu), g)$ is unique

Δ Minimality $X \leadsto (\underline{X}, D)$

log δ $\langle \delta = 0 \rangle$

1.2 $\overline{M}_{X,x} = \begin{cases} \mathbb{N} & x \in D \\ 0 & x \notin D \end{cases}$



> not markings

$\begin{array}{ccc} C & \xrightarrow{f} & X \\ \pi \downarrow & & \\ S & & \end{array}$ $\quad \underline{S} = pt$

locally near node p.

at q_i, $\overline{f}^b : f^* \overline{M}_X \to \overline{M}_C = \pi^* \overline{M}_S$ $\quad \langle \sigma \geq 0 \rangle$

$\log \delta \mapsto e_i$ $\quad$ what's the meaning ?

Note $e_i = 0 \Leftrightarrow f(Z_i) \not\subset D$. (**)

at Σ : $\overline{f}^b : \overline{M}_X \to \overline{M}_C = \pi^* \overline{M}_S \oplus \mathbb{N}$

$\delta \mapsto e_\Sigma + c \cdot \log \sigma$

Note: 1. $e_\Sigma = e_1$

2. $c \in \mathbb{Z}_{\geq 0}$ is the tangency

The most interesting part is at node p :

$\overline{f}^b : \overline{M}_X \to \overline{M}_C = \overline{M}_S \oplus_{\mathbb{N}} \mathbb{N}^2$ $\quad \langle \log x, \log y \rangle$

$\langle e_p \rangle$ $\quad e_p = \log x + \log y$

$\delta \mapsto e' + c_p \log x$

Note: 1 $e' = e_1$

2. $e_2 = e_1 + c_p \cdot \log x$ $\quad$ (i.e. h_p : $e_2 = e_1 + c_p e_p$)
(*)

G the marked gp (graphs)

1) $\underline{G}$ dual graph of $\underline{C}$

2) for each $v \in V(G) \rightsquigarrow e_v$

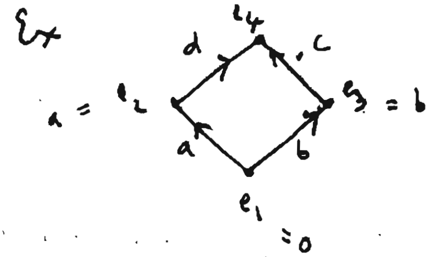
3) for each $l \in E(G) \rightsquigarrow e_l$

$\overline{M}(G) = \langle e_v, e_l \mid (*), (**) \rangle^{sat} / \text{torsion}$

Lemma: $\exists !$ $\overline{M}(G) \xrightarrow{\varphi} \overline{M}_S$:

Def $\begin{array}{c} C \to X \\ \downarrow \\ S \end{array}$ is minimal (or called Basic in Gross-Siebert) if φ is an isom

Ex

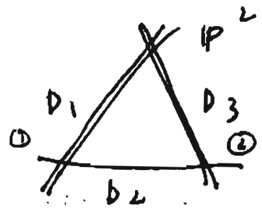


$\overline{M}(G) = \langle a, b, c, d, \underbrace{e_1, e_2, e_3, e_4}\rangle$

$e_2 = e_1 + 1 a$ with unique rel

$a + d = b + c$

Ex



G the marked graph

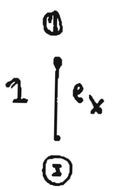
1) $\underline{G}$ dual graph of $\underline{C}$

2) for each $v \in V(G)$ ~> e_v

3) for each $\ell \in V(G)$ ~> e_ℓ

$D = \bigcup_{i=1}^{n} D_i$

$e_{v_1} \dots e_{v_i} \dots e_{v_k}$



$\langle e_1, e_3 \rangle^{\circ}$ (e_ℓ)

$\uparrow \quad \downarrow$

$\langle e_1', e_3' \rangle$

$\quad \shortparallel \quad \shortparallel$

$\quad 0 \quad\; e$

$\Rightarrow \overline{M}(G) = \mathbb{N} = \langle e_\ell \rangle$

for GW, the SNC case are all we need to considered.

Assume X with M_X, ~> D sm

Prop Minimality is an open condition.

Pf:

$$\begin{array}{ccc} C & \longrightarrow & X \\ \downarrow & & \\ S & & \end{array}$$

min at $s \in S$

shrink S, $\beta : \overline{M}(G_s) = \overline{M}_s \; S \to M_S$ is a chart.

marked graph of fibers is

Take $s \neq t \in S$,

$K_t := \{ a \in \overline{M}(G_s) \mid \beta(a) \text{ is inv at } t \}$.

$\overline{M}(G_s) \to (\underbrace{\overline{M}(G_s) + K_t^{gp}}_{\overline{M}}) \to \overline{M}/K_t^{gp} =: P$. sat. sharp

Construct G from G_s:

1) If $\varphi(e_\ell) = 0$ then contract ℓ

2) If $\varphi(e_v) = 0$ then set $e_v = 0$ in G

$\Big\} \Rightarrow$ marked graph G

Check ① $G = G_t$

shrink S further ② $\overline{M}(G) = P = \overline{M}_{t,S}$

$\Rightarrow$ nearby fiber still minimal

✕

The universality is a bit more complicated.

p. 4

given $\begin{matrix} C \xrightarrow{f} X \\ \pi \downarrow \\ S \end{matrix}$. $\exists$ a (*) $\begin{matrix} C_m \xrightarrow{f_m} X \\ \downarrow \\ S_m \end{matrix}$ with $S \xrightarrow{g} S_m$ st

(1) $\underline{g} = id_{\underline{S}}$, (2) diagram, (3) ((*), g) unique up to unique iso

Pf. Step 1 Assume $C_m \xrightarrow{\pi} S_m$, $S \longrightarrow S_m$ st.

1) $\overline{M}_{S_m, S} = \overline{M}(\mathcal{O}_S)$

2) $g^b_S : \overline{M}(\mathcal{O}_S) = \overline{M}_{S_m, S} \xrightarrow{\varphi} M_S$ is minimal map

3) $\begin{matrix} C & \longrightarrow & C_m \\ \downarrow & \circlearrowleft & \downarrow \\ S & \longrightarrow & S_m \end{matrix}$ Then

$$\begin{tikzcd} C \arrow[r] \arrow[d] \arrow[rr, bend left, "\exists !"] & C_m \arrow[r] \arrow[d] & X \\ S \arrow[r] & S_m \end{tikzcd}$$

Pf : $\begin{tikzcd} & f^*M_X \arrow[dl, "f_m"'] \arrow[dr, "f^b"] & \\ M_{C_m} \arrow[rr, "g_C"'] & & M_C \end{tikzcd}$ / C

observe ① $f^b_m(\delta)$ exist and unique

② $f^b_m(\delta)_1 = f^b_m(\delta)_2 + \log u$, $u \in \mathcal{O}^\times$

$\Rightarrow g_C(u) = 1 \Rightarrow$ lifting is unique

Step 2. $\beta : \overline{M}_{S,s} \longrightarrow M_S$

$\beta' : \overline{M}(\mathcal{O}_S) \to \overline{M}_{S,s} \xrightarrow{\beta} M_S \longrightarrow \mathcal{O}_S$

$M_{S_m} = \overline{M}(\mathcal{O}_S) \oplus_{\beta^{-1}\mathcal{O}^\times} \mathcal{O}^\times \longrightarrow M_S$

$S_m = (\underline{S}, M_{S_m}) \xleftarrow{g} S$

Note : $S \xrightarrow{g} S_m$ depends on β

but two different choices of lifting diff by a unique iso

Step 3. $\begin{tikzcd} & M^{\prime S} \arrow[dl, dashed, "\exists !"'] \arrow[dr] & \\ M_{S_m} \arrow[rr, "g"'] & & M_S \end{tikzcd}$

same as in step 1 #

Thm $\begin{matrix} LM \\ \downarrow \\ Sch \end{matrix} = \left\{ \begin{matrix} C \longrightarrow X \\ \downarrow \\ S \\ \downarrow \\ \underline{S} \end{matrix} \right\} / \sim$, $\mathcal{M}_{g,n}(X, \beta) \underset{\text{open}}{\hookrightarrow} LM$

Qile Chen 7/10

lecture 3 on log geometry / log GW

Δ Finiteness. Let $X = (\underline{X}, D)$

Thm $\mathcal{M}_{g,n}(X,\beta) \longrightarrow \mathcal{M}_{g,n}(\underline{X},\beta)$ is representable and finite / $\mathbb{C}$

(True also for $\mathcal{M}^{p.s.}$ pre-stable pt. Same pf but / clump not quite true)

pf: φ. quasi-finite: study fibers of φ

(ξ: log map / S, $\underline{\xi}$ underlying prestable map)

- φ: $\mathrm{Aut}_S(\xi) \xrightarrow{\sim} \mathrm{Aut}_{\underline{S}}(\underline{\xi})$
- φ satisfies valuative criterion (weak val)

quasi-finite: suffice to check for geo fiber

$$\begin{array}{ccc} \underline{\xi}: & \underline{C} \xrightarrow{f} \underline{X} \\ & \downarrow \\ & \underline{S} \quad pt \end{array} \qquad \begin{array}{c} \xi: C \xrightarrow{f} X \\ \downarrow \pi \\ S \end{array}$$

~~trivial mark~~ G

a log map over $\underline{\xi}$ minimal with marked graph G

$\overline{N}(G) \subset \overline{M}(G)$

$\overline{N}(G) = \langle e_\ell, e_v \rangle$

(fine)

$\overline{N}(G)$: finite gen int.

$\overline{N}(G)^{sat} = \overline{M}(G)$

Fix a chart:

$\beta \cdot \overline{M}(G) \to M_S$

observe 1. $\beta' : \overline{N}(G) \to \overline{M}(G) \xrightarrow{\beta} M_S$

$\Rightarrow$ $N_S = \overline{N}(G) \oplus_{(\beta')^{-1}\mathcal{O}^*} \mathcal{O}^* \subset M_S$ <u>sub log str</u>

does not depend on β.

$\Rightarrow$ $S_C = (\underline{S}, N_{S_C})$ fine, not saturated in general

2 $\exists!$ factorization

$$M_{\underline{S}}^{C/S} \longrightarrow M_S$$

$$\exists! \searrow N_{S_C} \nearrow$$

canonical log str of curves on $\underline{S}$

$\rightsquigarrow$ do base change:

$$\begin{array}{ccc} C_C & \longrightarrow & C_{can} \\ \downarrow & \square & \downarrow \\ S_C & \longrightarrow & S_{can} \end{array}$$

log curve with <u>fine</u> log str

only have to characteristic factors through

p. 2

$$\exists\quad \begin{tikzcd} \underline{f}^* M_X \arrow[rr, "f^b"] \arrow[dr, "\exists !"'] & \circlearrowleft & M_C \\ & M_{C_c} \arrow[ur, dashed] & \end{tikzcd} \qquad (2,3) \Rightarrow \begin{tikzcd} C \arrow[r, "f_c"] \arrow[d, "\xi_c"'] & X \\ S_c & \end{tikzcd} \qquad \underline{S} \xrightarrow{\underline{f}} \underline{X}$$

Def: ξ_c is called coarse map of ξ (ξ_c uniquely dep on ξ)

Lemma: Let ξ_1, ξ_2 be 2 diff log map / $\underline{\xi}$ with the same G, then $\xi_{1C} \simeq \xi_{2C}$

pf:

$$\begin{tikzcd} & N_{1C} \arrow[d, dashed, "h", "?"'] & \\ f^* M_X \arrow[ur] \arrow[dr] & & M_{\underline{C}}^{C/S} \arrow[ul] \arrow[dl] \\ & N_{2C} & \end{tikzcd}$$

It's enough to construct the map h

It requires a local argument (explicit)

$\Rightarrow h \Rightarrow$ descent to $N_{1S} \xrightarrow{\sim} N_{2S}$.

Lemma There are finite many log maps with fixed $(\underline{\xi}, G)$

pf: $\xi/(\underline{\xi}, G) \Rightarrow \exists!$ coarse map ξ_C

then get all others by saturation

$$\begin{tikzcd} \xi' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \xi_C \arrow[d] \\ S' \arrow[r] & S_C \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \xi' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \xi_C^{sat} \arrow[d] \\ S' \arrow[r] & S_C^{sat} \end{tikzcd}$$

$$\text{sat} \Rightarrow \begin{tikzcd} S' \arrow[rr] \arrow[dr] & \circlearrowleft & S_C \\ & S_C^{sat} \arrow[ur, "\text{fine}", "\text{finite}"'] & \end{tikzcd}$$

($S' \to S_C^{sat}$ has choices, may not unique)

Rmk: In general it's very difficult to compute the degree in saturation, but for GW theory, the virtual cycle always involve only very small saturation, hence can be computed.

Lemma $\mathrm{Aut}_S(\xi) \simeq \mathrm{Aut}_{\underline{S}}(\underline{\xi})$.

$$\begin{matrix} C \to X \\ \downarrow \\ S \\ \downarrow \\ \underline{S} \end{matrix}$$

pf $\mathrm{Aut}_{\underline{S}}(\underline{\xi}) \simeq \mathrm{Aut}_{\underline{S}}(\xi_C) \simeq \mathrm{Aut}_{\underline{S}}(\xi)$

This "proves" the representability

Valuative criterion

$\underline{S} = \operatorname{Spec} R$ $\quad R$: DVR , u : uniformizer

$$\begin{tikzcd} f^b(\delta) \quad C_\eta \arrow[r, hook] \arrow[d] \arrow[rr, bend left, "f_\eta"] & C \arrow[r, dashed, "f \ \exists ?"] \arrow[d] & X \\ \eta \arrow[r, hook] & S & \end{tikzcd}$$

$f^b(\delta) = a$ length $(\cdots)$
with $\underline{S} \to \underline{S}$ fixed, but the underlying log str can be changed

Sketch of the pf. $(\delta) = \overline{M}_X$, Extend $f^b(\delta)$ to the central fiber.

* with no poles $\rightsquigarrow$ extension exists
$\rightsquigarrow$ extension as minimal obj exists

notice that in classical approach we do blow-up to do the ext but now we simply adjust the log str on S

* with zeros $\rightsquigarrow$ controlled by minimality ✡

product structure

$\underline{X}$ variety , $D = \bigcup_{i=1}^{k} D_i$ SNC

$(\underline{X}, D_i) \rightsquigarrow X_i$ log scheme , $(\underline{X}, D) \rightsquigarrow X$ log scheme

then by definition: $X = X_1 \times_{\underline{X}} \cdots \times_{\underline{X}} X_k$

Fix $\Gamma_i = (g, n, \beta, \{c_{ij}\}_{j=1}^{n})$
$\Gamma = (g, n, \beta, \bigcup_i \{c_{ij}\}_{j=1}^{n})$

eg  or even 

Cor : $\mathcal{M}_\Gamma(X) = \mathcal{M}_{\Gamma_1}(X_1) \times_{\mathcal{M}_{g,n}(\underline{X},\beta)} \cdots \times_{\mathcal{M}_{g,n}(\underline{X},\beta)} \mathcal{M}_{\Gamma_k}(X_k)$

with can. min log str

In fact

$X = \varprojlim X_i \Rightarrow \mathcal{M}(X) = \varprojlim \mathcal{M}(X_i)$.

pf Denote $\mathcal{M}' = $ RHS then

$S \to \mathcal{M}'$ "$\Leftrightarrow$" $\begin{tikzcd} C \arrow[r] \arrow[dr] \arrow[d] \arrow[ddr] & X_1 \arrow[r] & \underline{X} \\ S & X_2 \arrow[ur] & \\ & \vdots & \\ & X_k \arrow[uuur] & \end{tikzcd}$ "$\Leftrightarrow$" $\begin{tikzcd} C \arrow[r] \arrow[d] & X \\ S & \end{tikzcd}$ "$\Leftrightarrow$" $S \to \mathcal{M}_\Gamma(X)$

" " means still need to check all tangencies are compatible ✡

p.4 △ Birational modification

Thm $Y \xrightarrow{g} X$ is a log ét bir'l modification of smooth log schemes. (i.e. locally a toric blow-up)

$\beta, g(\beta)$ discrete inv of Y, X resp.

Then ① $h : M_\beta(Y) \longrightarrow M_{g(\beta)}(X)$

② $h_* [M_\beta(Y)]^{vir} = [M_{g(\beta)}(X)]^{vir}$

pf. Say, $Y \longrightarrow X$ all log ét modification takes this form

simplified to the case

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "\text{strict}"] \\ \mathcal{Y} \arrow[r] & \mathcal{A}_P \\ & A_P \arrow[u] \end{tikzcd}$$

← call this $\mathcal{X}$

consider a thin stack:

$$M(\mathcal{Y}/\mathcal{X}) = \left\{ \begin{array}{c|ccc} \text{log} & C & \longrightarrow & \mathcal{Y} \\ \text{ét} & \downarrow & & \downarrow \\ & \bar{C} & \longrightarrow & \mathcal{X} \end{array} \right\} / \sim$$

get

$$\begin{tikzcd} M_\beta(Y) \arrow[r, "h"] \arrow[dd, bend right] & M_{g(\beta)}(X) \arrow[d] \\ & M(\mathcal{X}) \\ M(\mathcal{Y}) \arrow[ur] & \end{tikzcd}$$

however, $\boxtimes$

can't compare the perfect obs th.

so insert $M(\mathcal{Y}/\mathcal{X})$ into it

by Costello,

$h_* [M_\beta(Y)] = [M_{g(\beta)}(X)]$ ⟸

$$\begin{tikzcd} M_\beta(Y) \arrow[r, "h"] \arrow[d] \arrow[dr, phantom, "\square"] & M_{g(\beta)}(X) \arrow[d] \\ M(\mathcal{Y}/\mathcal{X}) \arrow[r] \arrow[d, "\text{ét}"] \arrow[r, phantom, bend right, "\hookrightarrow"] & M(\mathcal{X}) \\ M(\mathcal{Y}) \arrow[ur] & \end{tikzcd}$$

DM type of pure deg 1

log sm :

not log ét

← log sm

∗

△ Decomposition formula

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d, "\text{log sm flat proj}"] \\ 0 \in & B \end{tikzcd}$$

curve. log str by 0.

$$\begin{tikzcd} M_G \arrow[r] \arrow[d, "\text{split}"'] & M_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & M_{g,n}(X/B, \beta) \arrow[d] \\ J_G \arrow[r] \arrow[d, "\psi"'] & \mathrm{Log}_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Log}\,B \arrow[d] \\ e_v \in \mathbb{N} & 0 \arrow[r, phantom, "\in"] & B \end{tikzcd}$$

Olsson stack

the decomposition formula asks

$\mathrm{Log}_0 \leftarrow \bigsqcup J_G$

G splitting graph

Def G is called splitting if $\bar{m}(G) = \mathbb{N}$

$\psi = \bigsqcup_G \psi_G$

$\bar{\psi}_G^b|_{\text{generic}} : \mathbb{N} \longrightarrow \mathbb{N}$

$1 \longmapsto n_G$

n_G is hard to describe but easy in SNC case

Thm (ACGS, forthcoming paper)

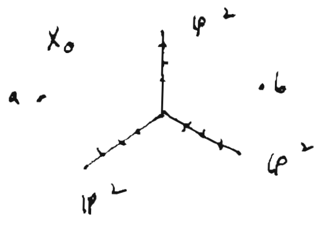
$$[M_0] = \sum_{G \text{ splitting}} n_G [M_G]$$

Ex X_y general cubic surface

$\langle pt, pt \rangle_{0,2}^{X_y, \beta} = 12$

β : plane rational cubic.

How: $\begin{matrix} X_0 & \to & X \\ \downarrow & & \downarrow \\ 0 & \in & B \end{matrix}$



but this is not log smooth since has base locus

$12 = 9 + 3$

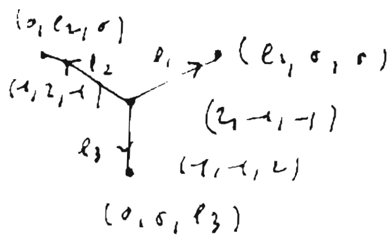
from base pt

from plane a, b, any pt not with the degenerate cubic

e.g. 

graph G

+ tangency



variable (a, b, c)

$\bar{m}(G) = \mathbb{N}$ splitting graph

n_G comes from writing $\mathbb{N} = \mathbb{N}\langle\pi\rangle$.

$\ell_1 = \ell_2 = \ell_3 = 3\pi \Rightarrow n_G = 3 \longrightarrow$ this gives $12 = 9 + 3$

$a = b = c = \pi$, $\ell_1 = \ell_2 = \ell_3 = \pi$

Rmk: the graph looks like tropical graph
n_G looks like weight in tropical graphs

End